Simple Proofs of Nakano’s Vanishing Theorem and Kazama’s Approximation Theorem for Weakly 1-Complete Manifolds

By
Osamu SUZUKI*

Introduction

Let $X$ be an $m$-dimensional complex manifold and let $E$ be a vector bundle on $X$. A hermitian inner product in $E$ is given as usual and is denoted by $H(\xi, \eta)$. In particular, when $\xi = \eta$, we write $H(\xi, \xi)$ as $|\xi|^2$. By $\mathcal{O}(E)$ we denote the sheaf of germs of holomorphic sections of $E$. $X$ is called a weakly 1-complete manifold when there exists a $C^\infty$-differentiable pseudoconvex function $\Psi$ on $X$ such that $X_\Psi = \{\Psi < c\}$ is relatively compact in $X$ for any real number $c$. We see that if $X$ is a weakly 1-complete manifold, $X_\Psi$ is also a weakly 1-complete manifold.

Now we consider a weakly 1-complete manifold with a positive vector bundle $E$ (see, Definition (1.4) in §1). Then the following theorems have been proved by S. Nakano [8] and H. Kazama [4] respectively:

**Theorem 1.** For any real number $c$, we have

$$H^q(X_\Psi, \mathcal{O}(E \otimes K)) = 0 \quad \text{for} \quad q \geq 1,$$

where $K$ denotes the canonical line bundle of $X$.

**Theorem 2.** Fix two constants $c$ and $d$ with $c > d$. Then for any holomorphic section $\varphi \in H^0(\overline{X}_d, \mathcal{O}(E \otimes K))$, $\overline{X}_d$ being the closure of $X_d$ in $X$ and for any positive constant $\varepsilon$, there exists a section $\tilde{\varphi} \in H^0(X_\Psi, \mathcal{O}(E \otimes K))$ such that $|\varphi - \tilde{\varphi}|^2 < \varepsilon$ everywhere in $\overline{X}_d$.

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* Graduate School, University of Tokyo, Tokyo.
Corollary. \[ H^q(X, \varphi(E \otimes K)) = 0 \quad \text{for} \quad q \geq 1. \]

This follows from Theorems 1 and 2 by a well known technique (see, Gunning and Rossi [2], p. 243, Theorem 14).

In this short note we shall give simple proofs of the above theorems by using the method due to K. Kodaira (see, Theorem 3 in §2) and a key lemma due to A. Andreotti and E. Vesentini (see, [1], p. 93, Proposition 5). The original proof of Theorem 1 is very complicated because of the choices of the metrics of \( E \) and \( X \) (see, the proof of (iii) in Proposition 1 in p. 172, Nakano [8]). Kazama's proof is very long.

Sections 1 and 2 are devoted to preliminaries and in section 3 our proofs will be done.

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§1. Hermitian Connections of Hermitian Vector Bundles

Let \( X \) be an \( m \)-dimensional complex manifold and let \( E \) be a hermitian vector bundle of rank \( r \) on \( X \). We cover \( X \) by locally finite coordinate neighborhoods \( \{ U_\lambda \} \) and denote local coordinates on \( U_\lambda \) by \( z_1^\lambda, z_2^\lambda, \ldots, z_m^\lambda \). With respect to this covering a hermitian inner product \( H \) is expressed by a system of positive definite hermitian matrixes \( \{ (h_{\lambda, kj}) \} \) on \( U_\lambda \): for \( C^\infty \)-sections \( \xi = \{ (\xi_1^\lambda, \xi_2^\lambda, \ldots, \xi_m^\lambda) \} \) and \( \eta = \{ (\eta_1^\lambda, \eta_2^\lambda, \ldots, \eta_m^\lambda) \} \) of \( E \) on \( X \),

\[
H(\xi, \eta) = \sum_{k,l} h_{\lambda, kl} \xi_k \eta_l.
\]

By \( (h_{\lambda, kj}^{\diamond}) \) we denote the inverse matrix of \( (h_{\lambda, kj}) \). By using \( H \), we can define a hermitian connection in a canonical manner: A system of matrix valued 1-forms \( \{ \omega_{\lambda}^x \} \), \( \omega_{\lambda}^x = \{ \omega_{\lambda, ij}^x \} \) on \( U_\lambda \) is called a hermitian connection if

\[
(1.2) \quad \omega_{\lambda, ij}^x = \sum_{a=1}^m \Gamma_{\lambda, ak}^x dz_a^\lambda \text{ where } \Gamma_{\lambda, ak}^x = \sum_{j=1}^r h_{\lambda, kj}^{\diamond} \frac{\partial h_{\lambda, kj}}{\partial z_a^\lambda}. 
\]

