On the Cohomology of the Classifying Spaces of $PSU(4n+2)$ and $PO(4n+2)$

By

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§0. Introduction

The quotients of $SU(m)$ and $SO(2m)$ by their centers $\Gamma_m = \left\{ e^{2\pi i J/m} \begin{pmatrix} 1 & \cdots & 0 \\ 0 & \ddots & 0 \\ \vdots & \cdots & 1 \end{pmatrix}; \ 0 \leq j < m \right\}$ and $\Gamma_2 = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ are denoted by $PU(m)$ and $PO(2m)$ respectively.

The purpose of this paper is to determine the module structure of the cohomology mod 2 of the classifying spaces $BPU(4n+2)$ and $BPO(4n+2)$.

The method is first to determine the $E_2$-term of the Eilenberg-Moore spectral sequence by constructing an injective resolution for $H^*(G; \mathbb{Z}_2)$, $(G = SU(4n+2)/\Gamma_2, PO(4n+2))$. Then by making use of naturality of the Eilenberg-Moore spectral sequence we show that the spectral sequence with $\mathbb{Z}_2$-coefficient collapses for these $G$.

Our results are

Theorem. As a module

$$H^*(BPU(4n+2); \mathbb{Z}_2) \cong \mathbb{Z}_2[a_2, a_3, x_{8i+8}, y(I)]/R,$$

where $1 \leq i \leq 2n$ and $R$ is an ideal generated by $a_3y(I), y(I)^2 + \sum_{j=1}^s x_{8i_1+8} \cdots a_3^2a_{4j+2} \cdots x_{8i_r+8}$ and $y(I)y(J) + \Sigma f(I)$.

Theorem. As a module

$$H^*(BPO(4n+2); \mathbb{Z}_2) \cong \mathbb{Z}_2[a_2, x_{4l+4}, y'(I)]/R,$$


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where \(1 \leq l \leq 2n\) and \(R\) is an ideal generated by \(a_2y'(I), y'(I)^2 + \Sigma x_{4l+4}\).

In the above theorems \(I\) runs over all sequences of integers \((i_1, \ldots, i_r)\) satisfying \(1 \leq r \leq 2n\) and \(1 \leq i_1 < \cdots < i_r \leq 2n\). (For details see §5.)

The paper is organized as follows:

In the first section we show that there exists a sort of "stability" in \(H^*(BG; \mathbb{Z}_2)\). §2 is used to calculate \(H^*(U(n)/\Gamma_p; \mathbb{Z}_p)\). In §3 we determine the \(E_2\)-term of the Eilenberg-Moore spectral sequence, Cotor \(H^*(G; \mathbb{Z}_2)(\mathbb{Z}_2, \mathbb{Z}_2)\), for \(G = PO(4n+2), PU(4n+2)\). In the next section, §4, we show that the Eilenberg-Moore spectral sequence (with \(\mathbb{Z}_2\)-coefficient) collapses for these \(G\). §5 is devoted to showing that the elements \(a_i\)'s in the above theorems, namely Theorems 4.9 and 4.12, are in the transgression image. In the last section, the generators \(x_{8l+8}\) and \(x_{4l+4}\) in Theorems 4.9 and 4.12 are shown to be represented by certain exterior power representations.

Throughout the paper the map \(BH \to BG\) induced from a homomorphism \(H \to G\) of groups is denoted by the same symbol.

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§1. Quotients of \(SU(n)\) and \(SO(n)\)

**Notation.** \(I_n = \begin{pmatrix} 1 & \cdots & 0 \\ 0 & \ddots & 0 \\ \vdots & \ddots & 1 \end{pmatrix} \in U(n)\) the identity matrix,

\[C(n) = \{xI_n; |x| = 1\} \quad \text{and} \quad x \in \mathbb{C},\]

\[\Gamma_m = \{wI_n; w^m = 1\} \quad \text{and} \quad w \in \mathbb{C} \subset C(n).\]

Then \(C(n)\) (resp. \(\Gamma_m\)) is the center of the unitary group \(U(n)\) (resp. \(SU(n)\)). In particular we have the inclusions

\[\Gamma_2 = \left\{ \pm \begin{pmatrix} 1 & \cdots & 0 \\ 0 & \ddots & 0 \\ \vdots & \ddots & 1 \end{pmatrix} \right\} \subset SO(2n) \subset SU(2n).\]

Hereafter we use the following

**Notation.**
\[ G_i(m) = SU(m)/\Gamma_i \] for a subgroup \( \Gamma_i \) of the center \( \Gamma_m \),

\[ G_m(m) = PU(m) = PSU(m) \cong U(m)/C(m) , \]

\[ G(2n) = G_2(2n) = SU(2n)/\Gamma_2 , \]

\[ PO(2n) = SO(2n)/\Gamma_2 . \]

Denote by \( \pi \) the natural projections \( SU(m) \to G_i(m) \) and \( SO(2n) \to PO(2n) \)

Consider the \( k \)-fold diagonal map:

\[ \Delta_k : SU(n) \to (SU(n))^k \to SU(nk) , \]

\[ \Delta_k : SO(n) \to (SO(n))^k \to SO(nk) , \]

where \( \Delta_k \) is the diagonal embedding:

\[ \Delta_k(A) = \begin{pmatrix} A & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A \end{pmatrix} . \]

