An Aspect of Quasi-Invariant Measures on $\mathbb{R}^\infty$

By

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Introduction

The study of translationally quasi-invariant measures $\mu$ on an infinite-dimensional vector space is essentially different from the same study on a finite-dimensional vector space. In the finite-dimensional case, we can characterize $\mu$ as the Lebesgue measure modulo equivalence of absolute continuity. However, in the infinite-dimensional case the situation is more complicated and difficult, owing to the fact that there exist many extremal measures. For example, in a rigged Hilbert space $E \subset H \subset E^*$, we can construct various kinds of ergodic quasi-invariant measures which are singular with respect to each other. See [6]. So we want to study an aspect of quasi-invariant measures. As a special but essential case we shall here discuss the translationally quasi-invariant measures on $\mathbb{R}^\infty$ which are of the type of countably infinite products of one-dimensional probability measures. The main result is a characterization of $l^2$-quasi-invariant measures in terms of its second moment. The author thanks Prof. H. Yoshizawa for the many valuable comments and thanks Prof. Y. Yamasaki for his useful suggestions.

§1. Preliminary Discussions

Throughout this paper, we shall only consider probability measures which are defined on the usual Borel $\sigma$-field $\mathcal{B}(\mathbb{R}^\infty)$ on $\mathbb{R}^\infty$. First we shall prepare some basic concepts and theorems for our later discussions. Let $\mu$ be a measure on $\mathcal{B}(\mathbb{R}^\infty)$ and $t=(t_1, t_2, \ldots, t_n, \ldots) \in \mathbb{R}^\infty$. We define a transformed measure $\mu_t$ such that,
\[ \mu(A) = \mu(A - t), \quad \text{for all } A \in \mathcal{B}(\mathbb{R}^\infty). \]

**Definition 1.1.** A measure \( \mu \) is called \( t \)-quasi-invariant, if and only if \( \mu_t \asymp \mu \) holds.

Here the symbol \( \asymp \) means the equivalence relation of the absolute continuity. We put \( T_\mu = \{ t \in \mathbb{R}^\infty | \mu_t \asymp \mu \} \).

**Definition 1.2.** Let \( \Phi \) be a subset of \( \mathbb{R}^\infty \). A measure \( \mu \) is called \( \Phi \)-quasi-invariant (strictly-\( \Phi \)-quasi-invariant), if and only if \( \mu^c = T_\mu (\mu = T_\mu) \) holds, respectively.

From now on, the measure \( \mu \) is always assumed to be the product-measure of one-dimensional probability measures. More exactly,

\[ d\mu(x) = \bigotimes_{j=1}^\infty f_j(x_j)dx_j, \quad x = (x_1, x_2, \ldots) \in \mathbb{R}^\infty, \] where \( f_j(u) > 0 \), for Lebesgue-a.e. \( u \) and \( \int_{-\infty}^\infty f_j(u)du = 1, \ldots, (II) \).

Clearly, \( T_\mu \supset \mathbb{R}^\infty_0 \) holds, where \( \mathbb{R}^\infty_0 = \{ x = (x_1, x_2, \ldots) \in \mathbb{R}^\infty | x_n = 0 \text{ except finite numbers of } n \} \). \( T_\mu \) forms an additive group, but does not necessarily form a vector space. We shall give a counter example for it in the last part of this section.

Now let \( f_n(u) \) be as in (II). Then \( \sqrt{f_n} \in L^2_{\mu}(\mathbb{R}^1) \), which is the class of all square summable functions with respect to the one-dimensional Lebesgue measure \( du \). Let \( \mathcal{F} \) be the Fourier transformation on \( L^2_{\mu}(\mathbb{R}^1) \), \( \mathcal{F}(f)(v) = \int \exp(2\pi i uv)f(u)du \), and we put \( \mathcal{F}(\sqrt{f_n}) = g_n \), for all \( n \). First we shall state a simple criterion for equivalence of measures.

**Theorem 1.1.** Let \( \mu \) be a measure as in (II), and \( d\mu_1(x) = \bigotimes_{j=1}^\infty f_j^1(x_j)dx_j \) be another measure also as in (II). Then in order that \( \mu_1 \asymp \mu \) holds, it is necessary and sufficient that,

\[ \sum_{j=1}^\infty \left\{ 1 - \int_{-\infty}^\infty \sqrt{f_j^1(u)} \sqrt{f_j(u)}du \right\} < \infty. \]

Especially, if \( \mu_1 = \mu_2 \) for some \( t = (t_1, t_2, \ldots) \in \mathbb{R}^\infty \), the above inequality becomes

\[ \sum_{j=1}^\infty \left\{ 1 - \int_{-\infty}^\infty \sqrt{f_j(u)} \sqrt{f_j(u - t_j)}du \right\} < \infty, \quad \text{or equivalently,} \]
\[ \sum_{j=1}^\infty \left\{ 1 - \int_{-\infty}^\infty \exp(2\pi it_j v)|g_j(v)|^2dv \right\} < \infty. \]

**Proof.** The assertion of the theorem is the special case in the
general equivalence criterion in [10]. It is a typical application of the martingale convergence theorem. We omit it.

If \( \mu \) is a measure as in (II) and for all \( n, f_n \) is identical with the same function \( f \), we say that \( \mu \) is a stationary measure with \( f \). Then,

**Proposition 1.1.** Let \( \mu \) be a stationary measure with \( f \). Then we conclude that \( T_\mu \subset l^2 \).

**Proof.** Let \( t = (t_1, t_2, \ldots) \in T_\mu \), and we put \( \mathcal{F}(\sqrt{f}) = g \). Then by the above theorem, we have

\[
\sum_{j=1}^{\infty} \left\{ 1 - \int \exp(2\pi it_j v) |g(v)|^2 dv \right\} < \infty.
\]

First we shall show that \( t_j \to 0 \), as \( j \to \infty \). For it, we put

\[
H(s) = \int (1 - \exp(2\pi isv)) |g(v)|^2 dv, \quad \text{for } s \in \mathbb{R}^1.
\]

Then \( H(0) = 0, \lim_{s \to \pm \infty} H(s) = 1 \) due to the Riemann-Lebesgue theorem, and \( 0 < H(s) < 1 \), for \( 0 < |s| < \infty \). It follows that, for an arbitrary \( \varepsilon > 0 \),

\[\inf_{|s| \geq \varepsilon} H(s) > 0.\]

Suppose \( \{t_j\} \) does not converge to 0. Then there exist some \( \varepsilon_0 > 0 \) and subsequence \( \{t_{j_n}\} \) such that, \( |t_{j_n}| > \varepsilon_0 \), for all \( n \). Consequently,

\[\infty > \sum_{n=1}^{\infty} \int (1 - \exp(2\pi it_{j_n} v)) |g(v)|^2 dv = \sum_{n=1}^{\infty} H(t_{j_n}) = \sum_{n=1}^{\infty} \inf_{|s| \geq \varepsilon_0} H(s) = \infty.\]

We reach to a contradiction.

As \( 1 - \cos(x) = O(x^2) \) at \( x = 0 \), and

\[\infty > \sum_{j=1}^{\infty} H(t_j) = \sum_{j=1}^{\infty} \int (1 - \cos 2\pi t_j v) |g(v)|^2 dv > \sum_{j=1}^{\infty} \int_{-K}^{K} (1 - \cos 2\pi t_j v) |g(v)|^2 dv, \text{for all } K > 0, \text{ so } \sum_{j=1}^{\infty} t_j^2 \int_{-K}^{K} v^2 |g(v)|^2 dv < \infty.\]

If we take \( K \) so large that, \( \int_{-K}^{K} v^2 |g(v)|^2 dv > 0 \), it follows that \( \sum_{j=1}^{\infty} t_j^2 < \infty \).

