The Principle of Limit Amplitude for Symmetric
Hyperbolic Systems of First Order
in the Half-Space \( \mathbb{R}^n_+ \)

By

Seiichiro WAKABAYASHI*

§ 1. Introduction

The present paper is concerned with the principle of limit amplitude for symmetric hyperbolic systems in a half-space. Our proof for the validity of the principle is based on the eigenfunction expansion theorem established in the preceding paper [3]. For the notation and terminology in this paper we refer the reader to [3]. We shall consider the following mixed initial-boundary value problem for hyperbolic systems:

\[
\begin{align*}
\frac{\partial}{\partial t} u(t, x) &= iAu(t, x) + \frac{1}{t} e^{ikt} f(x), \quad t > 0, \quad x \in \mathbb{R}^n_+, \\
Bu(t, x)|_{x_n=0} &= 0, \\
u(0, x) &= g(x),
\end{align*}
\]

where \( k (\neq 0) \) is a real number,

\[
A = \frac{1}{i} \sum_{j=1}^{n} A_j \frac{\partial}{\partial x_j},
\]

the \( A_j \) are \( 2m \times 2m \) constant Hermitian matrices, \( B \) is an \( m \times 2m \) constant matrix with rank \( m \) and \( u(t, x) \), \( g(x) \) and \( f(x) \) are vector-valued functions whose values lie in \( \mathbb{C}^{2m} \). Replacing \( u(t, x) \) in (1.1) and (1.2) by \( e^{ikt} v(x) \), we obtain the corresponding stationary problem:

\[
(A - kI)v(x) = f(x), \quad x \in \mathbb{R}^n_+,
\]
We assume the following conditions:

(L.1) \( L \equiv \left( I - \frac{\partial}{\partial t} - iA \right) \) is uniformly propagative.

(L.2) The operator \( A \) is elliptic.

(L.3) The multiplicity of the real roots of \( Q(\lambda, \eta)|_{\eta=(\xi, \tau)} = 0 \) with respect to \( \tau \) is not greater than two for every \( \xi \in \Xi \) and real \( \lambda \neq 0 \). Moreover the equation has at most only one couple of real double roots for every \( (\xi, \lambda) \neq (0, 0) \).

(B.1) The boundary matrix \( B \) is minimally conservative.

Under the above assumptions the following expansion theorem was proved in [3]:

**Theorem 1** (cf. Theorem 6.2 in [3]). Assume that the conditions (L.1)-(L.3) and (B.1) are satisfied and that \( f \in L^2(\mathbb{R}^n) \).

(i) The expansion formula

\[
Pf(x) = \sum_{j=1}^{2^p} \int_{\Xi^n} \Psi_j^+(x, \eta) f_j^+(\eta) d\eta
\]

\[+ \sum_{v=1}^{s} \sum_{j=1}^{2^p} \int_{D_v \times \Xi} \Psi_j^{+2,v}(x, \eta) f_j^{+2,v}(\eta) d\eta
\]

holds, where

\[
f_j^+(\eta) = \int_{\mathbb{R}^n} \Psi_j^+(x, \eta)^* f(x) dx, \quad 1 \leq j \leq 2^p,
\]

\[
f_j^{+2,v}(\eta) = \int_{\mathbb{R}^n} \Psi_j^{+2,v}(x, \eta)^* f(x) dx, \quad 1 \leq j \leq 2^p, 1 \leq v \leq s.
\]

Here the above integrals are taken in the sense of the limit in the mean and \( P \) is the orthogonal projection onto \( R(A)^{\perp} = N(A)^{\perp} \).

(ii) \( f \in D(A) \) if and only if \( \lambda_j(\eta) f_j^+(\eta) \in P_j(\eta)L^2(\Xi^n), k_v(\xi) f_j^{+2,v}(\eta) \in P_j(\eta)L^2(D_v \times \Xi), 1 \leq j \leq 2^p, 1 \leq v \leq s \). Then

\[
(Af)(x) = \sum_{j=1}^{2^p} \int_{\Xi^n} \lambda_j(\eta) \Psi_j^+(x, \eta) f_j^+(\eta) d\eta
\]

\[+ \sum_{v=1}^{s} \sum_{j=1}^{2^p} \int_{D_v \times \Xi} k_v(\xi) \Psi_j^{+2,v}(x, \eta) f_j^{+2,v}(\eta) d\eta,
\]
(1.11) \[(Af)^{j+\pm}(\eta) = \lambda_j(\eta) \chi^j(\eta), \quad 1 \leq j \leq 2\rho, \]

(1.12) \[(Af)^{j+2\nu+}(\eta) = k_{\nu}(\xi) \chi_{j+2\nu}(\eta), \quad 1 \leq j \leq 2\rho, \quad 1 \leq \nu \leq s. \]

Here the \(\lambda_j(\eta)\) are distinct roots of \(Q(\lambda, \eta) = 0\) and the \(k_{\nu}(\xi)\) are non-vanishing zeros of the Lopatinski determinant (see [3]).