For the identity matrix \( I_n \) then we have

\[ \Delta_k(I_n) = I_{nk} \text{ and } \Delta_k(-I_n) = -I_{nk} . \]

So for even \( n \) there exist maps \( G(n) \to G(nk) \) and \( PO(n) \to PO(nk) \) such that the following diagrams commute:

\[
\begin{array}{ccc}
SU(n) & \xrightarrow{\Delta_k} & SU(nk) \\
\downarrow{\pi} & & \downarrow{\pi} \\
G(n) & \xrightarrow{\pi} & G(nk)
\end{array}
\quad \quad
\begin{array}{ccc}
SO(n) & \xrightarrow{\pi} & SO(nk) \\
\downarrow{\pi} & & \downarrow{\pi} \\
PO(n) & \xrightarrow{\pi} & PO(nk)
\end{array}
\]

We denote them by the same symbol:

\[ \Delta_k : G(n) \to G(nk), \quad \Delta_k : PO(n) \to PO(nk). \]

**Notation.**

\[ C(n, k) = SU(nk)/\Delta_k SU(n) , \]

\[ R(n, k) = SO(nk)/\Delta_k SO(n) . \]
So we have fiberings:

(1.1) \[ SU(n) \xrightarrow{\Delta_k} SU(nk) \xrightarrow{p} C(n, k) \, . \]
(1.2) \[ SO(n) \xrightarrow{\Delta_k} SO(nk) \xrightarrow{p} R(n, k) \, . \]

**Remark 1.3.**

(1) \( C(n, k) \) is homeomorphic to \( G(nk)/\Delta_k G(n) \) for \( l|n \).

(2) \( R(2n, k) \) is homeomorphic to \( PO(2nk)/\Delta_k PO(2n) \).

Now recall from [4] and [5] the following

**Proposition 1.4.**

(1) \[ H^*(SU(n); \mathbb{Z}) \cong \Lambda(u_1, \ldots, u_{2n-1}) \, , \]
\[ H^*(U(n); \mathbb{Z}) \cong \Lambda(u_1, u_3, \ldots, u_{2n-1}) \, , \]
where \( \deg u_{2i-1} = 2i - 1 \) and \( u_{2i-1} \) is universally transgressive with \( \tau(u_{2i-1}) = c_i \) the \( i \)-th universal Chern class.

(2) \[ H^*(SO(n); \mathbb{Z}_2) \cong \Delta(v_1, \ldots, v_{n-1}) \, , \]
where \( \deg v_{i-1} = i - 1 \) and \( v_{i-1} \) is universally transgressive with \( \tau(v_{i-1}) = w_i \) the \( i \)-th universal Stiefel-Whitney class.

Then

**Proposition 1.5.** (1) For any integer \( k > 0 \) and any prime \( p \) with \( (k, p) = 1 \), we have
\[ H^*(C(n, k); \mathbb{Z}_p) \cong \Lambda(\bar{x}_{2n+1}, \ldots, \bar{x}_{2nk-1}) \]
where \( \deg \bar{x}_{2i+1} = 2i + 1 \) and \( \rho^* \bar{x}_{2i+1} = u_{2i+1} \).

(2) For any odd integer \( k > 0 \) we have
\[ H^*(R(n, k); \mathbb{Z}_2) \cong \Delta(\bar{z}_n, \ldots, \bar{z}_{mk-1}) \]
where \( \deg \bar{z}_i = i \) and \( \rho^* \bar{z}_i = v_i \).
Proof. (1) The map $\Delta_k: SU(n)\to SU(nk)$ induces a map $\Delta_k: BSU(n)\to BSU(nk)$ which gives the $k$-fold Whitney sum of complex vector bundles. Thus

$$
\Delta_k^* (c_i) = \sum_{i_1+ \cdots + i_k = i} c_{i_1} \cdots c_{i_k} = kc_i + \text{(decomposables)}.
$$

For the Serre cohomology spectral sequence with $\mathbb{Z}_p$-coefficient $\{E_{s,t}^*\}$ of the fibering

$$SU(nk) \to C(n, k) \to BSU(n),$$

we have

$$E_2^{s,t} = \mathbb{Z}_p[c_2, \ldots, c_n] \otimes \Lambda(u_2, \ldots, u_{2nk-1})$$

and

$$E_{\infty}^{s,t} \simeq \mathcal{G}(H^*(C(n, k); \mathbb{Z}_p)).$$

Then it follows from Proposition 1.4 and (1.6) that

$$d_2(1 \otimes u_{2i-1}) = kc_i \otimes 1 \quad \text{for} \quad 2 \leq i \leq n$$

and all other differentials are trivial. So we get

$$\mathcal{G}(H^*(C(n, k); \mathbb{Z}_p)) \simeq E_{\infty}^{*,*} \simeq E_{2n+1}^{*,*} \simeq \Lambda(u_{2n+1}, \ldots, u_{2nk-1}).$$

Since $(k, p) = 1$, (1.6) implies that $\Delta_k^*: H^*(SU(nk); \mathbb{Z}_p) \to H^*(SU(n); \mathbb{Z}_p)$ is epimorphic, and hence $SU(n)$ is totally non-homologous to zero in the fibering (1.1). Thus $\rho^*: H^*(C(n, k); \mathbb{Z}_p) \to H^*(SU(nk); \mathbb{Z}_p)$ is monomorphic.