The curvature tensor of the above connection is defined by
We also define
\[ K_{\lambda,ik\beta z} = \sum h_{\lambda,il} K_{\lambda,ik\beta z} \cdot \]
It is easily seen that \( K_{\lambda,ik\beta z} = \overline{K_{\lambda,ik\beta z}} \). This shows that \( (K_{\lambda,ik\beta z}) \) can be regarded as a hermitian matrix of type \((mr, mr)\).

**Definition (1.4).** \( E \) is called positive in the sense of S. Nakano [6] if there exists a hermitian inner product in \( E \) such that \((-K_{\lambda,ik\beta z})\) is positive definite everywhere.

Set \( K_{\lambda,il\beta z} = \sum_{l=1}^{r} K_{\lambda,il\beta z} \). Then \( K_{\lambda,il\beta z} = \partial_{\beta} \log h_{\lambda} \), where \( h_{\lambda} = \det (h_{\lambda,kl}) \).

The following is easily proved.

**Proposition (1.5).** If \( E \) is positive, then
\[ -\Sigma K_{\lambda,il\beta z} dz_{\lambda}^{l} \wedge d\bar{z}_{\lambda}^{\beta} \]
is positive definite \((1.1)\)-form on \( X \).

Then we see that a positive vector bundle induces a kähler metric on \( X \).

Now we shall restrict ourselves to a weakly 1-complete manifold with a positive vector bundle \( E \). The positive metric is denoted by \((1.1)\). Fix a real number \( c \) and consider \( X_{c} \). Then \( X_{c} \) is also a weakly 1-complete manifold with respect to a complete pseudoconvex function
\[ \psi = 1/(1 - \frac{\psi}{c}) \).

For a convex increasing function \( \Lambda \), set
\[ a_{\lambda} = h_{\lambda}^{-1} e^{\Lambda(\psi)} \).

Then we have a kähler metric
\[ ds^{2} = \sum_{\lambda} \frac{\partial^{2} \log a_{\lambda}}{\partial z_{\lambda}^{l} \partial z_{\lambda}^{\beta}} dz_{\lambda}^{l} \cdot d\bar{z}_{\lambda}^{\beta} \cdot \]
S. Nakano [7] proved
Proposition (1.7). If $\int_0^\infty \sqrt{A''(t)} \, dt = \infty$, then (1.6) is a complete kähler metric on $X_c$.

In what follows, we fix such a complete metric on $X_c$, which is denoted by

(1.8) \[ ds^2 = \sum g_{\lambda, \beta} dz^\lambda_\lambda d\bar{z}^\beta_\lambda. \]

We define the metric form by

(1.9) \[ \Omega = \sqrt{-1} \sum g_{\lambda, \beta} dz^\lambda_\lambda \wedge d\bar{z}^\beta_\lambda. \]

From this metric we can define a connection $\{\omega_\lambda\}, \omega_\lambda = (\omega_\lambda, \beta_\alpha)$ in a well known manner:

(1.10) \[ \omega_\lambda^{\beta_\alpha} = \sum_{\sigma=1}^m \Gamma^{\beta_\alpha}_{\lambda, \sigma} \, dz^\sigma_\lambda \] where $\Gamma^{\beta_\alpha}_{\lambda, \sigma} = \sum_{\sigma=1}^m g^{\beta_\sigma} \frac{\partial g_{\lambda, \gamma\sigma}}{\partial z^\gamma_\lambda}$,

where $(g^{\beta_\sigma})$ is the inverse of $(g_{\lambda, \beta\sigma})$. The Riemann curvature tensor is defined by

\[ R_{\lambda, \beta\gamma\delta} = \frac{\partial^2 \Gamma_{\lambda, \beta\gamma\delta}}{\partial z^\alpha_\lambda}, \]

and also we define

\[ R_{\lambda, \beta\gamma\delta} = \sum_{\rho=1}^m g_{\lambda, \rho\alpha} \, R_{\lambda, \beta\gamma\delta}. \]