Now we give an example for the fact that \( T_\mu \) is not necessarily a vector
Example 1.1. We put $f_c(u) = 1/4A_c \left(1 + \cos 2\pi u \right)^2 \exp \left(-4\pi^2 cu^2 \right)$, for a positive constant $c$, where $A_c$ is a normalizing constant such that,$$
 A_c = 1/16 \sqrt{\pi c} \left\{3 + \exp \left(-1/c \right) + 4 \exp \left(-1/4c \right) \right\}.$$Some calculations show that,$$g_c(v) = \mathcal{F} \left(\sqrt{f_c} \right)(v) = 1/4 \sqrt{2\pi cA_c} \left\{\exp \left(-v-1/2c \right) + \exp \left(-v+1/2c \right) + 2 \exp \left(-v^2/2c \right) \right\},$$and that$$\int \exp(iyv)g_c(v)^2 dv = 1/16 A_c \sqrt{\pi c} \exp \left(-cy^2/4 \right) \cos y + 2 \exp \left(-1/c \right) + 4 \exp \left(-1/4c \right) \cos (y/2) \right\}.$$Now we define a measure $\mu$ such that,$$d\mu(x) = \otimes \mathcal{F}_{\xi}(f_{c_j}(x)) dx_j, \text{ where } \{c_j\} \text{ is taken as } \sum_{j=1}^\infty c_j < \infty.$$Then in order that $t = (t_1, t_2, \ldots) \in T_\mu$, it is necessary and sufficient that,$$\sum_{j=1}^\infty 1/B_{c_j} \left\{3 + \exp \left(-1/c_j \right) + 4 \exp \left(-1/4c_j \right) - \exp \left(-9\pi^2 t_j^2 c_j \right) + \exp (-1/c_j) + 4 \exp \left(-1/4c_j \right) \cos (\pi t_j) \right\} < \infty,$$where we put $B_{c_j} = 16 \sqrt{\pi e_j A_{c_j}}$ (hence $\lim B_{c_j} = 3$).As $\sum_{j=1}^\infty c_j < \infty$ (hence $\sum_{j=1}^\infty \exp \left(-1/4c_j \right) < \infty$), so the above inequality is equivalent that,$$\sum_{j=1}^\infty \left\{3 - \exp \left(-\pi^2 t_j^2 c_j \right) \cos (2\pi t_j) \right\} < \infty.$$If we put $t_j = 1$ for all $j$, then it yields$$\sum_{j=1}^\infty 3 \left\{1 - \exp \left(-\pi^2 c_j \right) \right\} \leq \sum_{j=1}^\infty \pi^2 c_j < \infty.$$On the other hand if we put $t_j = 1/2$ for all $j$, then it yields$$\sum_{j=1}^\infty \left\{3 - \exp \left(-\pi^2 c_j/4 \right) \right\} = \infty.$$Therefore we assured that $e = (1, 1, \ldots, 1, \ldots) \in T_\mu$, while $1/2 e \notin T_\mu$.

§2. General Aspect of Quasi-Invariant Measures

In this section, we shall mostly consider the $l^p$-quasi-invariance
(0 < p ≤ ∞) of measures, where $l^p = \{ x = (x_1, x_2, \ldots) \mid \sum_{j=1}^{\infty} |x_j|^p < \infty \}$. It will be turned out that $l^p$-quasi-invariant measures actually exist, but strictly-$l^p$-quasi-invariant measure does not exist except for $0 < p \leq 2$. Further from another criterions (compactness, e.t.c.) we shall see that the case $p = 2$ is a worth special interest.

Let $\mu$ be a measure on $\mathcal{B}(\mathbb{R}^{\infty})$ as in (II), and put $\mathcal{F}(\sqrt{f_n}) = g_n$ for each $n$. Then $\int |g_n(\nu)|^2 d\nu = 1$, due to the Plancherel's theorem, and we can construct a probability measure $\nu$ on $\mathcal{B}(\mathbb{R}^{\infty})$ such that,

$$d\nu(x) = \otimes_{j=1}^{\infty} |g_j(x_j)|^2 dx_j.$$

The measure $\nu$ is called an adjoint measure of $\mu$. Formally $\mu$ and $\nu$ is related as follows. We use the duality bracket $\langle x, y \rangle = \sum_{j=1}^{\infty} x_j y_j$, for $x = (x_1, x_2, \ldots) \in \mathbb{R}^{\infty}$ and for $y = (y_1, y_2, \ldots) \in \mathbb{R}_1^{\infty}$, and $\frac{d\mu}{d\mu}(x)$ means the Radon-Nikodym derivative.

$$\int_{\mathbb{R}^{\infty}} \sqrt{\frac{d\mu(c\mu)}{d\mu}}(x)d\mu(x) = \int_{\mathbb{R}^{\infty}} \exp (2\pi i \langle x, t \rangle) d\nu(x).$$

Now let $a = (a_1, a_2, \ldots) \in \mathbb{R}^{\infty}$, and we set $H_a = \{ x \in \mathbb{R}^{\infty} \mid \sum_{j=1}^{\infty} x_j^2 a_j^2 < \infty \}$. Unless otherwise stated, we fix these symbols. The quasi-invariance of the measure $\mu$ concerns with the smoothness of each function $f_n$, while the support of the measure $\nu$ concerns with the decreasing order at infinity of each function $|g_n|^2$. As Fourier transformation reflects these two properties, we can settle this point in a following lemma.

**Lemma 2.1.** (Fundamental) Let $t = (t_1, t_2, \ldots) \in \mathbb{R}^{\infty}$. Then in order that $ct \in T_\mu$ holds for all $c \in \mathbb{R}^1$, it is necessary and sufficient that $\nu(H_t) = 1$.

**Proof.** First we shall assume that $ct \in T_\mu$, for all $c \in \mathbb{R}^1$. Then we can define a one-parameter unitary group $\{ U_c \}_{c \in \mathbb{R}^1}$ on $L_2^\mu(\mathbb{R}^{\infty})$ such that,

$$U_c: f(x) \in L_2^\mu(\mathbb{R}^{\infty}) \longrightarrow \sqrt{\frac{d\mu(c\mu)}{d\mu}}(x)f(x - ct).$$

$$\int \sqrt{\frac{d\mu(c\mu)}{d\mu}}(x)d\mu(x) = \prod_{j=1}^{\infty} \int \exp (2\pi i c t_j x_j) d\nu(x).$$
holds by the martingale convergence theorem and it is the continuous function of \( c \) by the well known theorem for a one-parameter group. As the each term of the above infinite product is positive for any \( c \), so 
\[
\sum_{j=1}^{\infty} \left\{ 1 - \int \exp\left(2\pi i c t_j x_j\right) dv(x) \right\}
\]
is again a continuous function of \( c \). We integrate it with the normalized Lebesgue measure on \([0, 1]\), then we get
\[
\sum_{j=1}^{\infty} \left( 1 - \frac{\sin \left(\frac{2\pi c t_j x_j}{2}\right)}{2\pi c t_j x_j} \right) dv(x) < \infty.
\]
It follows that
\[
\sum_{j=1}^{\infty} \left( 1 - \frac{\sin \left(\frac{2\pi c t_j x_j}{2}\right)}{2\pi c t_j x_j} \right) < \infty \quad \text{equivalently,} \quad \sum_{j=1}^{\infty} t_j^2 x_j^2 < \infty
\]
holds for \( \nu \text{-a.e.} x = (x_1, x_2, \ldots) \). It shows that \( \nu(H_t) = 1 \). From the above argument, we remark that \( \sum_{j=1}^{\infty} t_j x_j \) converges in law for \( \nu \), and that the independence of each \( x_j \) derives that it converges almost surely for \( \nu \). Conversely, suppose that \( \nu(H_t) = 1 \), then
\[
0 < \int_{\mathbb{R}} \exp \left( - \sum_{j=1}^{\infty} t_j^2 x_j^2 / 2 \right) dv(x) = \lim_{N} \prod_{j=1}^{N} \int \exp \left( - t_j^2 v^2 / 2 \right) |g_j(v)|^2 dv
\]
and therefore,
\[
\sum_{j=1}^{\infty} \left\{ 1 - \int \exp \left( - t_j^2 v^2 / 2 \right) |g_j(v)|^2 dv \right\} < \infty.
\]
Let \( n \) be the one-dimensional Gaussian probability measure with mean 0 and variance 1. Then the above inequality yields,
\[
\sum_{j=1}^{\infty} \left\{ 1 - \exp(it_j u v) \right\}|g_j(v)|^2 dvdu < \infty.
\]
It follows that,
\[
\sum_{j=1}^{\infty} \left\{ 1 - \exp(it_j u v) \right\}|g_j(v)|^2 dv < \infty, \quad \text{for} \ n\text{-a.e.} \ u.
\]
We set
\[
A = \left\{ u \in \mathbb{R}^1 \mid \sum_{j=1}^{\infty} \left\{ 1 - \exp(it_j u v) \right\}|g_j(v)|^2 dv < \infty \right\}.
\]
Then \( n(A) = 1 \), and clearly \( A \) forms an additive group. Suppose \( A \neq \mathbb{R}^1 \), then there exists some \( s \) such that \((s + A) \cap A = \emptyset\), and therefore \( n(A \cup (s + A)) = 2 \). It contradicts the total mass of \( n \), so \( A = \mathbb{R}^1 \). It follows that \( ct \in T_\mu \) holds for all \( c \).