(2) is proved quite similarly. Q.E.D.

**Theorem 1.7.** (1) Let $p$ be a prime, $k$ an integer with $(k, p) = 1$ and $l|n$. Then $\Delta_k^*: H^i(BG_l(nk); \mathbb{Z}_p) \to H^i(BG_l(n); \mathbb{Z}_p)$ is isomorphic for $i \leq 2n$ and monomorphic for $i \leq 2n + 1$.

(2) Let $k$ be an odd integer. Then $\Delta_k^*: H^i(BPO(2kn); \mathbb{Z}_2) \to H^i(BPO(2n); \mathbb{Z}_2)$ is isomorphic for $i \leq n - 1$ and monomorphic for $i \leq n$.

Proof. Proposition 1.5 applied with the Serre exact sequence (Proposition 5 of [12]) for the fiberings

$$C(n, k) \to BG_l(n) \to BG_l(nk)$$
\[ R(2n, k) \longrightarrow BPO(2n) \longrightarrow BPO(2nk) \]
gives the results. Q.E.D.

**Notation.** For each rational number \( k \), define \( v_p(k) \) to be the exponent of \( p \) when \( k \) is expressed as a product of powers of distinct primes.

**Corollary 1.8.** (1) If \( v_p(n) = v_p(m) \), then as algebras there hold
\[
H^*(BG(n); \mathbb{Z}_p) \cong H^*(BG(m); \mathbb{Z}_p) \quad \text{for} \quad * \leq 2\min(m, n).
\]
(2) If \( v_2(m) = v_2(n) \), then as algebras there hold
\[
H^*(BPO(2n); \mathbb{Z}_2) \cong H^*(BPO(2m); \mathbb{Z}_2) \quad \text{for} \quad * \leq \min(m, n).
\]

In the below we denote by \( \phi \) the diagonal map in \( H^*(G; \mathbb{Z}_p) \) induced from the multiplication on a group \( G \). Put \( \bar{\phi} = (\eta \otimes \eta) \circ \phi \), where \( \eta: H^*(G; \mathbb{Z}_p) \rightarrow \sum_{i \geq 0} H^i(G; \mathbb{Z}_p) \) is the natural projection.

Now we recall from [3] and [5] the following facts:

**Proposition 1.9.** Let \( n = p^rn' \) with \( (p, n') = 1 \) and \( l | n \). Then
\[
H^*(G(n); \mathbb{Z}_p) \cong \mathbb{Z}_p[y]/(y^{p^l}) \otimes A(x_1, \ldots, x_{2p^r-1}, \ldots, x_{2n-1}),
\]
where \( \deg y = 2 \) and \( \deg x_{2i-1} = 2i-1 \).

**Proposition 1.9'.** There exist generators \( y \in H^1(G(4n+2); \mathbb{Z}_2) \) and \( x_{2i+1} \in H^{2i+1}(G(4n+2); \mathbb{Z}_2) \), \( 2 \leq i \leq 4n+1 \), such that

(1) \[
H^*(G(4n+2); \mathbb{Z}_2) \cong A(y, y^2, x_5, \ldots, x_{8n+3}),
\]
(2) \[
\bar{\phi}(y) = 0, \quad \bar{\phi}(x_{4j+1}) = 0 \quad \text{for} \quad 1 \leq j \leq 2n,
\]
(3) \[
\bar{\phi}(x_{4j+3}) = x_{4j+1} \otimes y^2 \quad \text{for} \quad 1 \leq j \leq 2n,
\]
(4) \[
Sq^{2k}x_{2i-1} = (k, i-k-1)x_{2l+2k-1}.
\]

**Remark 1.9''.** \( \bar{\phi}(x_{4j+3} + \text{decomp.}) \neq 0 \).

**Proposition 1.10.** There exist generators \( y \in H^1(PO(4n+2); \mathbb{Z}_2) \) and
\[ z_i \in H^i(PO(4n+2); \mathbb{Z}_2), \ 2 \leq i \leq 4n+1, \text{ such that} \]

1. \[ H^*(PO(4n+2); \mathbb{Z}_2) \cong \Delta(y, z_2, \ldots, z_{4n+1}), \]

2. \[ \bar{\phi}(y) = 0, \quad \bar{\phi}(z_{2k}) = 0 \quad \text{for} \quad 1 \leq k \leq 2n, \]
   \[ \bar{\phi}(z_{2k+1}) = z_{2k} \otimes y \quad \text{for} \quad 1 \leq k \leq 2n, \]

3. \[ Sq^iz_k = (k-j)z_{j+k}. \]

**Notation.** \( PS(X; p) \) = the Poincaré series of \( X \) over \( \mathbb{Z}_p \), i.e.,
\[ PS(X; p) = \sum_{i=0}^{\infty} (\text{rank } H^i(X; \mathbb{Z}_p)) t^i. \]

Using this expression we obtain from Propositions 1.5, 1.9 and 1.10:
\[ PS(G_l(n); p) \cdot PS(C(n, k); p) = PS(G_l(nk); p) \quad \text{for} \quad (k, p) = 1, \]
\[ PS(PO(2n); 2) \cdot PS(R(n, k); 2) = PS(PO(2nk); 2). \]

Thus we have

**Proposition 1.11.** (1) The cohomology Serre spectral sequence with \( \mathbb{Z}_p \)-coefficient for the fibering \( G_l(n) \to G_l(nk) \to C(l, k) \) collapses if \( (k, p) = 1 \).