As for the conjugates of the above, we define

\[ \Gamma_{\lambda, \beta\gamma} = \Gamma_{\lambda, \beta\gamma}, \quad R_{\lambda, \beta\gamma\delta} = R_{\lambda, \beta\gamma\delta} \quad \text{and} \quad \overline{R_{\lambda, \beta\gamma\delta}} = R_{\lambda, \beta\gamma\delta}. \]

The Ricci form is defined by

\[ R_{\lambda, \beta\gamma} dz^\gamma_\lambda \wedge d\bar{z}^\beta_\lambda, \quad \text{where} \quad R_{\lambda, \beta\gamma} = \sum_{\rho=1}^m R_{\lambda, \beta\gamma\rho}. \]

We infer that $\Gamma_{\lambda, \beta\gamma} = \Gamma_{\lambda, \beta\gamma}$, since the connection is induced from a kähler metric. The canonical line bundle $K$ of $X$ is defined to be

$K = \{ J_{\lambda, \mu} \}$, where $J_{\lambda, \mu} = \frac{\partial (z^1_\mu, z^2_\mu, \ldots, z^m_\mu)}{\partial (z^1_\lambda, z^2_\lambda, \ldots, z^m_\lambda)}$ on $U_\lambda \cap U_\mu$. 

We see that

\[ |f_{\lambda\mu}|^2 = \frac{g_{\lambda\lambda}}{g_{\mu\mu}} \quad \text{on } U_\lambda \cap U_\mu \] where \( g_{\lambda\lambda} = \det (g_{\lambda,\alpha \bar{\beta}}) \).

Therefore

\[ \{g_{\lambda}^{-1}\} \]

determines a metric of \( K \) on \( X_c \). The following is well known:

\[ R_{\lambda,\bar{\beta}\alpha} = \partial_a \partial_{\bar{\beta}} \log g_{\lambda\lambda} \]

In what follows we choose \( \{g_{\lambda}^{-1}\} \) as a metric of \( K \) and fix once for all. By using (1.1) and (1.10), we define a hermitian inner product in \( E \otimes K \).

\[ \langle h_{\lambda,kj} \rangle \text{ where } h_{\lambda,kj} = g_{\lambda}^{-1} h_{\lambda,kj} \]

Also for a convex increasing function \( \chi \), we take

\[ \langle e^{-\chi} (\tilde{h}_{\lambda,kj}) \rangle \]

Then we get another inner product in \( E \otimes K \). The Riemann curvature tensor induced from (1.13) is denoted by

\[ K_{\lambda,\bar{\beta}\alpha} \]

We see

\[ K_{\lambda,\bar{\beta}\alpha} = K_{\lambda,\bar{\beta}\alpha} - \delta_{j}^{\mu} \partial_a \partial_{\bar{\beta}} \chi(\psi) - \delta_{j}^{\mu} \partial_a \partial_{\bar{\beta}} \log g_{\lambda\lambda} \]

\[ \S 2. \text{ Differential and Integral Calculus of } E \otimes K\text{-valued Forms} \]

We recall differential and integral calculus of \( E \otimes K \)-valued forms on \( X_c \). Let \( C_{p,q}(X_c, E \otimes K) \) denote the space of \( C^\infty \)-differentiable \( E \otimes K \)-valued \((p, q)\)-forms on \( X_c \) and let \( D_{p,q}(X_c, E \otimes K) = \{ \varphi \in C_{p,q}(X_c, E \otimes K) \text{ the support of } \varphi \text{ is compact} \} \). We express \( \varphi = (\varphi^j) \in C_{p,q}(X_c, E \otimes K) \) as

\[ \varphi^j = -\frac{1}{p! q!} \sum_{a_1, \ldots, a_p, \beta_1, \ldots, \beta_q} \sum (\varphi^j)_{\lambda,\bar{\beta}_1, \ldots, \bar{\beta}_q} d z_{\lambda_{a_1}} z_{\lambda_{\beta_1}} \wedge \cdots \wedge d z_{\lambda_{\beta_q}}^q \]

For \( \varphi \in C_{p,q}(X_c, E \otimes K) \), we set
Particularly when \( \varphi \) is a \((0, q)\)-form, we write \((\varphi)_{\lambda}^{\alpha}...^{\beta_{q}}\) and \((\varphi)_{\lambda}^{\beta_{1}...^{\beta_{q}}}\).