**Remark.** The differential operator \(A\) defines an unbounded linear operator \(\mathcal{A}\) in \(L^2(\mathbb{R}^n)\) with domain

\[D(\mathcal{A}) = \{v(x) \in C_0^\infty(\overline{\mathbb{R}^n}); Bv(x)|_{x_n=0} = 0\}.\]

Thus we denoted by \(A\) the closure of \(\mathcal{A}\) in [3], which is a self-adjoint operator in \(L^2(\mathbb{R}^n)\).

In [3] it was proved that \(\sigma(A) = \mathbb{R}^1\). Hence in general there exist no solutions of (1.5) and (1.6) belonging to \(L^2(\mathbb{R}^n)\) for real \(k\) and \(f \in L^2(\mathbb{R}^n)\). Then the question arises of determining uniquely a solution of (1.5) and (1.6). There are three important approaches of deriving a unique solution of (1.5) and (1.6), i.e., the radiation principle, the limiting absorption principle and the limiting amplitude principle. The limiting absorption principle for (1.5) and (1.6) was justified in M. Matsumura [2] under more restrictive assumptions than ours.

In this paper the following theorems will be proved:

**Theorem 2.** Let \(f(x) = 0\) in (1.1), \(g(x) \in D(A)\) in (1.3) and \(\text{Supp}\,g(x)\) be bounded. Suppose that at least one of the following conditions holds:

a) \(E^+(\xi; 0) \cap \mathcal{B} = \{0\}\) for \(|\xi| \neq 0\), b) \(g(x)\) belongs to \(C^\infty(\mathbb{R}^n)\). Then for the solution \(u(t, x)\) of the equations (1.1)–(1.3) and any compact set \(K \subset \mathbb{R}^n\)

\[
(1.13) \quad \frac{d^l}{dt^l} u(t, x) \longrightarrow \delta_{0,l}g_2(x) \text{ in } L^2(K) \text{ as } t \longrightarrow \infty, \quad l = 0, 1,
\]

hold, where \(\delta_{0,l} = 1\) for \(l = 0, = 0\) for \(l = 1\), and \(g = g_1 + g_2, g_1 \in N(A) = R(A)^e, g_2 \in N(A)\). Moreover if a) holds, then \(g_2 = 0\).

**Theorem 3** (the limiting amplitude principle). Let \(f \in L^2(\mathbb{R}^n), g \in D(A)\) and \(k \neq 0\) be real in (1.1)–(1.3) and \(\text{Supp}\,f(x)\) and \(\text{Supp}\,g(x)\)
be bounded. Suppose that at least one of the following conditions holds: (a) $E^+(\xi; 0) \cap \mathcal{B} = \{0\}$ for $|\xi| \neq 0$; (b) $f(x)$ and $g(x)$ belong to $C^\infty(\mathbb{R}_+^n)$. Then for the solution $u(t, x)$ of the equations (1.1)–(1.3) and any compact set $K \subset \mathbb{R}_+^n$.

\[
(1.14) \quad e^{-ikt} \left[ \frac{d^l}{dt^l} u(t, x) - \delta_{0,t} \left( \frac{1}{k} f_2 + g_2 \right) \right] \rightarrow (ik)^l v(x) \quad \text{in} \quad L^2(K)
\]

as $t \rightarrow \infty$, $l = 0, 1,$

hold, where $v(x) \in L^2_{loc}(\mathbb{R}_+^n)$ and $f = f_1 + f_2, f_1 \in N(A)^1, f_2 \in N(A)$. Moreover, $\alpha_R(x) v(x) \in D(A)$ and $v(x)$ satisfies (1.5) in the sense of distribution and also satisfies (1.5) and (1.6) in the following sense: There exists a sequence $\{v_j\} \subset H^1(\mathbb{R}_+^n)$ such that $B v_j|_{x_n=0} = 0$, $v_j \rightarrow v(x)$ in $L^2_{loc}(\mathbb{R}_+^n)$ and $(A - kI) v_j \rightarrow f$ in $L^2_{loc}(\mathbb{R}_+^n)$ as $j \rightarrow \infty$. Here $\alpha_R(x) = 1$ for $|x| \leq R$, $= 0$ for $|x| \geq R + 1$ and belongs to $C^\infty(\mathbb{R}_+^n)$.

**Remark 1.** a) $E^+(\xi; 0) \cap \mathcal{B} = \{0\}$ for $|\xi| \neq 0$ implies the coercivity of $B$ for the elliptic operator $A$. Thus if a) holds, then

\[
(1.14)' \quad e^{-ikt} \frac{d^l}{dt^l} u(t, x) \rightarrow (ik)^l v(x) \quad \text{in} \quad H^1_{loc}(\mathbb{R}_+^n) \quad \text{as} \quad t \rightarrow \infty, \quad l = 0, 1,
\]

and $v(x)$ is a solution of (1.5) and (1.6) belonging to $H^1_{loc}(\mathbb{R}_+^n)$. 2)

**Remark 2.** From Lemma 3 it is easy to see that

\[
(A - \lambda I)^{-1} f(x) \rightarrow v(x) \quad \text{in} \quad L^2_{loc}(\mathbb{R}_+^n) \quad \text{as} \quad \lambda \rightarrow k - i0.
\]

In order to prove the above theorems we state some lemmas and propositions in §2, following D. M. Eidus [1]. In §3 we shall give their proofs.