(2) The cohomology Serre spectral sequence with \( \mathbb{Z}_2 \)-coefficient for the fibering \( PO(2n) \to PO(2nk) \to R(2n, k) \) collapses.

Now we choose generators in \( H^*(G(4n+2); \mathbb{Z}_2) \) and \( H^*(PO(4n+2); \mathbb{Z}_2) \) appropriately.

**Lemma 1.12.** In Proposition 1.9' we may choose generators \( y, x_{2i+1}, 2 \leq i \leq 4n+1, \) of \( H^*(G(4n+2); \mathbb{Z}_2) \) by using the correspondent generators in \( H^*(G(4n-2); \mathbb{Z}_2) \) and in \( H^*(C(4n-2, 2n+1); \mathbb{Z}_2) \) as follows:
\[ y = A_{2n-1}^* \circ A_{2n+1}^*-1(y), \]
\[ x_{2i+1} = A_{2n-1}^* \circ A_{2n+1}^*-1(x_{2i+1}), \ 2 \leq i \leq 4n-3, \]
\[ x_{2i+1} = A_{2n-1}^* \circ p^*(x_{2i+1}), \ 4n-2 \leq i \leq 4n+1. \]
Proof. This is clear from Proposition 1.11. Q.E.D.

Similarly

**Lemma 1.13.** In Proposition 1.10 we may choose generators $y$, $z_i$, $2 \leq i \leq 4n+1$, of $H^*(PO(4n+2); \mathbb{Z}_2)$ by using the correspondent generators in $H^*(PO(4n-2); \mathbb{Z}_2)$ and in $H^*(R(4n-2, 2n+1); \mathbb{Z}_2)$ as follows:

$$y = \Delta_{2n-1}^* \circ \Delta_{2n+1}^{-1}(y),$$

$$z_i = \Delta_{2n-1}^* \circ \Delta_{2n+1}^{-1}(z_i), \quad 2 \leq i \leq 4n-3,$$

$$z_i = \Delta_{2n-1}^* \circ p^*(z_i), \quad 4n-2 \leq i \leq 4n+1.$$

**Proposition 1.14.** (1) In $H^*(C(4n-2, 2n+1); \mathbb{Z}_2) \cong \Lambda(\bar{x}_{8n-3}, \bar{x}_{8n-1}, \bar{x}_{8n+1}, \bar{x}_{8n+3}, \ldots)$ there hold $Sq^2 \bar{x}_{8n-3} = \bar{x}_{8n+1}$ and $Sq^2 \bar{x}_{8n-1} = \bar{x}_{8n+3}$.

(2) In $H^*(R(4n-2, 2n+1); \mathbb{Z}_2) \cong \Lambda(\bar{z}_{4n-2}, \bar{z}_{4n-1}, \bar{z}_{4n}, \bar{z}_{4n+1}, \ldots)$ there hold $Sq^2 \bar{z}_{4n-2} = \bar{z}_{4n}$ and $Sq^2 \bar{z}_{4n-1} = \bar{z}_{4n+1}$.

Proof. (1) and (2) follow from (3) of Proposition 1.9' and (3) of Proposition 1.10 respectively. Q.E.D.

**Remark.** See [9] for the results of the symplectic case.

§2. Quotients of $U(n)$

In this section let $p$ be a prime and $n$ an integer such that $(n, p) = 1$. Then obviously

(2.1) $H^*(BP\mu(n); \mathbb{Z}_p) \cong H^*(BU(n); \mathbb{Z}_p)$.

The following are easily obtained:

(2.2) $H^*(BC(n); \mathbb{Z}_p) \cong \mathbb{Z}_p[\alpha]$ with $\deg \alpha = 2$.

(2.3) $H^*(B\Gamma_p; \mathbb{Z}_p) \cong \begin{cases} \mathbb{Z}_2[t] & \text{with } \deg t = 1 \text{ for } p = 2 \\ \mathbb{Z}_p[\mu] \otimes \Lambda(\lambda) & \text{with } \deg \mu = 2, \deg \lambda = 1, \delta \lambda = \mu \text{ for } p: \text{odd}, \end{cases}$
where \( \delta \) is the Bockstein operator.