Q. E. D.

Using the above fundamental lemma, we can actually construct \( l^p \)-quasi-invariant measures as follows. First we put for \( u > 0 \),

\[
K_0(u) = \int_0^\infty \exp(-u \cosh t) dt ,
\]

which is the modified Bessel function. And for each \( 1 \leq p \leq \infty \), we take and fix a sequence \( \beta = \{\beta_j\} \in l^p \), whose all components are positive numbers. Further let \( q \) be a conjugate exponent of \( p \), \( 1/p + 1/q = 1 \). Now we define a measure \( \mu_q \) such that,

\[
d\mu_q(x) = \prod_{j=1}^\infty 4 \beta_j / \pi \ K_0^2(2\pi \beta_j |x_j|) dx_j , \quad \text{for each } (p, q) \text{ and } \beta \in l^p .
\]

Noting that
\[
\mathcal{F}(2K_0(2\pi a|u|))(v) = (v^2 + a^2)^{-1/2} , \text{ for an arbitrary real constant } a ,
\]
we get for the adjoint measure \( v_q \) of \( \mu_q \),

\[
dv_q(x) = \prod_{j=1}^\infty \beta_j / \pi \ (x_j^2 + \beta_j^2)^{-1} dx_j .
\]

**Proposition 2.1.** If \( a = (a_1, a_2, \ldots) \in l^q \), then \( v_q(H_a) = 1 \).

**Proof.** \( v_q(H_a) = 1 \) is equivalent that,

\[
\sum_{j=1}^\infty \left\{ 1 - \exp\left(-a_j^2 \beta_j^2 u^2\right) \right\} 1/\pi (1 + u^2)^{-1} du < \infty .
\]

In order to assure (1), we put for \( s \in \mathbb{R}^1 \),

\[
w(s) = 1/\pi \left\{ 1 - \exp\left(-s^2 u^2\right) \right\} (1 + u^2)^{-1} du ,
\]

and estimate the order of \( w(s) \) at \( s = 0 \). Then after some calculations, we can derive that \( w(s) = O(|s|) \) at \( s = 0 \), so the convergence of (1) is equivalent to \( \sum_{j=1}^\infty |a_j| \beta_j < \infty \). Clearly this inequality is satisfied by the assumptions for \( a \) and for \( \beta \).

Q. E. D.
Combining Proposition 2.1 together with Lemma 2.1, $\mu_q$ is actually the $l^q$-quasi-invariant measure for each $q \geq 1$. Later we shall give examples of strictly-$l^q$-quasi-invariant measures for $0 < p \leq 2$.

The following definition and lemma are essentially due to L. Shwarz, [7]. We list them here in a partially different but special form of the original one.

**Definition 2.1.** Let $m$ be a measure on $\mathcal{B}(R^\infty)$ such that, $m = \bigotimes_{j=1}^\infty m_j$, where $m_j$ is the probability measure on $\mathcal{B}(R^1)$ for each $j$, and $\Phi$ be a subspace of $R^\infty$.

(a) If for an arbitrary element $t = (t_1, t_2, \ldots) \in \Phi$, $\sum_{j=1}^\infty t_j x_j$ converges for $m$-a.e. $x = (x_1, x_2, \ldots)$, then we say that $m$ is a type $\Phi$.

(b) Conversely, if a following assertion holds, we say that $m$ is a cotype $\Phi$.

Let $t = (t_1, t_2, \ldots) \in R^\infty$. If $\sum_{j=1}^\infty t_j x_j$ converges for $m$-a.e. $x$, then it follows that $t \in \Phi$.

(c) If (a) and (b) are both satisfied, we say that $m$ is a special type $\Phi$.

After these definitions, we can state the following corollary of Lemma 2.1.

**Corollary.** Let $\mu$ be a measure as in (II), and $\nu$ be the adjoint measure of $\mu$. Then,

(a) $T_\mu \supset l^p$ is equivalent that $\nu$ is a type $l^p$.

(b) if $T_\mu = l^p$, then $\nu$ is a special type $l^p$.

(c) if $\nu$ is a special type $l^p$, and $T_\mu$ forms a vector space, then $T_\mu = l^p$.

**Proof** is derived from the consideration and the remark of Lemma 2.1.

Now if $m$ is a type $l^p$ ($p > 0$), we can define a following operator $T$ from $l^p$ to $\text{Mes}(R^\infty, m, R^1)$ such that,

$$T: t = (t_1, t_2, \ldots) \in l^p \longrightarrow \sum_{j=1}^\infty t_j x_j \in \text{Mes}(R^\infty, m, R^1),$$

where the last symbol means the class of all real-valued $m$-measurable functions defined on $R^\infty$. 
Lemma 2.2. Let $m$ be a measure on $\mathfrak{B}(\mathbb{R}^\infty)$ as in Definition 2.1. For $1 \leq p \leq \infty$,

(a) if $m$ is a type $l^p$, then the above mapping $T$ is the continuous operator from $l^p$ to $\text{Mes}(\mathbb{R}^\infty, m, \mathbb{R}^1)$ equipped with the topology of convergence in probability.

(b) if $m$ is a special type $l^p$, and is the symmetric measure, then the map $T$ is the homeomorphic operator from $l^p$ to $\text{Mes}(\mathbb{R}^\infty, m, \mathbb{R}^1)$ equipped with the same one.

Proof is stated in [7], [8]. So we omit it. But it is an application of Baire's theorem and closed graph theorem.

Here we shall discuss strictly-$l^p$-quasi-invariant measures $(0 < p \leq \infty)$ on $\mathfrak{B}(\mathbb{R}^\infty)$.

Proposition 2.2. There does not exist any strictly-$l^\infty$-quasi-invariant measure as in (II) on $\mathfrak{B}(\mathbb{R}^\infty)$.

Proof. Suppose the contrary case, namely let $\mu$ be a measure as in (II), and be the strictly-$l^\infty$-quasi-invariant measure. Then the adjoint measure $v$ of $\mu$ is the special type $l^\infty$ in virtue of the corollary of Lemma 2.1. Applying Theorem 1.1 for an element $(s, s, \ldots, s, \ldots) \in l^\infty (s \in \mathbb{R}^1)$, we get

$$\sum_{j=1}^{\infty} \left\{1 - \int \exp(2\pi isx_j)dv(x)\right\} < \infty.$$ 

Therefore,

$$\lim_{j} \int \exp(2\pi isx_j)dv(x) = 1.$$ 

It follows that for $v$, $\{x_j\}$ converges in law to the Dirac measure, equivalently it converges to 0 in probability. Let $T$ be the same meaning as in Lemma 2.2, in which we shall put $v$ for $m$. Then an element $e_j=(0, 0, \ldots, 0, 1, 0, \ldots)$ corresponds to $x_j$ by the map $T$, and it is a homeomorphic operator in virtue of Lemma 2.2. Therefore by the above argument, $e_j$ must tend to $0=(0, 0, \ldots)$, which is a contradiction. Q.E.D.