### §2. Preliminaries

Let $\mathcal{H}$ be a separable Hilbert space, $\mathcal{H}_0$ a subspace of $\mathcal{H}$, $P$ the orthogonal projection onto $\mathcal{H}_0$, and $A$ a self-adjoint operator in $\mathcal{H}$.  

1) $L^2_{loc}(\mathbb{R}_+^n)$ denotes the space of vector-valued functions $f$ such that $f \in L^1(K)$ for any compact set $K \subset \mathbb{R}_+^n$.

2) $H^1_{loc}(\mathbb{R}_+^n)$ denotes the space of vector-valued functions $f$ such that $\frac{\partial^\alpha}{\partial x^\alpha} f(x) \in L^1(\mathbb{R}_+^n)$ for $|\alpha| \leq 1$ and any $R > 0$. 

Definition 1. An element $f \in \mathcal{H}$ is said to satisfy the condition $T_1$ if for all real $a$ and $b$ ($a \cdot b > 0, a < b$) there exists a constant $C$ such that for any complex $\lambda = k + \varepsilon i$ ($\varepsilon \neq 0, k \in [a, b]$) the inequality

$$\|\overline{\mathcal{P}}R(\lambda)f\| \leq C$$

holds.

Lemma 1. Suppose that $f$ satisfies $T_1$. Then for almost all $k \in \mathbb{R}^1$ there exist the weak limits

$$w \lim_{\lambda \rightarrow k \pm i0} \overline{\mathcal{P}}R(\lambda)f = v^{(\pm)}(k).$$

This lemma can be proved by Fatou's theorem and Riemann's mapping theorem.

Lemma 2. Let $f$ satisfy $T_1$. Suppose that $F(k)$ is a complex-valued function, continuous and bounded on $(−\infty, 0) \cup (0, \infty)$. Then for any real $a$ and $b$ ($a \cdot b > 0, a < b, \text{ including } a = -\infty \text{ and } b = \infty$)

$$\overline{\mathcal{P}} \int_b^a F(k) dE(k)f = \frac{1}{2\pi i} \int_b^a F(k) \theta(k) dk$$

holds, where $\{E(k)\}_{−\infty < k < \infty}$ denotes the right-continuous spectral family of $A$ and

$$\theta(k) = v^+(k) - v^-(k).$$

Here the integral on the right-hand side of (2.3) is taken in the sense of Bochner's integral.

We first consider the Cauchy problem for $t > 0$

$$\frac{d}{dt} u(t) = iAu(t) + \frac{1}{i} e^{ikt} f,$$

(2.6) $u(0) = g \in D(A)$.

The following proposition can be easily verified.

3) $\lambda \rightarrow k \pm i0$ means that $\lambda \rightarrow k$ along any path that does not cross the real axis and is not tangent to it.
Proposition 1. Let \( f = 0 \) in (2.5), \( g \) satisfy \( T_1 \) and \( g \in D(A) \). Let \( u(t) \) be the solution of (2.5), (2.6). Then

\[
\lim_{t \to \infty} \frac{d}{dt} \delta_{0,l} g_2 \quad \text{in} \quad \mathcal{H} \quad \text{as} \quad t \to \infty, \ l = 0, 1,
\]

hold, where \( \delta_{0,l} = 1 \) for \( l = 0 \), \( = 0 \) for \( l = 1 \), and \( g = g_1 + g_2, \ g_1 \in N(A)^+, \ g_2 \in N(A) \).

Definition 2. An element \( f \in \mathcal{H} \) is said to satisfy the condition \( T_2 \) if it satisfies \( T_1 \) and if \( \theta(k) \) determined by (2.4) satisfies a Hölder condition on any interval \([a, b] \subset \mathbb{R}\{0\} \).

Lemma 3. Let \( f \in N(A)^\perp \) satisfy \( T_2 \). Then replacing the weak limits by the strong limits in (2.2) Lemma 1 holds. Moreover

\[
v(\pm)(k) = \pm \frac{\theta(k)}{2} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\theta(k_1)}{k_1 - k} dk_1,
\]

where the integral is taken in the sense of the principal value. \( v(\pm)(k) \) satisfy a Hölder condition on any interval \([a, b] \subset \mathbb{R}\{0\} \).

This lemma follows from Lemma 2.

Proposition 2. Let \( f \) satisfy \( T_2, g = 0 \) and \( k \neq 0 \) be real in (2.5), (2.6), and \( u(t) \) be the solution of (2.5), (2.6). Then

\[
e^{-ikt} \lim_{t \to \infty} \delta_{0,l} \frac{d}{dt} \left( \frac{1}{k} f \right) \quad \text{in} \quad \mathcal{H}
\]

hold.