Consider the cohomology Serre spectral sequence with \( \mathbb{Z}_p \)-coefficient associated with the fibering:

\[
BC(n) \rightarrowtail BU(n) \rightarrow BPU(n),
\]

where \( i' \) is induced from the natural inclusion \( C(n) \subset U(n) \). The map \( i'^* \) is epimorphic since the spectral sequence collapses by (2.2) and by the fact that \( H^3(BPU(n); \mathbb{Z}_p) = H^3(BSU(n); \mathbb{Z}_p) = 0 \). Let \( j: \Gamma_p \subset C(n) \) be the inclusion. Then

\[
\text{Im} j^* \cong \begin{cases} 
\mathbb{Z}_p[\mu] & \text{for } p \text{: odd} \\
\mathbb{Z}_2[t^2] & \text{for } p = 2.
\end{cases}
\]

Putting \( i = i'oj \) and choosing \( \mu \) (or \( t \)) suitably we get

\[
i^*(c_1) = \begin{cases} 
\mu & \text{for } p \text{: odd} \\
t^2 & \text{for } p = 2.
\end{cases}
\]

Let \( \{E_r^{*,*}\} \) be the cohomology Serre spectral sequence with \( \mathbb{Z}_p \)-coefficient associated with the fibering \( U(n) \rightarrowtail U(n)/\Gamma_p \rightarrow B\Gamma_p \). Since the generators in \( H^*(U(n); \mathbb{Z}_p) \cong \Lambda(u_1, u_3, \ldots, u_{2n-1}) \) are universally transgressive, they are transgressive with respect to this fibering. In particular we have

\[
\tau(u_1) = i^*(c_1)
\]

where \( \tau \) is the transgression.

Therefore \( E_2^{a,b} = 0 \) if \( a \geq 2 \), and hence

\[
E_3 \cong E_\infty \cong \begin{cases} 
\Lambda(\lambda) \otimes \Lambda(u_3, u_5, \ldots, u_{2n-1}) & \text{for } p \text{: odd} \\
\Lambda(t) \otimes \Lambda(u_3, u_5, \ldots, u_{2n-1}) & \text{for } p = 2.
\end{cases}
\]

**Proposition 2.8.** \( H^*(U(n)/\Gamma_p; \mathbb{Z}) \) is \( p \)-torsion free and hence it is torsion free.

Proof is left to the reader.

It follows from this proposition
Theorem 2.9. Let \((n, p) = 1\). Then

\[ H^*(U(n)/\Gamma_p; \mathbb{Z}_p) \cong \Lambda(\lambda, u_3', \ldots, u_{2n-1}') \]

such that

(1) \(\lambda\) and \(u_{2i-1}'\) are universally transgressive (and hence they are primitive),

(2) \(\deg \lambda = 1\) and \(\deg u_{2i-1}' = 2i - 1\),

(3) \(\pi^*(u_{2i-1}') = u_{2i-1}'\) for the projection \(\pi: U(n) \to U(n)/\Gamma_p\).

Proof. (1) and (2) follow from (2.7) and the Borel's theorem (Theorem 13.1 of [4]). (3) is clear, since \(\pi^*(u_{2i-1}') \neq 0\) by (2.7) and since \(\pi^*(u_{2i-1}')\) are universally transgressive. Q. E. D.

§3. The \(E_2\)-term of the Eilenberg-Moore Spectral Sequence

Put \(A = H^*(G(4n + 2); \mathbb{Z}_2)\) for simplicity and regard \(A\) as a coalgebra over \(\mathbb{Z}_2\), where the coalgebra structure \(\bar{\phi}\) is given by Proposition 1.9'.

Let \(L\) be a \(\mathbb{Z}_2\)-submodule of \(A^+ = \bigoplus_{i>0} H^i(G(4n+2); \mathbb{Z}_2)\) generated by \(\{y, y^2, x_{4i+1}, x_{4i+3}\}, 1 \leq i, j \leq 2n\). Let \(s: L \to sL\) be the suspension. We express the corresponding elements as \(sL = \{a_2, a_3, a_{4j+2}, b_{4i+4}\}, 1 \leq i, j \leq 2n\). Let \(\iota: L \to A\) be the inclusion and \(\theta: A \to L\) the projection

\[
\begin{array}{ccc}
L & \rightarrow & A \\
\downarrow \iota & & \downarrow \theta \\
S & \rightarrow & L
\end{array}
\]

such that \(\theta \circ \iota = 1_L\). Define \(\bar{\theta}: A \to sL\) by \(\bar{\theta} = \iota \circ \theta\) and \(\gamma: sL \to A\) by \(\bar{\gamma} = \iota \circ S^{-1}\). Consider the tensor algebra \(T(sL)\). Denote by \(I\) the ideal of \(T(sL)\) generated by \(\text{Im}(\{\bar{\theta} \otimes \bar{\gamma}\} \circ \overline{\phi}) \circ \text{Ker} \bar{\theta}\). Put \(\overline{X} = T(sL)/I\). Then \(\overline{X} \cong \mathbb{Z}_2[a_2, a_3, a_{4j+2}, b_{4i+4}], 1 \leq i, j \leq 2n\).

The map \(\overline{d} = (\bar{\theta} \otimes \bar{\gamma}) \circ \overline{\phi} \circ \overline{\gamma}\) on \(sL\) can be extended over \(\overline{X}\), since \(\overline{d}(I) \subset I\). Further, \(\overline{d}\) satisfies \(\overline{d} \circ \overline{d} = 0\) on \(\overline{X}\). So \(\overline{X}\) is a differential algebra.