Lemma 2.3. Let $\alpha$ be a probability measure on $\mathfrak{B}(\mathbb{R}^1)$ and put
\[ \Phi(s) = \int_{-\infty}^{\infty} (1 - \exp(-s^2 u^2)) d\pi(u), \quad \text{for } s \in \mathbb{R}^1. \]

Then,
\[ \Phi(\lambda s) \geq \lambda^2 \Phi(s), \quad \text{for all } |\lambda| \leq 1. \]

**Proof.** It is derived from the following elementary inequality.
\[ 1 - \exp(-c v) \geq c(1 - \exp(-v)), \quad \text{for } 0 \leq c \leq 1 \text{ and for } \forall v \geq 0. \]
Q.E.D.

**Proposition 2.3.** For \( 2 < p < \infty \), there does not exist any strictly-\( l^p \)-quasi-invariant measure as in (II) on \( \mathcal{B}(\mathbb{R}^\infty) \).

**Proof.** Suppose the contrary case, namely let \( \mu \) be a measure as in (II) and be the strictly-\( l^p \)-quasi-invariant measure. Then from Lemma 2.1, \( \sum_{j=1}^{\infty} (1 - \exp(-b_j x_j^2)) d\nu(x) < \infty \) holds if and only if \{\( b_j \)\} \( \in l^p \). If necessary, we divide each \( b_j \) by a suitable normalizing constant, and apply Lemma 2.3. Then it follows that,
\[ \sum_{j=1}^{\infty} b_j^2 \int (1 - \exp(-x_j^2)) d\nu(x) < \infty, \quad \text{for all } \{b_j\} \in l^p. \]

Suppose \( \inf \int (1 - \exp(-x_j^2)) d\nu(x) = 0 \). Then a suitable subsequence \{\( j_n \)\} exists such that,
\[ \sum_{n=1}^{\infty} \int (1 - \exp(-x_{j_n}^2)) d\nu(x) < \infty. \]

So putting \( e = (0, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots) \), the above inequality shows that \( \mu \) is \( e \)-quasi-invariant. But \( e \) does not belong to any \( l^p \) \( (\infty > p > 0) \). So it contradicts the assumption of quasi-invariance. Consequently \( \inf \int (1 - \exp(-x_j^2)) d\nu(x) > 0 \), and from (3) we get \( \sum_{j=1}^{\infty} b_j^2 < \infty \) for all \( \{b_j\} \in l^p \). Again we reach to a contradiction. Q.E.D.

On the other hand, in the case of \( 0 < p \leq 2 \), strictly-\( l^p \)-quasi-invariant measures actually exist as follows. Let \( r \) be a real number such that \( 2r > 1 \) and we put \( f_r(u) = \frac{4\pi}{\gamma_r F^2(r/2)} |\pi u|^{-r-1} K_{\frac{r}{2}}^2(2\pi|u|), \) where \( K_{\frac{r}{2}}^{-1} \) is again
the modified Bessel function and \( \gamma_r \) is the normalizing constant.

\[
\gamma_r = \sqrt{\pi} \Gamma(\frac{r-1}{2})/\Gamma(r).
\]

We define a measure \( \mu_r \) on \( \mathcal{B}(\mathbb{R}^\infty) \) such that,

\[
d\mu_r(x) = \otimes_{j=1}^\infty f_r(x_j)dx_j,
\]

namely \( \mu_r \) is a stationary measure with \( f_r \). By the well known formula for the Fourier transformation, we obtain for the adjoint measure \( \nu_r \)

\[
d\nu_r(x) = \otimes_{j=1}^\infty 1/\gamma_r (1 + x_j^2)^{-r}dx_j.
\]

Therefore from Theorem 1.1, \( t = (t_1, t_2, \ldots) \in T_\mu \) holds, if and only if

\[
\sum_{j=1}^\infty \frac{1 - \cos 2\pi t_j u}{\gamma_r (1 + u^2)^r} du < \infty.
\]

Using the result of the stationary case (Proposition 1.1), it is necessary that \( \lim_{j} \gamma_j = 0 \) for (4). So putting

\[
W(s) = \int_{-\infty}^{\infty} 1/\gamma_r (1 - \cos (2\pi su))(1 + u^2)^{-r} du,
\]

and estimating the order of it at \( s = 0 \), we get

(a) \( W(s) = O(|s|^{2r-1}) \), if \( 2r-1 < 2 \)

(b) \( W(s) = O(|s|^2 \log |s|) \), if \( 2r-1 = 2 \)

(c) \( W(s) = O(s^2) \), if \( 2r-1 > 2 \).

Finally we have the following result

(A) \( T_{\mu_r} = l^{2r-1} \), if \( 2r-1 < 2 \)

(B) \( T_{\mu_r} = l^{2-} \), if \( 2r-1 = 2 \), where \( l^{2-} = \{ x \in \mathbb{R}^\infty | \sum_{j=1}^\infty x_j^2 < \infty \text{ and } \sum_{j=1}^\infty |x_j|^2 \log |x_j| | < \infty \} \)

(C) \( T_{\mu_r} = l^2 \), if \( 2r-1 > 2 \).

**Lemma 2.4.** For \( 1 \leq p \leq \infty \), let \( \mu \) be a measure as in (II) and be the \( l^p \)-quasi-invariant measure. Then for \( a = (a_1, a_2, \ldots) \in l^p \),

\[
\sum_{j=1}^p \left\{ 1 - \sqrt{f_j(u-a_j)f_j(u)} \right\}
\]

is a continuous function of \( a \) with respect to the natural topology of
Proof. Let $v$ be the adjoint measure of $\mu$. $(dv(x) = \otimes_{j=1}^{\sigma} g_j(x_j)^2 dx_j)$. Then $v$ is the type $l^p$ and a function

$$v(a) = \sum_{j=1}^{\sigma} \exp(2\pi i \sum_{j=1}^{\sigma} a_j x_j) dv(x) = \prod_{j=1}^{\sigma} \exp(2\pi i a_j g_j(v)^2 dv$$

is the continuous function of $a \in l^p$, in virtue of Lemma 2.2. As $v(a)$ is always positive, and the infinite product converges uniformly in a neighbourhood of each point,

$$\sum_{j=1}^{\sigma} \left\{1 - \int \exp(2\pi i a_j g_j(v)^2 dv\right\} = \sum_{j=1}^{\sigma} \left\{1 - \int \sqrt{f_j(u-a_j)} \sqrt{f_j(u)} du\right\}$$

is also continuous. The assertion of the last part is an easy consequence of the above argument. Q.E.D.

Now we shall discuss the compactness of the set \{\sqrt{f_j(u)}\} in $L^2_{du}(\mathbb{R}^1)$. The following proposition is found in [9]. But we list it for reference.

**Proposition 2.4.** Let $d\xi$ be the Lebesgue measure on $\mathbb{R}^N$, and $L^p_{d\xi}(\mathbb{R}^N)$ ($1 \leq p < \infty$) be the Banach space of (classes of) functions $f$ such that $|f|^p$ is Lebesgue integral. Then a subset $A$ of $L^p_{d\xi}(\mathbb{R}^N)$ is totally bounded if and only if it has the following three properties.

(a) $A$ is bounded in $L^p_{d\xi}(\mathbb{R}^N)$ (in the sense of the $L^p$ norm).

(b) $A$ is equismall at infinity, i.e., to every $\varepsilon > 0$, there is $\rho > 0$, such that, for all $f \in A$

$$\int_{\|\xi\| > \rho} |f(\xi)|^p d\xi < \varepsilon.$$

(c) To every $\varepsilon > 0$, there is $\delta > 0$ such that, for all $a \in \mathbb{R}^N$ such that $\|a\| < \delta$ and for all $f \in A$,

$$\int |f(\xi - a) - f(\xi)|^p d\xi \leq \varepsilon.$$

Applying it to the present case, from Lemma 2.4,

**Proposition 2.5.** For $1 \leq p \leq \infty$, let $\mu$ be a measure as in (II),
and be the \( l^p\)-quasi-invariant measure. If \( \{\sqrt{f_j(u)}\} \) is equismall at infinity, then it is the totally bounded set in \( L^{2u} \).