Using Lemmas 2 and 3, we can easily prove the above proposition.

Lemma 4. Let \( f \in \mathcal{H} \) be such that for all \( a \) and \( b \) with \( a \cdot b > 0, \ a < b, \) and any \( h_0 \in \mathcal{H}_0 \) there exists \( C_0 > 0 \) such that for all complex \( \lambda = k + \varepsilon i \) \((k \in [a, b], \varepsilon \neq 0) \) the inequality

\[ |(R(\lambda)f, R(\lambda)h_0)| \leq C_0 \]

(2.10)
holds. Then \( f \) satisfies \( T_2 \). Here \( v^\pm(k) \) satisfy a H"older condition with exponent 1 on an arbitrary interval \([a, b] \subset \mathbb{R}\setminus\{0\}\).

The above lemma follows from the equality

\[
\frac{d}{d\lambda} (R(\lambda)f, h_0) = (R(\lambda)^2f, h_0) = (R(\lambda)f, R(\lambda)h_0).
\]

\section*{§3. Proof of the Theorems}

We first state the following

\textbf{Lemma 5.} Let \( f(r) \) be a complex-valued function and \( f(r) \in L^1(0, \infty) \cap C^{1+a}(0, \infty), a > 0 \). Then for all \( a \) and \( b \) \((a \cdot b > 0, a < b)\) there exists a constant \( C_0 > 0 \) such that for any \( \gamma \in [a, b] \)

\[
\left| \int_{0}^{\infty} \frac{f(r)}{(r-\lambda)^{2\gamma}} \, dr \right| \leq C_0
\]

holds, where \( \lambda = k + i\varepsilon, \varepsilon \neq 0 \).

Now let \( f, h \in L^2(\mathbb{R}^+), \text{Supp} f(x) \text{ and Supp} h(x) \) be bounded and \( A \) the self-adjoint operator associated with (1.5) and (1.6). Then it follows from Theorem 1 that for \( \text{Im} \lambda \neq 0 \)

\[
(R(\lambda)Pf, R(\lambda)h) = \sum_{j=1}^{p} \int_{E} \frac{1}{(\lambda_j - \lambda)^{2\gamma}} f^{\pm}_j(\eta) \cdot \bar{h}^{\pm}_j(\eta) d\eta
\]

hold. We can replace \( f \) by \( h \) in (3.2) and (3.3). Moreover we have \( PR(\lambda)f = R(\lambda)Pf \). Thus by Theorem 5.10 in [3] we obtain

\[
(R(\lambda)Pf, R(\lambda)h) = \sum_{j=1}^{p} \int_{E} \frac{1}{(\lambda_j - \lambda)^{2\gamma}} f^{\pm}_j(\eta) \cdot \bar{h}^{\pm}_j(\eta) d\eta
\]

\[
+ \sum_{j=1}^{p} \sum_{s=1}^{k} \int_{D_v \times \mathbb{R}} \frac{1}{(k_v(\xi) - \lambda)^{2\gamma}} f^{\pm}_{j+2\nu p}(\eta) \cdot \bar{h}^{\pm}_{j+2\nu p}(\eta) d\eta.
\]

\[\text{4) } C^{1+a}(0, \infty) \text{ consists of continuous functions whose derivatives satisfy a H"older condition with exponent } a \text{ on any bounded closed subinterval of } (0, \infty).\]
Let us consider the case $1 \leq j \leq \rho$. Put $S_j = \{ \eta \in \mathbb{R}^n; \lambda_j(\eta) = 1 \}, r = \lambda_j(\eta)$ and $\omega = \frac{\eta}{\lambda_j(\eta)}, 1 \leq j \leq \rho$. Then

$$(3.5) \quad \int_{\mathbb{R}^n} \frac{1}{(\lambda_j(\eta) - \lambda)^2} \hat{f}_j^+ (\eta) \cdot \hat{h}_j^+ (\eta) d\eta$$

$$= \int_0^\infty \frac{r^{n-1}}{(r - \lambda)^2} \int_{S_j} (\omega \cdot n_j(\omega)) \hat{f}_j^+ (r \omega) \cdot \hat{h}_j^+ (r \omega) dS_j$$

holds, where $n_j(\omega)$ denotes the outward unit normal to $S_j$ at $\omega$ and $dS_j$ the surface element on $S_j$. Put

$$(3.6) \quad I_j(r) = \int_{S_j} (\omega \cdot n_j(\omega)) \hat{f}_j^+ (r \omega) \cdot \hat{h}_j^+ (r \omega) dS_j.$$