With respect to (1.8) and (1.12) we define a hermitian inner product in \(C_{p,q}(X_{c}, E \otimes K)\) as follows: For \( \varphi \) and \( \psi \in C_{p,q}(X_{c}, E \otimes K) \)

\[
H_{\lambda}(\varphi, \psi) = \sum_{j,k} h_{j,k}^{\lambda}(\varphi)_{\lambda}^{\alpha}...^{\beta_{p}}(\psi)_{\lambda}^{\alpha}...^{\beta_{q}}.
\]

Also we define

\[
H_{\lambda}(\varphi, \psi) = e^{-x(\varphi)}H_{\lambda}(\varphi, \psi).
\]

We define

\[
(\varphi, \psi)_{\lambda_{k}} = \int_{X_{c}} H_{\lambda}(\varphi, \psi) dV,
\]

\[
(\varphi, \psi)_{\lambda} = \int_{X_{c}} H_{\lambda}(\varphi, \psi) dV \quad \text{for} \quad \varphi, \psi \in \mathcal{O}_{p,q}(X_{c}, E \otimes K)
\]

where \(dV = \frac{1}{m!} \Omega \wedge \Omega \wedge \cdots \wedge \Omega(\text{m-times})\). Particularly when \( \varphi = \psi \), we denote \((\varphi, \varphi)_{\lambda_{k}}\) (resp. \((\varphi, \varphi)_{\lambda}\)) by \(\|\varphi\|^{2}_{\lambda_{k}}\) (resp. \(\|\varphi\|^{2}_{\lambda}\)). \(\delta: C_{p,q}(X_{c}, E \otimes K) \rightarrow C_{p,q+1}(X_{c}, E \otimes K)\) is defined as usual. With respect to (2.2) (resp. (2.1)) the formally adjoint operator is defined, which is denoted by \(\vartheta_{\lambda}\) (resp. \(\vartheta_{\lambda_{k}}\)). The Laplace-Beltrami operator \(\square_{\lambda}\) is defined by \(\square_{\lambda} = \delta \vartheta_{\lambda} + \vartheta_{\lambda} \delta\).

Let \(\mathcal{F}_{p,q}(X_{c}, E \otimes K)\) denote \(E \otimes K\)-valued covariant tensor fields of type \((p, q)\). We write the \((a_{1}...a_{p}, \beta_{1}...\beta_{q})\)-component of \(\varphi \in \mathcal{F}_{p,q}(X_{c}, E \otimes K)\), \((\varphi)_{\lambda}^{a_{1}...a_{p}, \beta_{1}...\beta_{q}}\). The connections (1.2) and (1.9) derive covariant differentiations \(\nabla_{\alpha}^{(x)}\) of type \((1, 0)\) and \(\nabla_{\beta}^{(x)}\) of type \((0, 1)\) in \(\mathcal{F}_{p,q}(X_{c}, E \otimes K)\) respectively:

\[
(\nabla_{\alpha}^{(x)}\varphi)_{\lambda}^{a_{1}...a_{p}, \beta_{1}...\beta_{q}} = \frac{\partial (\varphi)_{\lambda}^{a_{1}...a_{p}, \beta_{1}...\beta_{q}}}{\partial x^{\alpha}}
\]

\[
+ \sum_{i=1}^{p} \Gamma_{\lambda, \alpha}^{(x)}(\varphi)_{\lambda}^{a_{1}...a_{p}, \beta_{1}...\beta_{q}} - \sum_{i=1}^{q} \sum_{\tau=1}^{m} \Gamma_{\lambda, a_{i}}^{\alpha}(\varphi)_{\lambda}^{a_{1}...\tau a_{i}...\beta_{1}...\beta_{q}}
\]

where \(\Gamma_{\lambda, \alpha}^{(x)}\) denotes the connection coefficients defined from (1.12) as in (1.2), and
Then we obtain the commutation formula: For \( \varphi \in \mathcal{F}_{p,q}(X, E \otimes K) \),

\[
(\nabla^x_\beta \varphi)_l^{i, a_1 \ldots a_p, \bar{b}_1 \ldots \bar{b}_q} = \frac{\partial (\varphi)_l^{i, a_1 \ldots a_p, \bar{b}_1 \ldots \bar{b}_q}}{\partial \bar{z}^\beta},
\]

\[ - \sum_{l=1}^q \sum_{r=1}^m \Gamma^r_{\lambda, \bar{b}_r} (\varphi)_l^{i, a_1 \ldots a_p, \bar{b}_1 \ldots \bar{b}_q}.
\]