Now we construct the twisted tensor product \(X = A \otimes \overline{X}\) with respect to \(\theta\) following Brown (cf. [7], [8] or [13]). Then \(X = A \otimes \overline{X}\) is a dif-
ferential $A$-comodule with the differential operator $d = 1 \otimes \bar{d} + (1 \otimes \psi) \circ (1 \otimes \theta \otimes 1) \circ \phi \otimes 1$, where $\phi$ is the diagonal structure in $A$ and $\psi$ is the multiplication in $\bar{X}$. More concretely,

\[
dy = a_2, \quad dy^2 = a_3,  \\
dx_{4j+1} = a_{4j+2}, \quad 1 \leq j \leq 2n,  \\
dx_{4i+3} = b_{4i+4} + x_{4i+1}a_3, \quad 1 \leq i \leq n.
\]

Now we define weight in $X$ as follows:

\[
A: \quad y \quad y^2 \quad x_{4j+1} \quad x_{4i+3}  \\
\phi \quad  \\
\bar{X}: \quad a_2 \quad a_3 \quad a_{4j+2} \quad b_{4i+4} \\
weight \quad 0 \quad 0 \quad 0 \quad 1
\]

The weight of a monomial is a sum of the weight of each element. Put $F_i = \{x \mid \text{weight } x \leq i\}$. Then

\[
E_0X = \sum_i F_i/F_{i-1}  \\
\cong A(y, y^2, x_{4j+1}, x_{4i+3}) \otimes \mathbb{Z}_2[a_2, a_3, a_{4j+2}, b_{4i+4}],
\]

where the induced differential operator $d_0$ is given by

\[
d_0y = a_2, \quad d_0y^2 = a_3, \quad d_0x_{4j+1} = a_{4j+2}, \quad d_0x_{4i+3} = b_{4i+4}.
\]

Thus $E_0X$ is acyclic and hence $X$ is acyclic. Namely $X = A \otimes \bar{X}$ is an injective resolution for $A$ over $\mathbb{Z}_2$. Therefore by definition

\[
H^\ast(\bar{X}; \bar{d}) = \text{Cotor}^A(\mathbb{Z}_2, \mathbb{Z}_2).
\]

As described above the differential operator $\bar{d}$ in $\bar{X} = \mathbb{Z}_2[a_2, a_3, a_{4j+2}, b_{4i+4}]$ is given by

\[
\bar{d}a_i = 0 \quad \text{for } i = 2, 3, 4j+2 \quad (1 \leq j \leq 2n),  \\
\bar{d}b_{4i+4} = a_{4i+2}a_3 \quad (1 \leq i \leq 2n).
\]

For simplicity we put $P = \mathbb{Z}_2[a_{4j+2}; 1 \leq j \leq 2n]$ and $Q = \mathbb{Z}_2[b_{4i+4}; 1 \leq i \leq 2n]$. 
Let $C$ be a submodule of $\mathcal{X}$ generated by \{b_{51} \ldots b_{5n+4}^i = 0 \text{ or } 1\}. Then as a module

\[ \mathcal{X} \cong \mathbb{Z}_2[a_2] \otimes Q \otimes \mathbb{Z}_2[a_3] \otimes P \otimes C. \]

We remark that as a chain complex, $\mathcal{X}$ may be thought of as a tensor product of $(\mathbb{Z}_2[a_2] \otimes Q)$ with a trivial differential operator $d_0$ and $(\mathbb{Z}_2[a_3] \otimes P \otimes C)$ with a differential operator $d_1$ such that $d_1(a_3) = d_1(a_{4j+2}) = 0$ and $d_1(b_{4i+4}) = a_3 a_{4i+2}$. Therefore

\[ H(\mathcal{X}; d) \cong H(\mathbb{Z}_2[a_2] \otimes Q; d_0) \otimes H(\mathbb{Z}_2[a_3] \otimes P \otimes C; d_1). \]

For $f \in P \otimes C$ there hold $d_1(f) = a_3 f$ for some $\tilde{f} \in P \otimes C$. Then we define $\tilde{d}_1 : P \otimes C \rightarrow P \otimes C$ by $\tilde{d}_1(f) = \frac{d_1(f)}{a_3}$.

\textbf{Lemma 3.1.} The chain complex $(P \otimes C; \tilde{d}_1)$ is acyclic.

\textbf{Proof.} Consider the Koszul resolution of the exterior algebra $\Lambda(b_{4i+4}; 1 \leq i \leq 2n)$. Q.E.D.

\textbf{Proposition 3.2.} Let $f \in \mathbb{Z}_2[a_3] \otimes P \otimes C$. Then $d_1 f = 0 \text{ iff }$ there exists an element $g \in \mathbb{Z}_2[a_3] \otimes P \otimes C$ such that $d_1(g) = a_3 f$, or else $f = 1 \otimes 1 \otimes 1$.

\textbf{Proof.} Sufficiency is clear, since $\mathcal{X}$ is a polynomial algebra.

(Necessity) It suffices to prove necessity for an element $f \in \mathbb{Z}_2 \otimes P \otimes C \cong P \otimes C$. Suppose $d_1(f) = 0$. Then $a_3 \tilde{f} = 0$, and hence $\tilde{f} = 0$. So by definition $\tilde{d}_1(f) = 0$, from which we deduce that $f = 1 \otimes 1 \otimes 1$ or else by Lemma 3.1 that there is an element $g \in P \otimes C$ such that $\tilde{d}_1(g) = f$. Thus $a_3 f = d_1(g)$. Q.E.D.