Let us prove below that $I_j(r) \in C^\infty(0, \infty)$. By Theorem 6.1 in [3] we have a local representation

$$\Psi_j^+(x, \eta) = \begin{cases} 0 & \text{(when there exists } l_j, 1 \leq l_j \leq \rho, \text{ such that} } \\
\tau_j^+(\xi; \lambda_j(\eta) \pm i0) = \sigma) \\
(2\pi)^{-n} e^{ix \cdot \eta} P_j(\eta) + (2\pi)^{-n} \sum_{\mu=1}^p e^{ix \cdot \xi} e^{-i\xi_2 (\xi_1 \lambda_j(\eta) + i0) x_n} \\
\times C_{j\mu}(\eta) + (2\pi)^{-n} e^{ix \cdot \xi} \sum_{\mu(l_j)=l_j} \frac{\partial \lambda_j(\eta)}{\partial \xi} \\
\times \det (Bh_1^+(\xi; \lambda_j(\eta) \mp i0), \ldots, Bh_4(x_n, \xi; \lambda_j(\eta) \mp i0), \ldots, Bh_4^*(\xi; \lambda_j(\eta) \mp i0))^{*} \\
\text{(when there exists } l_j, 1 \leq l_j \leq \rho, \text{ such that} } \\
\tau_j^+(\xi; \lambda_j(\eta) \mp i0) = \sigma), \end{cases}$$

5) $\lambda_1(\eta) > \lambda_2(\eta) > \cdots > \lambda_p(\eta) > 0 > \lambda_{p+1}(\eta) > \cdots > \lambda_{2n}(\eta) \text{ for } \eta \neq 0.$
where the $C_{j\mu}(\eta)$ are $2m \times 2m$ matrices whose elements are positively homogeneous of degree 0. Noting that the $h_+^j(\xi; \lambda_j(\eta) \mp i0)$, $\frac{\partial \lambda_j(\eta)}{\partial \tau}$ and $A(\xi; \lambda_j(\eta) \mp i0)$ are positively homogeneous of degree 0 and that the $\tau^+_n(\xi; \lambda_j(\eta) \mp i0)$ are positively homogeneous of degree 1, we obtain

$$\frac{\partial^k \Psi^\pm_j(x, \eta)}{\partial r^k}$$

$$= \begin{cases} 0 & (\tau^+_j(\xi; \lambda_j(\eta) \mp i0) = \sigma), \\ (2\pi)^{-\frac{n}{2}} \left( \frac{i x \cdot \eta}{r} \right)^k e^{i x \cdot \eta} P_j(\eta) + (2\pi)^{-\frac{n}{2}} \sum_{\mu=1}^P b_{\mu}^j & \\ \left( \frac{i x \cdot \xi + i \tau^+_n(\xi; \lambda_j(\eta) \mp i0) x_n}{r} \right)^k e^{i x \cdot \xi} e^{i \xi \cdot \tau^+_n(\xi; \lambda_j(\eta) \mp i0)} x_n C_{j\mu}(\eta) \right) \\ + (2\pi)^{-\frac{n}{2}} e^{i x \cdot \xi} \sum_{p(l)=1}^{m} \left[ \frac{\partial \lambda_j(\eta)}{\partial \tau} \right] \left[ \frac{1}{A(\xi; \lambda_j(\eta) \mp i0)} \right] \sum_{h=0}^k \left( \frac{k}{h} \right) \left( \frac{i x \cdot \xi}{r} \right)^k \\ \times \det(Bh_1^j, \ldots, \frac{\partial^k \tau^+}{\partial r^k} Bh_n^j, B\eta(x, \xi; \lambda_j(\eta) \mp i0), \ldots, Bh_n^j) \\ \tau^+_j(\xi; \lambda_j(\eta) \mp i0)^* \right)_{v=1, \ldots, 2m} (\tau^+_j(\xi; \lambda_j(\eta) \mp i0) = \sigma), \\ k=1, 2, \ldots. \end{cases}$$

(3.8)

Here we have used the relation $\frac{\partial}{\partial r} = \sum_{k=1}^n \frac{\eta_k}{r} \frac{\partial}{\partial \eta_k}$ and Euler's identity.

It follows from the estimates of $\Psi^\pm_j(x, \eta)$ in [3] that for fixed $a$ and $b$, $0 < a < b$, and $r \in [a, b]$

$$\left| \left( \frac{\partial^k \Psi^\pm_j}{\partial r^k} (x, \eta) \right)^* f(x) \right| \leq F^j_k(x, \omega)$$

holds, where $F^j_k(x, \omega)$ satisfies the inequality

(3.9)

$$\int_{S_j} (\omega \cdot n_j(\omega)) \left( \int_{\mathbb{R}^n} |F^j_k(x, \omega)| dx \right)^2 dS_j < + \infty,$$