**Remark.** The equismall property of \( \{\sqrt{f_j(u)}\} \) is the same as the uniform tightness of the set of the measures \( \{f_j(u)du\} \). It depends on the continuity of \( u \). For example, if \( \mu \) can be regarded as a continuous cylindrical measure on \( l^p \) \( (1 \leq p < \infty) \), then the uniform tightness condition is satisfied.

Conversely,

**Proposition 2.6.** Let \( u \) be a measure as in (II), and \( T_\mu \) be a vector space. Suppose that \( \{\sqrt{f_j(u)}\} \) is a totally bounded set in \( L^2_\mu(\mathbb{R}^1) \), then we conclude that \( T_\mu \subset 1^2 \).

**Proof.** Let \( dv(x) = \otimes_{j=1}^p |g_j(x_j)|^2 dx_j \) be the adjoint measure of \( \mu \). Suppose that \( a = (a_1, a_2, ...) \in T_\mu \). Then in virtue of Lemma 2.1, \( (5) \quad \sum_{j=1}^p \int (1 - \exp(-s^2a_j^2v^2)) |g_j(v)|^2 dv < \infty \), for all \( s \in \mathbb{R}^1 \).

First we shall assume that \( \sup a_j = \infty \). Then, as \( 1 - \exp(-u^2) \) is a monotone increasing function, there exist subsequence \( \{j_n\} \) such that,

\[
\sum_{n=1}^\infty \int (1 - \exp(-v^2)) |g_j(v)|^2 dv < \infty ,
\]

hence

\[
(6) \quad \lim_n \int (1 - \exp(-v^2)) |g_j(v)|^2 dv = 0 .
\]

By the assumption, \( \{g_j(v)\} \) is the totally bounded set, and therefore if necessary taking a subsequence of \( \{j_n\} \), we can suppose that \( g_{j_n}(v) \) converges to some \( g_0(v) \) in \( L^2_{d\nu}(\mathbb{R}^1) \). Then from (6), we get

\[
\int (1 - \exp(-v^2)) |g_0(v)|^2 dv = 0 ,
\]

and we reach to a contradiction. So \( \sup_j |a_j| < \infty \). Using Lemma 2.3, from (5) it follows that
\[
\sum_{j=1}^{n_j} a_j^2 \int (1 - \exp(-v^2)) |g_j(v)|^2 dv < \infty.
\]

Again the compactness of \(\{g_j(v)\}\) implies

\[
\inf_j \int (1 - \exp(-v^2)) |g_j(v)|^2 dv > 0, \text{ so } \sum_{j=1}^{n_j} a_j^2 < \infty.
\]

Q.E.D.

Even if a measure \(\mu\) is strictly-\(l^2\)-quasi-invariant, \(\{\sqrt{f_j(u)}\}\) is not necessarily a totally bounded set.

**Example 2.1.** We set \(\mu\) such that,

\[
d\mu(x) = \otimes_{j=1}^{n_j} 1/\sqrt{2\pi} \exp(- (x_j - r_j)^2/2) dx_j,
\]

where the positive sequence \(\{r_j\}\) is taken such that \(\lim r_j = \infty\).

Then \(T_\mu = l^2\) holds but the set \(\{\exp(-(u - r_j)^2/4)\}\) is not totally bounded.

**Example 2.2.**

\[
d\mu(x) = \otimes_{j=1}^{n_j} 1/\sqrt{2\pi} \exp(- (x_j^2 + r_j^2) + \cosh(r_j x_j)) dx_j,
\]

where \(\{r_j\}\) is a positive sequence such that \(\sum_{j=1}^{n_j} \exp(- r_j^2/2 + r_j) < \infty\).

\(\mu\) has the same properties as in above, but we remark that it is a symmetric measure.

Roughly speaking, the compact case is possible to arise only in that of \(T_\mu \subset l^2\). On the other hand,

**Proposition 2.7.** For \(p > 2\), let \(\mu\) be a measure as in (II), and be the \(l^p\)-quasi-invariant measure. Then \(\{\sqrt{f_j(u)}\}\) is a discrete set in \(L_{2\mu}^p(\mathbb{R}^1)\).

**Proof.** Suppose the contrary case, namely we shall assume that a suitable subsequence \(\{\sqrt{f_{j_n}(u)}\}\) converges to some \(\sqrt{f_0(u)} \in L_{2\mu}^p(\mathbb{R}^1)\).

If necessary, again we take a subsequence of \(\{j_n\}\) such that,

\[
\sum_{n=1}^{\infty} \| \sqrt{f_{j_n}(u)} - \sqrt{f_0(u)} \|_{L^2} < \infty.
\]

Then by the assumption, also for \(\{j_n\},\)

\[
\sum_{n=1}^{\infty} \left\{ 1 - \sqrt{f_{j_n}(u - a_n)} \sqrt{f_{j_n}(u)} \right\} du < \infty, \text{ for all } a = (a_1, a_2, \ldots) \in l^p.
\]

As
we have

\[ \sum_{j=1}^{\infty} \left\{ 1 - \sqrt[\alpha]{f_0(u)} \sqrt[\alpha]{f_0(u - a_j)} du \right\} < \infty, \text{ for all } a = (a_1, a_2, \ldots) \in l^\alpha. \]

Even if \( \sqrt[\alpha]{f_0(u)} \) vanishes on a set with positive Lebesgue measure, the proof of Proposition 1.1 is still valid for the present case. Therefore we conclude that \( a \in l^2 \). But it contradicts the fact, \( l^2 \subseteq l^\alpha \). Q.E.D.

§ 3. Characterization of \( l^2 \)-Quasi-Invariant Measures

In the former sections, we have discussed the aspect of (mainly, \( l^\alpha \)-)quasi-invariant measures, and showed that the case \( p>2 \) and \( p\leq2 \) present the different situations. So we wish to consider the case of \( T_\mu \supseteq l^2 \) (especially \( T_\mu = l^2 \)) and to characterize it in terms of \( \{ \sqrt[\alpha]{f_\mu(u)} \} \).

First we shall consider a stationary measure.

Theorem 3.1. Let \( \mu \) be a measure as in (II), and be the stationary measure with \( f \). We put \( \mathcal{F}(\sqrt[\alpha]{f}) = g \). Then in order that \( T_\mu \supseteq l^2 \) holds (automatically, \( T_\mu = l^2 \) holds due to Proposition 1.1), it is necessary and sufficient that

\[ \int v^2 |g(v)|^2 dv < \infty. \]

(It is equivalent to \( \left\| \frac{d}{du} \sqrt{f(u)} \right\|_{L^2} < \infty \).)

Proof. Let \( v \) be the adjoint measure of \( \mu \),

\[ dv(x) = \bigotimes_{j=1}^{\infty} |g(x_j)|^2 dx_j. \]

First we shall prove sufficiency. Assume that (7) is satisfied. Then for an arbitrary element \( a = (a_1, a_2, \ldots) \in l^2 \), we have

\[ \infty > \sum_{j=1}^{\infty} a_j^2 \int v^2 |g(v)|^2 dv = \int \sum_{j=1}^{\infty} a_j^2 x_j^2 dv(x), \]
and it follows that \( v(H_{\omega}) = 1 \). Consequently, from Lemma 2.1, we conclude that \( T_{\mu} \subseteq L^2 \).