Then we obtain the commutation formula: For \( \varphi \in \mathcal{F}_{p,q}(X, E \otimes K) \),

\[
(2.3) \quad \left[ (\nabla^x_\alpha \nabla^x_\beta - \nabla^x_\beta \nabla^x_\alpha) \varphi \right]_l^{i, a_1 \ldots a_p, \bar{b}_1 \ldots \bar{b}_q} = - \sum_{l=1}^q \sum_{r=1}^m R_{\lambda, \bar{b}_r} (\varphi)_l^{i, a_1 \ldots a_p, \bar{b}_1 \ldots \bar{b}_q} + \sum_{s=1}^r K_{\lambda, \bar{b}_s} (\varphi)_l^{i, a_1 \ldots a_p, \bar{b}_1 \ldots \bar{b}_q}.
\]

In the same manner as in Kodaira and Morrow [5] (see, p. 110, Proposition (5.3) and Theorem (5.2), and p. 112, Proposition (6.7)), we get for \( \varphi \in \mathcal{F}_{p,q}(X, E \otimes K) \),

\[
(2.4) \quad \left\{ \begin{array}{l}
(\partial^x_\alpha \varphi)_l^{i, a_1 \ldots a_p, \bar{b}_0 \ldots \bar{b}_q} = \sum_{\mu=0}^q (-1)^{p+\mu} \nabla^x_\beta (\varphi)_l^{i, a_1 \ldots a_p, \bar{b}_0 \ldots \bar{b}_q},
\end{array} \right.
\]

Therefore

\[
(\Box^x_\varphi)_l^{i, a_1 \ldots a_p, \bar{b}_1 \ldots \bar{b}_q} = - \sum_{\alpha, \beta} g^{\bar{b}_0 \bar{b}_\alpha} (\nabla^x_\alpha \nabla^x_\beta (\varphi)_l^{i, a_1 \ldots a_p, \bar{b}_1 \ldots \bar{b}_q}) - \sum_{\alpha, \beta} (-1)^{\mu} g^{\bar{b}_0 \bar{b}_\beta} \nabla^x_\alpha (\nabla^x_\beta (\varphi)_l^{i, a_1 \ldots a_p, \bar{b}_1 \ldots \bar{b}_q}).
\]

In what follows we consider only \((0, q)\)-forms. Then by (2.3)

\[
(\Box^x_\varphi)_l^{i, a_1 \ldots a_p, \bar{b}_1 \ldots \bar{b}_q} = - \sum_{\alpha, \beta} g^{\bar{b}_0 \bar{b}_\alpha} (\nabla^x_\alpha \nabla^x_\beta (\varphi)_l^{i, a_1 \ldots a_p, \bar{b}_1 \ldots \bar{b}_q}) - \sum_{\alpha, \beta} (-1)^{\mu} g^{\bar{b}_0 \bar{b}_\beta} \nabla^x_\alpha (\nabla^x_\beta (\varphi)_l^{i, a_1 \ldots a_p, \bar{b}_1 \ldots \bar{b}_q}).
\]

where

\[
R_{\beta, \mu}^\alpha = \sum_{\lambda} g^{\bar{b}_\lambda \bar{b}_\mu} R_{\lambda, \bar{b}_\mu}^\alpha.
\]

Then
As in Kodaira and Morrow [5] (see, p. 126), we can prove

\[-\sum \int_{X_c} e^{-x(\psi)} g^{\beta\gamma} \delta_{\lambda} R_{\beta\mu} (\phi)_{\lambda,\beta_1 \ldots \beta_q}^\mu \frac{\partial}{\partial \gamma} + \frac{\partial}{\partial \gamma} dV \geq 0.\]

Thus we obtain

\[(\Box_x \phi, \phi)_x \geq -\int_{X_c} \sum \int_{X_c} e^{-x(\psi)} g^{\beta\gamma} \delta_{\lambda} R_{\beta\mu} (\phi)_{\lambda,\beta_1 \ldots \beta_q}^\mu \frac{\partial}{\partial \gamma} + \frac{\partial}{\partial \gamma} dV

\sum \int_{X_c} e^{-x(\psi)} g^{\beta\gamma} \delta_{\lambda} K^{x}_{\lambda,\beta_1 \ldots \beta_q}^\mu \frac{\partial}{\partial \gamma} + \frac{\partial}{\partial \gamma} dV.\]