(3.10)
since \( \frac{|\eta|}{r} \leq C \), \( \text{Supp}(f(x)) \) is bounded and \( \frac{\partial^{i}}{\partial r^{i}} Bq_{r}(x_{n}, \zeta; \lambda, (\eta) \mp i0) \) is continuous in \((x_{n}, \eta)\). Also we can replace \( f(x) \) and \( F_{j}(x, \omega) \) by \( h(x) \) and \( H_{j}(x, \omega) \) in (3.9) and (3.10), respectively. Therefore we have

\[
\frac{\partial^{k}}{\partial r^{k}} \tilde{f}_{j}(r) = \iint_{R^{2}} \left( \frac{\partial^{k} \Psi^{+}}{\partial r^{k}}(x, \eta) \right)^{*} f(x) dx,
\]

\[
\frac{\partial^{k}}{\partial r^{k}} \tilde{H}_{j}(r) = \iint_{R^{2}} \left( \frac{\partial^{k} \Psi^{+}}{\partial r^{k}}(x, \eta) \right)^{*} h(x) dx.
\]

Thus we have

\[
\frac{\partial^{k}}{\partial r^{k}} I_{j}(r) = \iint_{S_{j}} (\omega \cdot n_{j}(\omega)) \sum_{k=0}^{k} \left( \frac{k}{h} \right) \frac{\partial^{k}}{\partial r^{k}} \tilde{f}_{j}(r) \cdot \frac{\partial^{k-h}}{\partial r^{k-h}} \tilde{H}_{j}(r) dS_{j},
\]

since

\[
\left| \frac{\partial^{k}}{\partial r^{k}} \tilde{f}_{j}(r) \cdot \frac{\partial^{k-h}}{\partial r^{k-h}} \tilde{H}_{j}(r) \right| \leq C \left\{ \left( \iint_{R^{2}} |F_{j}(x, \omega)| dx \right)^{2} + \left( \iint_{R^{2}} |H_{j}^{+}(x, \omega)| dx \right)^{2} \right\}.
\]

From this it follows that \( I_{j}(r) \in C^{\omega}[a, b] \), i.e., \( I_{j}(r) \in C^{\omega}(0, \infty) \). Put \( S_{v} = \{ \eta \in \mathbb{R}^{n}; k_{v}(\xi) = 1, \xi \in D_{v} \}, D_{v}^{+} = \{ \xi \in D_{v}; k_{v}(\xi) > 0 \} \) and \( D_{v}^{-} = D_{v} \setminus D_{v}^{+}, 1 \leq v \leq s \). Moreover put \( r = k_{v}(\xi) \) and \( \omega = \frac{\eta}{k_{v}(\xi)} \) for \( \xi \in D_{v}^{+} \). Then

\[
\int_{D_{v} \times S_{v}} \frac{1}{(k_{v}(\xi) - \lambda)^{2}} \tilde{f}_{j+2\nu_{v}}(\eta) \cdot \tilde{H}_{j+2\nu_{v}}(\eta) d\eta
\]

\[
= \int_{D_{v}^{-} \times S_{v}} \frac{1}{(k_{v}(\xi) - \lambda)^{2}} \tilde{f}_{j+2\nu_{v}}(\eta) \cdot \tilde{H}_{j+2\nu_{v}}(\eta) d\eta
\]

\[
+ \int_{0}^{\infty} \frac{r^{n-1}}{(r - \lambda)^{2}} dr \iint_{S_{v}} |\omega \cdot \tilde{n}_{v}(\omega)| \tilde{f}_{j+2\nu_{v}}(r, \omega) \cdot \tilde{H}_{j+2\nu_{v}}(r, \omega) dS_{v}
\]

holds, where \( \tilde{n}_{v}(\omega) \) denotes the unit normal to \( S_{v} \) at \( \omega \) and \( dS_{v} \) denotes the surface element on \( S_{v} \). Put

\[
I_{j+2\nu_{v}}(r) = \iint_{S_{v}} |\omega \cdot \tilde{n}_{v}(\omega)| \tilde{f}_{j+2\nu_{v}}(r, \omega) \cdot \tilde{H}_{j+2\nu_{v}}(r, \omega) dS_{v}.
\]

By the definition of \( \Psi_{j+2\nu_{v}}^{+}(x, \eta) \) in [3] we see that
(3.16) \[ \Psi_{\mu}^{+2\nu}(x, \eta) = (2\pi)^{-\frac{n}{2}} e^{i\tau \cdot \xi} \sum_{\mu=1}^{\frac{n}{2}} e^{i\tau \cdot \xi} (\xi; k_v(\xi) \mp i0) x_n C_{\mu}^*(\eta) \]

\[ + (2\pi)^{-\frac{n}{2}} e^{i\tau \cdot \xi} \sum_{\mu=1}^{\frac{n}{2}} \left[ \left( \frac{\lambda - k_v(\xi)}{A(\xi; \lambda)} \right) \text{det}(Bh_{\mu}^1(\xi; \lambda)) \right] \]

\[ \cdots, Bq_{\mu}(x_{m}, \xi; \lambda), \cdots, Bh_{\mu}^m(\xi; \lambda) |_{\lambda = k_v(\xi) \pm i0} \]

\[ \times \gamma_{\mu}^*(\eta) \]

where the \( C_{\mu}^*(\eta) \) are \( 2m \times 2m \) matrices and positively homogeneous of degree 0, and the \( \gamma_{\mu}^*(\eta) \) are \( 1 \times 2m \) matrices and positively homogeneous of degree -1. Thus we have