Conversely, suppose to be \( T_{\mu} = L^2 \). Then from Lemma 2.4, for an arbitrary \( \varepsilon_0 > 0 \), there exists \( \delta_0 > 0 \) such that,

\[
\sum_{j=1}^{\infty} (1 - \cos(\theta_j)) |g(v)|^2 dv < \varepsilon_0, \quad \text{for all } \|a\|^2 = \sum_{j=1}^{\infty} a_j^2 \leq \delta_0^2.
\]

In this inequality, we shall put \( a_j = \delta_0/\sqrt{n} \) for \( 1 \leq j \leq n \) and \( a_j = 0 \) for \( j \geq n + 1 \). Then for any \( n \),

\[
n \int (1 - \cos(\delta_0 \sqrt{n})) |g(v)|^2 dv < \varepsilon_0.
\]

So, letting \( n \) tend to infinity and applying Lebesgue-Fatou's lemma, it follows that,

\[
\delta_0^2/2 \int v^2 |g(v)|^2 dv \leq \varepsilon_0, \text{ which shows the necessity.} \quad \text{Q. E. D.}
\]

Let \( \mu \) be a measure as in \((\Pi)\) i.e.,

\[
d\mu(x) = \otimes_{j=1}^{\infty} f_j(x_j) dx_j, \quad \text{and} \quad \mathcal{F}(\sqrt{f_j}) = g_j.
\]

Assume that \( \mu \) is the \( L^2 \)-quasi-invariant measure. Then from the result of the above stationary case, it seems that,

\[
\sup_j \int v^2 |g_j(v)|^2 dv < \infty.
\]

But it is false for the general \( \mu \). Even in the case of \( T_{\mu} = L^2 \), we have a following example.

**Example 3.1.** Let \( K_0(u) \) be the modified Bessel function, and \( \gamma \) be a constant such that \( 0 < \gamma < 1 \). We put

\[
f_j(u) = 1/n(\gamma) \left\{ (1 - \gamma)2/\sqrt{\pi} K_0(2\pi|u|) + \gamma/\sqrt{2} \exp(-|x|/2) \right\}^2,
\]

where \( n(\gamma) \) is the normalizing constant. And we define a measure \( \mu \) on \( \mathcal{B}(L^\infty) \) such that,

\[
d\mu(x) = \otimes_{j=1}^{\infty} f_j(x_j) dx_j,
\]

where \( \{\gamma_j\} \) is taken such that \( \sum_{j=1}^{\infty} (1 - \gamma_j)^2 < \infty \). Then some calculation shows that \( d\mu(x) \equiv \otimes_{j=1}^{\infty} 1/2 \exp(-|x_j|) dx_j \), and the later measure is strictly-\( L^2 \)-quasi-
invariant. Therefore the same holds for $\mu$. On the other hand, as
\[ g_\gamma(v) = \mathcal{F}(\sqrt{f}_{\gamma})(v) = 1/\sqrt{n(\gamma)} \{(1-\gamma)/\sqrt{\pi (1+v^2)^{-1/2}} + 2\sqrt{2} \gamma (1+16\pi^2 v^2)^{-1}\}, \]
and
\[ |g_\gamma(v)|^2 \geq (1-\gamma)^2/\pi n(\gamma) \cdot (1+v^2)^{-1}, \]
so
\[ \int v^2 |g_\gamma(v)|^2 dv = \infty, \quad \text{for any } \gamma > 0. \]

However the above conjecture is modified as a following theorem.

**Theorem 3.2.** Let $\mu$ be a measure as in (II). Then in order that $\mu$ is an $l^2$-quasi-invariant measure, it is necessary and sufficient that there exists some measure $M$ on $\mathcal{B}(\mathbb{R}^\infty)$, which has following three properties.

(a) $dM(x) = \otimes_{j=1}^\infty F_j(x_j) dx_j$, $F_j(u) > 0$ for Lebesgue-a.e. $u$
(b) $M \simeq \mu$ (in the sense of Definition 1.1)
(c) $\sup_j \int v^2 |G_j(v)|^2 dv < \infty$, where $G_j = \mathcal{F}(\sqrt{F_j})$.

**Proof.** First we shall prove sufficiency. Clearly the equivalence of measures does not change the set $T_\mu$, namely if $\mu \simeq M$ holds, then $T_\mu = T_M$. So we have only to check that $M$ itself is the $l^2$-quasi-invariant measure. Now using (c) in place of (7) in Theorem 3.1, we reach to the desired conclusion in a similar way with it. The necessity of the proof is derived from the following two lemmas. From now on we shall use a symbol $*$ for convolution operation. Let $\mu$ be a measure as in (II), and $a = (a_1, a_2, \ldots) \in \mathbb{R}^\infty$. We put
\[ \sqrt{h_j(u)} = \begin{cases} 1/n(a_j) \{(\sqrt{f_j(u)}*1/\sqrt{2\pi |a_j| \exp(-u^2/2a_j^2)}) \}, & \text{if } a_j \neq 0 \\
\sqrt{f_j(u)}, & \text{if } a_j = 0, \end{cases} \]
where $n(a_j)$ is the normalizing constant such that
\[ \int h_j(u) du = 1. \]
Using the above \( \{h_j(u)\} \), we define a measure such that,

\[
d\mu^a(x) = \bigotimes_{j=1}^\infty h_j(x_j)dx_j.
\]

Then,

**Lemma 3.1.** If \( \mu \) is \( l^2 \)-quasi-invariant and \( a= (a_1, a_2, \ldots) \in l^2 \), then \( \mu \simeq \mu^a \).

**Proof.** First we shall put \( w_j(u) = n(a_j)\sqrt{h_j(u)} \). Then,

\[
\|w_j - \sqrt{f_j}\|_{L^2} \leq 1/\sqrt{2\pi} \int |\sqrt{f_j(u - a_j s)} - \sqrt{f_j(u)}|^2 \exp(-s^2/2)duds = 2\sqrt{2\pi} \int (1 - \exp(2\pi a_j sv))|g_j(v)|^2 \exp(-s^2/2)duds = 2\int (1 - \exp(-2\pi^2 a_j^2 v^2))|g_j(v)|^2 dv.
\]

So \( \sum_{j=1}^\infty \|w_j - \sqrt{f_j}\|_{L^2}^2 < \infty \), in virtue of Lemma 2.1. Especially

\[
\sum_{j=1}^\infty (1 - n(a_j))^2 = \sum_{j=1}^\infty (1 - \|w_j\|_{L^2})^2 < \infty.
\]

Therefore,

\[
\sum_{j=1}^\infty \|\sqrt{h_j} - \sqrt{f_j}\|_{L^2}^2 \leq \sum_{j=1}^\infty (\|\sqrt{h_j} - w_j\|_{L^2} + \|w_j - \sqrt{f_j}\|_{L^2})^2 \leq 2\sum_{j=1}^\infty ((1 - n(a_j))^2 + \|w_j - \sqrt{f_j}\|_{L^2}^2) < \infty.
\]

It follows that \( \mu \) and \( \mu^a \) are equivalent with each other. Q.E.D.

We note that

\[
\mathcal{F}(\sqrt{h_j}(v)) = 1/n(a_j)g_j(v)\exp(-2\pi^2 a_j^2 v^2).
\]

**Lemma 3.2.** Let \( m \) be a measure on \( \mathcal{B}(\mathbb{R}^\infty) \) such that \( m = \bigotimes_{j=1}^\infty m_j \), where each \( m_j \) is the probability measure on \( \mathcal{B}(\mathbb{R}^1) \). Suppose that \( m(H_a) = 1 \) for all \( a \in l^2 \). Then there exists some \( \delta = \{\delta_j\} \in l^2 \) such that

\[
\sup_j \int u^2 \exp(-\delta_j^2 u^2)dm_j(u) < \infty.
\]

**Proof.** As \( \sum_{j=1}^\infty \int (1 - \exp(-a_j^2 u^2))dm_j(u) < \infty \), so we can put
\[ W(a) = \sum_{j=1}^{\infty} \left( 1 - \exp(-a_j^2 u^2) \right) dm_j(u), \quad \text{for all } a = (a_1, a_2, \ldots) \in l^2. \]

First, we shall claim that \( W \) is continuous and is bounded on the unit sphere of \( l^2 \). For, let \( n_x \) be the canonical Gaussian measure on \( \mathcal{B}(R^n) \). Namely, \( dn_x(x) = \otimes_{j=1}^{\infty} 1/\sqrt{2\pi} \exp(-x_j^2/2)dx_j \), and we put \( dn(u) = 1/\sqrt{2\pi} \exp(-u^2/2)du \) as for the measure on \( \mathcal{B}(R^1) \). Then,

\[ W(a) = \sum_{j=1}^{\infty} \left( 1 - \exp(\sqrt{2} i a_j u) \right) dm_j(u)dn(u). \]