Referring to (1.11) and (1.14), the second term of the right side of (2.5) becomes

\[-\int_{X_c} \sum \int_{X_c} e^{-x(\psi)} g^{\beta\gamma} \delta_{\lambda} K^{x}_{\lambda,\beta_1 \ldots \beta_q}^\mu \frac{\partial}{\partial \gamma} + \frac{\partial}{\partial \gamma} dV

+ \int_{X_c} \sum \int_{X_c} e^{-x(\psi)} g^{\beta\gamma} \delta_{\lambda} \delta_{\lambda} (\psi)_{\lambda,\beta_1 \ldots \beta_q}^\mu \frac{\partial}{\partial \gamma} + \frac{\partial}{\partial \gamma} dV.\]

Here note that since \(\delta(\psi) \geq 0\), the last term in (2.6) is non-negative and that the first term in (2.5) and the second term in (2.6) cancel each other. Finally we obtain

**Theorem 3.** For \(\phi \in \mathcal{D}_{0,e}(X_c, E \otimes K)\), we have

\[(\Box_x \phi, \phi)_x \geq -\int_{X_c} \sum \int_{X_c} e^{-x(\psi)} g^{\beta\gamma} \delta_{\lambda} K^{x}_{\lambda,\beta_1 \ldots \beta_q}^\mu \frac{\partial}{\partial \gamma} + \frac{\partial}{\partial \gamma} dV.\]

§3. Proofs of Theorems 1 and 2

First we prove Theorem 1. Making completion of \(\mathcal{D}_{0,e}(X_c, E \otimes K)\)
with respect to $\|\varphi\|_2^2$ (resp. $\|\varphi\|_2^2$) we obtain a Hilbert space $L^2_{0,q}(X_c, E\otimes K, \chi)$ (resp. $L^2_{0,q}(X_c, E\otimes K, \chi)$). We extend $\bar{\varphi} : \mathcal{D}_{0,q}(X_c, E\otimes K) \to \mathcal{D}_{0,q+1}(X_c, E\otimes K)$ (resp. $\bar{\varphi} : \mathcal{D}_{0,q+1}(X_c, E\otimes K) \to \mathcal{D}_{0,q+2}(X_c, E\otimes K)$) to the differential operator in the sense of distribution, which is denoted by $T : L^2_{0,q}(X_c, E\otimes K, \chi) \to L^2_{0,q+1}(X_c, E\otimes K, \chi)$ (resp. $S : L^2_{0,q+1}(X_c, E\otimes K, \chi) \to L^2_{0,q+2}(X_c, E\otimes K, \chi)$). Then $T$ (resp. $S$) is a densely defined closed operator, so the adjoint operator $T^*$ (resp. $S^*$) can be defined.

Consider

$$L^2_{0,q}(X_c, E\otimes K, \chi) \xrightarrow{T} L^2_{0,q+1}(X_c, E\otimes K, \chi) \xrightarrow{S} L^2_{0,q+2}(X_c, E\otimes K, \chi).$$

First we infer that by the completeness of $\psi$, there exists a convex increasing function $\chi$ such that $\|\varphi\|_2^2 < +\infty$ for any $\varphi \in C_{0,q}(X_c, E\otimes K)$. Then in view of Dolbault isomorphism and a lemma on $L^2$-estimate (see, Hörmander [3], p. 78, Lemma 4.1.1), it is sufficient for the proof of Theorem 1 to prove the following

**Theorem 4.** There exists a positive constant $C_0$ which does not depend on the choice of $\chi$ such that

\[
\|\varphi\|_2^2 \leq C_0 (\|T^\ast \varphi\|_2^2 + \|S\varphi\|_2^2) \quad \text{for} \quad \varphi \in D(T^\ast) \cap D(S),
\]

where

\[
D(T^\ast) = \{ \varphi \in L^2_{0,q+1}(X_c, E\otimes K, \chi) : T^\ast \varphi \in L^2_{0,q}(X_c, E\otimes K, \chi) \},
\]

\[
D(S) = \{ \varphi \in L^2_{0,q+1}(X_c, E\otimes K, \chi) : S\varphi \in L^2_{0,q+2}(X_c, E\otimes K, \chi) \}.
\]