(3.17) \[ \frac{\partial^k \Psi_{\mu}^{+2\nu}(x, \eta)}{\partial \tau^k} = (2\pi)^{-\frac{n}{2}} \sum_{\mu=1}^{\frac{n}{2}} \left( \frac{i\tau \cdot \xi + i\tau_n(\xi; k_v(\xi) \pm i0) x_n}{r} \right)^k \]

\[ \times e^{i\tau \cdot \xi} e^{-i\tau_n(\xi; k_v(\xi) \pm i0) x_n} C_{\mu}^*(\eta) + (2\pi)^{-\frac{n}{2}} e^{i\tau \cdot \xi} \]

\[ \times \sum_{h=0}^{k} \sum_{\nu=0}^{\frac{m}{2}} \left[ \left( \frac{k}{h} \right) \left( \frac{i\tau \cdot \xi}{r} \right)^h \frac{1}{A(\xi; \lambda)} \text{det}(Bh_{\nu}^1(\xi; \lambda)) \cdots, \right] \]

\[ \cdots, \frac{\partial^{k-h}}{\partial \tau^{k-h}} Bq_{\mu}(x_{m}, \xi; k_v(\xi \mp i0)), \cdots, Bh_{\mu}^n(\xi; \lambda) |_{\lambda = k_v(\xi) \pm i0} \]

\[ \times \gamma_{\mu}^*(\eta) \] \[ \mu = 1, \ldots, 2m; \quad \eta \in D_\nu, \quad k = 0, 1, 2, \ldots. \]

Here we have used the fact that

\[ \frac{\lambda - k_v(\xi)}{A(\xi; \lambda)} \text{det}(Bh_{\nu}^1(\xi; \lambda)), \cdots, e_{\nu}, \cdots, Bh_{\mu}^n(\xi; \lambda) |_{\lambda = k_v(\xi) \pm i0} \]

is positively homogeneous of degree 1, where \( e_{\nu} = t(0, \ldots, 1, \ldots, 0) \). We put in [3]

(3.18) \[ q_{\mu}(x_{m}, \xi; k_v(\xi \mp i0)) \text{ is equal to the } \mu \text{-th column vector of} \]

\[ \left( \frac{1}{2\pi i} \right) \gamma_\nu e^{-i\tau_n(\xi; M(\xi; k_v(\xi)))^{-1} A_{\nu}^1 d\tau_1} \right). \]
Thus

\[
\frac{\partial^h}{\partial \tau^h} q_\mu(x_n, \xi; \beta, L(\xi) \mp i0) \text{ is equal to the } \mu-\text{th column vector of}
\]

\[
\left( \frac{1}{2\pi i} \int_{\gamma} \left[ \frac{\mu}{\tau^h} \left( \tau_1 \frac{d}{d\tau_1} \right)^h e^{-i\lambda_n \tau_1} \right] (\tau_1 I - M(\xi; \beta))^{-1} A_n^{-1} d\tau_1 \right).
\]

Hence it follows from (3.19) that

\[
\frac{\partial^h q_\mu}{\partial \tau^h} (x_n, \xi; \beta, L(\xi) \mp i0) \leq C_h (1 + |\xi| x_n)^h
\]

holds, where the \( C_h \) are independent of \((x_n, \xi)\). Let us consider the case where \( E'(\xi; 0) \cap \emptyset = \emptyset \) for \(|\xi| \neq 0\). Then there exists \( \delta > 0 \) such that for all \( \xi \) with \(|\xi| = 1\) and \( \xi \in D_n, |k_s(\xi)| > \delta \). Thus \( \mathcal{K}_s \) are bounded, \( 1 \leq s \leq s \). Therefore, by the same argument as for \( I_j(r), 1 \leq j \leq \rho \), it follows from the estimates of \( \Psi^{j+2, \nu}_f(x, \eta) \) in [3] that \( I_{j+2, \nu}(r) \in C^\infty[a, b] \) for any interval \([a, b], 0 < a < b < \infty\), i.e., \( I_{j+2, \nu}(r) \in C^\infty(0, \infty) \). Next let us consider the case where \( f(x) \) belongs to \( C^\infty(R^n) \). Then we have

\[
|\mathcal{F}_r [p(x)f(x)](\xi, x_0)| = |(1 + |\xi|^2)^{-l} \mathcal{F}_r [(1 - A_x)^l
\]

\[
(p(x)f(x))(\xi, x_0)| \leq C_l (1 + |\xi|^2)^{-l}
\]

for arbitrary non-negative integer \( l \) and \( p(x) \in C^\infty \). Since we have

\[
|C_{j, \mu}(\eta)| \leq \frac{C(1 + |\xi|)}{1 + |\sigma|},
\]

\[
|V_{j, \mu}(\eta)| \leq \frac{C}{1 + |\sigma|},
\]

for sufficiently large \( R, |\eta| > R \) and \( k_s(\xi) \in [a, b], \)

\[
\int_{a \leq k_s(\xi) \leq b} (1 + |\xi|^2)^{-\frac{n}{2} - h - 1} \left| \frac{\partial^h \Psi^{j+2, \nu}_f(x, \eta)}{\partial \tau^h} \right|^2 d\eta < + \infty,
\]