Let $I = (i_1, \ldots, i_r)$ be a sequence of integers satisfying

\[ 1 \leq r \leq 2n \text{ and } 1 \leq i_1 < \cdots < i_r \leq 2n. \]

We put $y(I) = \frac{1}{a_3} \tilde{d}(b_{4i_1+4} \cdots b_{4i_r+4})$.

It follows from Proposition 3.2 that a system of generators of
Ker $d$ over $\mathbb{Z}_2[a_2, a_3, a_{4j+2}, b_{2i+4}]$ is $\{1, y(I)\}$, where $I$ runs over all sequences satisfying (3.3).

**Theorem 3.4.** For $A = H^*(G(4n+2); \mathbb{Z}_2)$

$$\text{Cotor}^4(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2[a_2, a_3, x_{8i+8}, y(I)]/R,$$

where $x_{8i+8} = \{b_{2i+4}\}$ for $1 \leq i \leq 2n$ and $I$ runs over all sequences satisfying (3.3). Further $R$ is the ideal generated by $a_3y(I), y(I)^2 + \sum_{j=1}^5 x_{8i+8}$ $a_{4j+2} \cdots x_{8i+8}$ and $y(I)y(J) + \sum_i f_i y(I_i)$, where $f$ is a polynomial of $a_2, a_3, x_{8i+8}$'s.

**Remark 3.5.** $y\{i\} = a_{4i+2}$. For $I = (i_1, \ldots, i_r)$, $(r \geq 2)$, $y(I)$ can be defined inductively. Put $I' = (i_1, \ldots, i_{r-1})$. Suppose that $y(I') = \{\frac{1}{a_3}d(b_{4i_1+4} \cdots b_{4i_{r-1}+4})\}$ is defined. Then $y(I) = y(I') = (b_{4i_1+4} \cdots b_{4i_{r-1}+4}a_{4i_r+2} + y(I')b_{4i_r+4} = \langle y(I'), a_3, a_{4i_r+2} \rangle$, the Massey product.

**Remark 3.6.** The relation $y(I)y(J) + \Sigma f_i y(I_i)$ can be obtained by calculation on the cochains, since $\{1, y(I)\}$ is a system of generators over $\mathbb{Z}_2[a_2, a_3, x_{8i+8}]$.

Now we consider the case $A = H^*(PO(4n+2); \mathbb{Z}_2)$. By a similar argument to the before we have $X = \mathbb{Z}_2[a_2, a_{2j+1}, b_{2i+2}]/R, 1 \leq i, j \leq 2n$, where $R$ is the ideal generated by $[a_{2k+1}, b_{2k+2}] + a_{4k+1}a_2, 1 \leq k \leq n$ and $[r, s]$ for other pairs of generators $(r, s)$ ($[x, y] = xy + yx$). We define weight in $X = A \otimes X$, the twisted tensor product with respect to $\theta$:

<table>
<thead>
<tr>
<th>$A$</th>
<th>$y$</th>
<th>$z_{2j}$</th>
<th>$z_{2i+1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$</td>
<td>$a_2$</td>
<td>$a_{2j+1}$</td>
<td>$b_{2i+2}$</td>
</tr>
</tbody>
</table>

weight 0 0 1

Put $F_i = \{x | \text{weight } x \leq i\}$ as before. Then

$$E_0 X = \Sigma F_i / F_{i-1}$$

$$\cong A(y) \otimes A(z_{2j}, z_{2i+1}) \otimes \mathbb{Z}_2[a_2, a_{2j+1}, b_{2i+2}].$$
where the induced differential operator is given by $d_0y = a_2$, $d_0z_{2j} = a_{2j+1}$ and $d_0z_{2i+1} = b_{2i+2}$. It shows that $E_0X$ and hence $X$ is acyclic.

The differential operator $\delta$ in $X$ is given by $\delta a_j = 0$ for any $j$ and $\delta b_{2i+2} = a_{2i+1} a_2$. By a similar, although a little bit complicated, calculation to the before, we obtain the following.

For a sequence of integers $I = (i_1, \ldots, i_r)$ satisfying (3.3) we put $y'(I) = \frac{1}{a_2} \delta(b_{2i_1+2} \cdots b_{2i_r+2})$.

**Theorem 3.7.** For $A = H^*(PO(4n+2); \mathbb{Z}_2)$

\[
\operatorname{Cotor}^4(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2[a_2, x_{4l+4}, y'(I)]/R,
\]

where $x_{4l+4} = b_{2l+2} + a_2 b_{4l+2}$ for $1 \leq l \leq n$ and $b_{2l+2} = a_{2n+1}$ for $n+1 \leq l \leq 2n$ and $I$ runs over all sequences satisfying (3.3). Further $R$ is the ideal generated by $a_2 y'(I)$, $y'(I)^2 + \sum_{j=1}^{l} x_{4l+4} \cdots a_{2l+1} \cdots x_{4l+4}$ and $y'(I)y'(J) + \sum_{I} f_I y'(I)$.

**Remark 3.8.** $y'(\{i\}) = a_{2i+1}$, For $I = (i_1, \ldots, i_r)$, $y'(I)$ can also be defined inductively, i.e.,

\[
y'(I) = \langle y'(I'), a_2, a_{2i_r+1} \rangle, \quad \text{where } I = (I', i_r).
\]

The following results can easily be obtained.