**Proof.** By the choice of the base metric, it is a complete metric. So referring to a key lemma which is due to A. Andreotti and E. Vesentini [1] (see, p. 93, Proposition 5), we have only to prove (*) for $\varphi \in D_{0,q+1}(X_c, E\otimes K)$. Let $C$ denote the minimum of the eigen values of $(-K_{x,ij})$ on $X_c$, then we see that $C > 0$. Thus by Theorem 3 we have

\[
(\Box \chi \varphi, \varphi)_x \geq C_0 \|\varphi\|_x^2,
\]

where $C_0 = (q+1)C$,

which proves (*).
Next we prove Theorem 2. We follow the proof given in the approximation theorem on Stein manifolds (see, L. Hörmander [3], p. 89–90). For $E \otimes K$-valued forms $\varphi$ and $\psi$, we set

\[
(\varphi, \psi)_{\bar{\partial}id} = \int_{X_d} H_{\bar{\partial}}(\varphi, \psi)dV.
\]

To prove Theorem 2 it is sufficient to show that if $u \in \mathcal{L}_{0,0}^2(X_\partial, E \otimes K, \chi)$ satisfies $(u, \varphi)_{\bar{\partial}id} = 0$ for any $\varphi \in H^0(X_c, \mathcal{O}(E \otimes K))$, then $(u, \varphi)_{\bar{\partial}id} = 0$ for any $\varphi \in H^0(X_\partial, \mathcal{O}(E \otimes K))$. Take such a $u$. We extend $u$ by setting 0 outside of $X_d$ and denote it by the same latter $u$. Let $N_T$ be the null space of $T: \mathcal{L}_{0,0}^2(X_c, E \otimes K, \chi) \rightarrow \mathcal{L}_{0,1}^2(X_c, E \otimes K, \chi)$, then we see that

\[
N_T^\perp = H^0(X_c, \mathcal{O}(E \otimes K)) \cap \mathcal{L}_{0,0}^2(X_c, E \otimes K, \chi),
\]

where $N_T^\perp$ denotes the orthogonal complement of $N_T$. So $u e^{x(\psi)}$ is contained in $N_T^\perp$. By a lemma due to L. Hörmander [3] (see, p. 79, Lemma 4.1.2) we see that there exists an $f \in \mathcal{L}_{0,1}^2(X_c, E \otimes K, \chi)$ such that

\[
(3.1) \quad u e^{x(\psi)} = T^* f \quad \text{and} \quad \|f\|^2_2 \leq C_0 \|u\|^2_2.
\]

Set $g = e^{-x(\psi)}f$. Then we have by (3.1)

\[
u = \partial_h g.
\]

Now we choose a sequence of functions $\{\chi_v\}$ such that (1) $\chi_v$ is a convex increasing function, (2) $\chi_v \geq \chi_1$ for each $v$, (3) $\chi_v(t) = 1$ if $t \leq d$ and (4) for any $t \in (d, c)$ $\chi_v(t) \rightarrow \infty$ ($v \rightarrow \infty$).

For each $\chi_v$ we get $g^{(v)}$. By (3.1) and (3) there exists a positive constant $M$ which does not depend on $v$ such that

\[
\int_{X_c} e^{x(\psi)}H_{\bar{\partial}}(g^{(v)}), g^{(v)})dV \leq M.
\]

Then $g^{(v)} \in \mathcal{L}_{0,0}^2(X_c, E \otimes K, -\chi_1)$ and $g^{(v)}$ is bounded. Therefore there exists a subsequence which converges weakly to a limit $g_0$. By (4) we see that $g_0 = 0$ on $X_c \cap X_\partial$. Also by the continuity of differentiation in the sense of distribution, we have $u = \partial_k g_0$. Therefore, $(u, \alpha)_{\bar{\partial}id} = (g_0, \bar{\partial} \alpha)_{\bar{\partial}id}$ for $\alpha \in \mathcal{D}_{0,0}(X_c, E \otimes K)$. Take $\tilde{\varphi} \in H^0(X_\partial, \mathcal{O}(E \otimes K))$ and extend $\tilde{\varphi}$ to $\tilde{\varphi}^*$ such that $\tilde{\varphi}^* \in \mathcal{D}_{0,0}(X_c, E \otimes K)$. Then we see that $(u, \varphi)_{\bar{\partial}id} = 0$, which proves Theorem 2.
References