\( h = 0, 1, 2, \ldots \),

hold. Thus for \( r \in [a, b] \) and \( h = 0, 1, 2, \ldots \),

\[
\left| \int_{R^n} \left( \frac{\partial^h \Psi^{j+2, \nu}_f(x, \eta)}{\partial \tau^h} \right)^* f(x) dx \right| \leq (1 + |\xi|^2)^{-1 + \frac{n}{2} + \frac{h}{2} + \frac{1}{2} F_{j+2, \nu}(a),}
\]

\[
E'(\xi; 0) \cap \emptyset = \emptyset.
\]
hold, where

\[(3.25) \quad \int_{S} \omega \tilde{n}_{\nu}(\omega)\|H_{j+2\nu}^h(\omega)\|^{2} dS_{\nu} < + \infty , \]

Choosing \(l\) more than \(n + \frac{k}{2} + 1\), we have \(I_{j+2\nu}(r) \in C^\infty(0, \infty)\). Therefore \(I_{j+2\nu}(r) \in C^\infty(0, \infty)\).

**Proof of Theorems.** Define an orthogonal projection \(\tilde{P}\) on \(L^2(\mathbb{R}^n)\) by \(\tilde{P}f(x) = f(x)\) for \(|x| \leq R\), \(= 0\) for \(|x| > R\), \(f(x) \in L^2(\mathbb{R}^n)\). Let \(f \in L^2(\mathbb{R}^n)\) and \(f = f_1 + f_2, f_1(x) = (Pf)(x) \in N(A)^\perp, f_2 \in N(A)\). Then \(f\) satisfies \(T_2\) if and only if \(f_1 = Pf\) satisfies \(T_2\). Suppose that \(\text{Supp } f(x)\) is bounded and that at least one of the following conditions holds: a) \(E^*(\xi; 0) \cap \emptyset = \{0\}\) for \(|\xi| \neq 0\). b) \(f(x)\) belongs to \(C^\infty(\mathbb{R}^n)\). The above arguments and Lemmas 4 and 5 imply that \(f\) satisfies \(T_2\). Thus Theorem 2 follows from Proposition 1. Similarly the first assertion of Theorem 3 follows from Proposition 2. Here although \(\tilde{P}R(\lambda)f \rightarrow v(\gamma)(k)\) in \(L^2(\mathbb{R}^n)\) as \(\lambda \rightarrow k - i0\) and \(\text{Supp } x v(\gamma)(k) \subset \{x; |x| \leq R\}\), moving \(R\) to \(+ \infty\), we can define \(v(\gamma)(k)\) as follows:

\(R(\lambda)f \rightarrow v(\gamma)(k)\) in \(L^2_{loc}(\mathbb{R}^n)\) as \(\lambda \rightarrow k - i0\).

Let \(x_R(x) \in C^\infty(\mathbb{R}^n)\) and \(x_R(x) = 1\) for \(|x| \leq R\), \(= 0\) for \(|x| \geq R + 1\). Then,

\[(A - k)x_R(x)R(\lambda)f = \alpha_R(x)f(x) + (\lambda - k)x_R(x)R(\lambda)f(x) + (Ax_R(x))(R(\lambda)f(x) \rightarrow \alpha_R(x)f(x) + (Ax_R(x))v(\gamma)(x) \text{ in } L^2(\mathbb{R}^n) \text{ as } \lambda \rightarrow k - i0.\]

Since \(\alpha_R(x)(R(\lambda)f)(x) \rightarrow \alpha_R(x)v(\gamma)(k)\) in \(L^2(\mathbb{R}^n)\) as \(\lambda \rightarrow k - i0\), the closedness of the operator \(A\) implies that \(\alpha_R(x)v(\gamma)(k) \in D(A)\) and that

\[(A - k)x_R(x)v(\gamma)(k) = \alpha_R(x)f(x) + (Ax_R(x))v(\gamma)(k),\]

that is,
Moreover suppose that $E^+(\xi; \theta) \cap \mathcal{A} = \{0\}$ for $|\xi| \neq 0$. Then $N(A) = \{0\}$ and the coerciveness inequalities for the operator $A$,

$$
(3.28) \quad \frac{\partial u}{\partial x_j} \leq C(\|Au\| + \|u\|), \quad j = 1, \ldots, n,
$$

hold for $u \in D(A)$. Thus from (3.26) it follows that

$$
(3.29) \quad R(\lambda)f \rightarrow v^{(-\lambda)}(k) \quad \text{in} \quad H^1_{loc}(\mathbb{R}^n_+) \quad \text{as} \quad \lambda \rightarrow k - i0.
$$

This completes the proof of Theorem 3.

References