Now, from \( m_j \) and \( n \), we define a new probability measure \( \lambda_j \) on \( \mathcal{B}(R^1) \) such that,

\[ \lambda_j(A) = n \otimes m_j \{(s, u) \in R^2 | su \in A \}, \quad \text{for all } A \in \mathcal{B}(R^1). \]

And we define a measure \( \lambda \) on \( \mathcal{B}(R^n) \) such that, \( \lambda = \otimes_{j=1}^{\infty} \lambda_j \). Then \( \lambda \) is the symmetric measure and the above equality can be written as

\[ W(a) = \sum_{j=1}^{\infty} \left( 1 - \exp(\sqrt{2} i a_j x_j) \right) d\lambda(x). \]

From this it follows that \( \lambda \) is the type \( l^2 \), and therefore \( W(a) \) is the continuous function by Lemma 2.4. Consequently, for any given \( \varepsilon > 0 \), there exists \( 0 < \delta < 1 \) such that, \( W(a) < \varepsilon \), for all \( ||a|| \leq \delta \). So \( W(\delta a) < \varepsilon \) for all \( ||a|| \leq 1 \). Applying Lemma 2.3 for \( W \), we conclude that, \( W(a) \leq \varepsilon/\delta^2 \), for all \( ||a|| \leq 1 \). From now on we put \( R = \varepsilon/\delta^2 \). Let \( K > 2 \) and we put for each \( j \)

\[ t_j = \inf \left\{ t > 0 \left| \int_{(-t, t)} u^2 dm_j(u) > KR \right. \right\} \quad \text{and} \quad s_j = 1/t_j. \]

If the above set is empty, we put \( t_j = \infty \) and \( s_j = 0 \). Then if \( s_j \neq 0 \), we have

\[ \int_{[-t_j, t_j]} u^2 dm_j(u) \geq KR \quad \text{and} \quad \int_{(-t_j, t_j)} u^2 dm_j(u) \leq KR. \]  

Secondly, we shall claim that \( \sum_{j=1}^{\infty} s_j^2 \leq 1 \). For, suppose the contrary case, namely there exists some \( n \) such that \( \sum_{j=1}^{n} s_j^2 > 1 \). Without loss of generality, we can assume that \( s_j \neq 0 \) for \( 1 \leq j \leq n \). In the definition of \( W \), we put \( a_j = s_j(s_1^2 + \ldots + s_n^2)^{-1/2} \) for \( 1 \leq j \leq n \) and \( a_j = 0 \) for \( j \geq n + 1 \). Then it yields
Using Lemma 2.3, it follows that

$$
\sum_{j=1}^{n} \int (1 - \exp(-s_j^2 u^2)) dm_j(u) \leq R \left( s_j^2 + \cdots + s_n^2 \right).
$$

Therefore,

$$
\sum_{j=1}^{n} s_j^2/2 \int_{[-t_j, t_j]} u^2 dm_j(u) \leq \sum_{j=1}^{n} \int_{[-t_j, t_j]} (1 - \exp(-s_j^2 u^2)) dm_j(u) \\
\leq R \left( s_j^2 + \cdots + s_n^2 \right).
$$

From (8), it follows that $1/2 \, KR \leq R$, which contradicts the choice of $K$.

As $\{s_j\} \in l^2$, so we get

$$
\sum_{j=1}^{n} \int (1 - \exp(-s_j^2 u^2)) dm_j(u) < \infty,
$$

and therefore

$$
\sum_{j=1}^{n} \int_{|u| \geq t_j} dm_j(u) < \infty.
$$

Lastly, we put for each $j \, \delta_j = \left\{ \int_{|u| \geq t_j} dm_j(u) \right\}^{1/2}$. Then $\{\delta_j\} \in l^2$ and,

$$
\int u^2 \exp(-\delta_j^2 u^2) dm_j(u) = \int_{(-t_j, t_j)} u^2 \exp(-\delta_j^2 u^2) dm_j(u) \\
+ \int_{|u| \geq t_j} u^2 \exp(-\delta_j^2 u^2) dm_j(u) \leq KR + 1/\delta_j^2 \int_{|u| \geq t_j} dm_j(u) = KR + 1.
$$

Q. E. D.

**Proof of the necessity of Theorem 3.2.** Let $\mu$ be an $l^2$-quasi-invariant measure and $v$ be the adjoint measure of it. Then in virtue of Lemma 2.1, $v(H_a) = 1$ holds for all $a \in l^2$. So we can take a sequence $\delta = \{\delta_j\}$ such that,

$$
\sup_j \int v^2 \exp(-\delta_j^2 v^2) |g_j(v)|^2 dv < \infty.
$$

In the notation of Lemma 3.1, putting $a_j = \delta_j / \sqrt{2} \pi$ and constructing a
measure \( d\mu^t_i(x) = \otimes_{j=1}^n h_j(x_j)dx_j \), then \( \mu^t_i \equiv \mu \) holds and

\[
\sup_j \int v^2 |\mathcal{F}(\sqrt{h_j})(v)|^2 dv = \sup_j 1/n(a_j)^2 \int v^2 \exp(-\delta_j^2 v^2) |g_j(v)|^2 dv < \infty,
\]
as

\[
\lim_j n(a_j)^2 = \lim_j n(\delta_j/\sqrt{2\pi})^2 = 1.
\]

Therefore \( \mu^t_i \) is the desired one \( M \). Q. E. D.

The case \( T_n = l^2 \) is settled in the following theorem.

**Theorem 3.3.** Let \( \mu \) be a measure as in (II), and we put \( \mathcal{F}(\sqrt{h_j}) = g_j \). Then in order that \( \mu \) is the strictly-\( l^2 \)-quasi-invariant measure, it is necessary and sufficient that

(a) \( T_\mu \) is a vector space

(b) there exists some measure \( M \) on \( \mathcal{B}(\mathbb{R}^\infty) \) which has the three properties in Theorem 3.2.

(c) \( \inf_j \int (1 - \exp(-v^2)) |g_j(v)|^2 dv > 0. \)

**Proof.** First, we shall consider the necessity. Then (a) is trivial, and (b) is the consequence of Theorem 3.2. While (c) is shown in a quite similar way with Proposition 2.3. Conversely, suppose that the three conditions are satisfied. Then \( T_\mu = l^2 \) holds by Theorem 3.2. Now if there exists \( a = (a_1, a_2, \ldots) \in T_\mu \setminus l^2 \), then from (a),

\[
\sum_{j=1}^\infty \int (1 - \exp(-a_j^2 v^2)) |g_j(v)|^2 dv < \infty.
\]

It follows that,

\[
\inf_j \int (1 - \exp(-v^2)) |g_j(v)|^2 dv \sum_{j=1}^\infty \min(a_j^2, 1) < \infty,
\]

and therefore \( \sum_{j=1}^\infty a_j^2 < \infty \). But it contradicts the choice of \( a \). Q. E. D.

**Remark.** Even if (a) and (b) are satisfied for \( \mu \), the condition \( \inf_j \int v^2 |g_j(v)|^2 dv > 0 \) is necessary but not sufficient for \( T_\mu = l^2 \). We have a counter example for it.
Lastly, we shall discuss the relation of the support to the quasi-invariance.

**Lemma 3.3.** For $1 \leq p < \infty$, let $\mu$ be a measure as in (II), and be the $l^p$-quasi-invariant measure. If $\mu(H_a) = 1$ for some $a = (a_1, a_2, \ldots) \in \mathbb{R}^\infty$, then we conclude that $a \in l^q$. ($q$ is the conjugate exponent of $p$).

**Proof.** Let $\bar{\mu}$ be a measure on $\mathcal{B}(\mathbb{R}^\infty)$ such that,

$$\bar{\mu}(A) = \mu(-A) \quad \text{for all } A \in \mathcal{B}(\mathbb{R}^\infty).$$

The convolution of $\mu$ with $\bar{\mu}$ defines a new measure $\mu^S$, namely

$$\mu^S(A) = \int \mu(A - x) d\bar{\mu}(x), \quad \text{for all } A \in \mathcal{B}(\mathbb{R}^\infty).$$

We can easily check that $\mu^S$ is also the product-measure and symmetric one. Moreover, $\mu^S(H_a) = 1$ and $T_{\mu^S} \supset T_\mu \supset I^p$ hold. For,

$$\mu^S(H_a) = \int_{\mathbb{R}} \mu(H_a - x) d\bar{\mu}(x) = \int_{H_a} \mu(H_a - x) d\bar{\mu}(x) = \mu(H_a) = 1.$$

And if $t = (t_1, t_2, \ldots) \in T_\mu$ and $\mu^S(A) = 0$ for some $A \in \mathcal{B}(\mathbb{R}^\infty)$, then it follows that $\mu(A - x) = 0$ (hence, $\mu(A - x - t) = 0$) holds for $\bar{\mu}$-a.e. $x$. Therefore,

$$\mu^S(A - t) = \int \mu(A - t - x) d\bar{\mu}(x) = 0.$$

As the converse assertion holds in a similar way, we conclude that

$$T_{\mu^S} \supset T_\mu.$$  

Now, $\mu^S(H_a) = 1$ is equivalent that $\sum_{j=1}^{\infty} a_j x_j^2 < \infty$ for $\mu^S$-a.e. $x$, which yields that $\sum_{j=1}^{\infty} a_j x_j$ converges for $\mu^S$-a.e. $x$ due to the Kolmogorov-Khintchine's theorem and the symmetry of $\mu^S$. Therefore if $\mu$ is $l^p$-quasi-invariant, then we conclude that for all $\{h_j\} \in I^p$, both $\sum_{j=1}^{\infty} a_j x_j$ and $\sum_{j=1}^{\infty} a_j(x_j + h_j)$ converges for $\mu^S$-a.e. $x$. It follows that, for any $\{h_j\} \in I^p$, $\sum_{j=1}^{\infty} a_j h_j$ converges, which shows $\{a_j\} \in l^q$. Q.E.D.

According to the above lemma, we are specially interested in a following measure $\mu$.  

(\ast) \mu \text{ is as in (II), and } l^2\text{-quasi-invariant}

(\ast\ast) \text{ for any } a=(a_1, a_2, \ldots) \in l^2, \mu(H_a)=1.

For example, a measure which can be regarded as a continuous cylindrical measure on \( l^2 \) satisfies (\ast\ast) due to Minlos.

**Theorem 3.4.** Let \( \mu \) be a measure as in (II). Then in order that \( \mu \) has the properties (\ast) and (\ast\ast), it is necessary and sufficient that there exists a measure \( M \) on \( \mathcal{B}(\mathbb{R}^\infty), dM(x)=\otimes_{j=1}^\infty F_j(x_j)dx_j, \mathcal{F}(\sqrt{F_j})=G_j \) such that,

(a) \( M \) has the three properties in Theorem 3.2

(b) \( \sup_j \int u^2 F_j(u)du < \infty. \)

(We can characterize it in terms of the uniform boundness of \( \left\| \frac{d}{du}\sqrt{F_j(u)} \right\|_{L^2} \) and \( \| u \sqrt{F_j(u)} \|_{L^2}. \)

Further, \( T_\mu = l^2 \) holds under the condition (\ast) and (\ast\ast).

**Proof.** First we shall prove that \( T_\mu = l^2 \). The proof \( T_\mu \subseteq l^2 \) is derived as below from the consideration in Lemma 3.3. Using the same notation in it, we can assure that for all \( a=(a_1, a_2, \ldots) \in l^2, \sum_{j=1}^\infty a_j x_j \) converges for \( \mu^a \)-a.e. \( x \). Therefore if \( t=(t_1, t_2, \ldots) \in T_\mu \), then both \( \sum_{j=1}^\infty a_j x_j \) and \( \sum_{j=1}^\infty a_j (x_j+t_j) \) converges for \( \mu^a \)-a.e. \( x \). Hence \( \sum_{j=1}^\infty a_j f_j \) converges for any \( a \in l^2 \), which shows that \( t \in l^2 \). Combining it with the assumption \( T_\mu \subseteq l^2 \), we conclude that \( T_\mu = l^2 \). Secondly, we shall prove sufficiency of the former part of this assertion. For (\ast), it is a consequence of Theorem 3.2. Since \( M \cong \mu \), for (\ast\ast) we have only to check that \( M(H_a)=1 \) for all \( a \in l^2 \).

Now

\[
\sum_{j=1}^\infty a_j^2 x_j^2 dM(x) = \sum_{j=1}^\infty a_j^2 \int u^2 F_j(u) \, du \leq \sup_j \int u^2 F_j(u) \, du \sum_{j=1}^\infty a_j^2 < \infty .
\]

It follows that \( \sum_{j=1}^\infty a_j^2 x_j^2 < \infty \) for \( M \)-a.e. \( x \), which shows \( M(H_a)=1 \). Lastly, we shall prove the necessity. Using Lemma 3.2 for \( \mu \), it asserts that there exists a sequence \( \sigma=\{\sigma_n\} \in l^2 \) such that,

\[
(9) \quad \sup_j \int u^2 \exp(-\sigma_j^2 u^2) f_j(u)du < \infty .
\]

Putting \( k_j(u)=1/n(\sigma_j) \exp(-\sigma_j^2 u^2)f_j(u) \), where \( n(\sigma_j) \) is the normalizing constant such that,
we define a new measure \( \mu_1 \) on \( \mathcal{B}(\mathbb{R}^\infty) \) such that,
\[
d\mu_1(x) = \otimes_{j=1}^n k_j(x_j) dx_j.
\]
As \( \mu(H_a)=1 \) derives that \( \mu_1 \simeq \mu \), so \( \mu_1 \) is also \( l^2 \)-quasi-invariant. Using Theorem 3.2 for \( \mu_1 \), it follows that there exists a sequence \( \delta = \{\delta_j\} \in l^2 \) such that,
\[
\sup_j \int |v^2 G_j(v)|^2 dv < \infty, \quad \text{where} \quad G_j = \mathcal{F}(\sqrt{F_j}),
\]
\[
\sqrt{F_j(u)} = 1/n(\delta_j) \left\{ \sqrt{k_j(u)} * 1/ \sqrt{2\pi \delta_j} \exp \left( -u^2/2\delta_j^2 \right) \right\}, \quad \text{and}
\]
\( n(\delta_j) \) is the normalizing constant such that \( \int F_j(u) du = 1 \) (and becomes \( \lim n(\delta_j) = 1 \)).
Now the measure \( dM(x) = \otimes_{j=1}^n F_j(x_j) dx_j \) is the desired one. Because we have only to check that
\[
(10) \quad \sup_j \int u^2 F_j(u) du < \infty.
\]
Some calculations derive that
\[
\int u^2 F_j(u) du \leq 1/n(\delta_j)^2 \left\{ \int u^2 k_j(u) du + \delta_j^2 \right\}.
\]
Combining it with (9), we can assure (10). Q.E.D.

As for the stationary case,

**Theorem 3.5.** Let \( \mu \) be a stationary measure with \( f \), and we put \( \mathcal{F}(\sqrt{f})=g \). Then \( \mu \) is \( l^2 \)-quasi-invariant and \( \mu(H_a)=1 \) for all \( a \in l^2 \), if and only if
\[
\int u^2 f(u) du < \infty \quad \text{and} \quad \int u^2 |g(v)|^2 dv < \infty.
\]

**Proof.** First we shall consider the necessity. The last inequality has already proven, so we have only to check the first one. Applying Lemma 3.2, there exist \( \{\delta_j\} \in l^2 \) and a positive constant \( R \) such that,
\[ \int u^2 \exp(-\delta ju^2)f(u)du \leq R, \quad \text{for all } j. \]

Letting \( j \) tend to infinity and using Lebesgue-Fatou's lemma, we get
\[ \int u^2 f(u)du \leq R. \]

The proof of the sufficiency is carried out in a quite similar way with it in Theorem 3.4. Q. E. D.

References