**Proposition 3.9.**

1. $\operatorname{Cotor}^\ast(U(2n+1); \mathbb{Z}_2)(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2[\tilde{c}_1, \ldots, \tilde{c}_{2n+1}],$

with $\deg \tilde{c}_i = 2i$.

2. $\operatorname{Cotor}^\ast(U(2n+1)/F_2; \mathbb{Z}_2)(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2[a_2', \tilde{c}_2', \ldots, \tilde{c}_{2n+1}'],$

with $\deg a_2' = 2$ and $\deg \tilde{c}_i' = 2i$.

3. $\operatorname{Cotor}^\ast(SO(4n+2); \mathbb{Z}_2)(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2[\tilde{w}_2, \tilde{w}_3, \ldots, \tilde{w}_{4n+2}]$, with $\deg \tilde{w}_i = i$.

4. $\operatorname{Cotor}^\ast(Sp(2n+1); \mathbb{Z}_2)(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2[\tilde{q}_1, \ldots, \tilde{q}_{2n+1}]$, with $\deg \tilde{q}_i = 4i$. 
(5) \( \text{Cotor} H^{*}(PSp(2n+1); \mathbb{Z}/2) \otimes (\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}[2a_1', a_2', a_3', \ldots, \tilde{q}_{2n+1}'], \)
with \( \text{deg} a_i' = i \) and \( \text{deg} \tilde{q}_i' = 4i. \)

(6) \( \text{Cotor} H^{*}(SU(4n+2); \mathbb{Z}/2) \otimes (\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}_2[\bar{c}_2', \ldots, \bar{c}_{2n+1}'], \)
with \( \text{deg} \bar{c}_i' = 2i. \)

§ 4. Collapsing of the Eilenberg-Moore Spectral Sequence

Let \( G \) be a topological group. In 1959 Eilenberg-Moore constructed a new type of spectral sequence \( \{E_r(G), d_r\} \) such that

(1) \( E_2(G) \cong \text{Cotor} H^{*}(G, \mathbb{Z}/p)(\mathbb{Z}/p, \mathbb{Z}/p), \)

(2) \( E_n(G) \cong \varnothing_r H^{*}(BG; \mathbb{Z}/p). \)

Furthermore, this spectral sequence satisfies naturality for a homomorphism \( f: G \to G' \). We denote by \( f^*: E_r(G') \to E_r(G) \) the induced homomorphism.

In this section we will show that the Eilenberg-Moore spectral sequence collapses for various \((G, p)\). In particular, we will show that for \( G = G(4n + 2) \) and \( PO(4n + 2) \) the Eilenberg-Moore spectral sequence with \( \mathbb{Z}_2 \)-coefficient collapses.

The following directly follows from Theorem 2.9:

**Proposition 4.1.** Let \((n, p) = 1\). Then the Eilenberg-Moore spectral sequence collapses for \((G, p) = (U(n)/\Gamma_p, p)\).

By Kono [9] \( H^{*}(PSp(2n + 1); \mathbb{Z}/2) \) is transgressively generated and hence we have

**Proposition 4.2.** The Eilenberg-Moore spectral sequence collapses for \((G, p) = (PSp(2n + 1), 2)\).

The following result will be used below. The proof is easy and left to the reader.

**Proposition 4.3.** (1) The Eilenberg-Moore spectral sequence col-
lapses for \( G = U(2n + 1)/\Gamma_2, SO(4n + 2), U(2n + 1), PSp(2n + 1), SU(4n + 2) \) and \( Sp(2n + 1) \).

(2) The elements \( \tilde{c}_i \) and \( \tilde{w}_i \) in Proposition 3.9 represent \( c_i \) and \( w_i \) respectively. The elements \( \tilde{q}_i \) and \( \tilde{c}_i \) do \( q_i \) and \( c_i \) in \( H^*(BG; \mathbb{Z}_2) \) such that \( \pi^*(c_i) = c_i + (\text{decomp.}) \) and \( \pi^*(q_i) = q_i + (\text{decomp.}) \), where \( \pi \) is the covering homomorphism.

For simplicity we use the following

**Notation.**

\[
A_1 = H^*(U(2n + 1); \mathbb{Z}_2),
\]

\[
A_2 = H^*(U(2n + 1)/\Gamma_2; \mathbb{Z}_2),
\]

\[
A_3 = H^*(SO(4n + 2); \mathbb{Z}_2),
\]

\[
A_4 = H^*(PO(4n + 2); \mathbb{Z}_2),
\]

\[
B_1 = H^*(Sp(2n + 1); \mathbb{Z}_2),
\]

\[
B_2 = H^*(PSp(2n + 1); \mathbb{Z}_2),
\]

\[
B_3 = H^*(SU(4n + 2); \mathbb{Z}_2),
\]

\[
B_4 = H^*(G(4n + 2); \mathbb{Z}_2).
\]

**Case I.** \( H^*(PO(4n + 2); \mathbb{Z}_2) \).

Consider the commutative diagram

\[
\begin{array}{ccc}
U(2n + 1) & \overset{i}{\longrightarrow} & SO(4n + 2) \\
\downarrow\pi & & \downarrow\pi \\
U(2n + 1)/\Gamma_2 & \overset{i}{\longrightarrow} & PO(4n + 2)
\end{array}
\]

where \( \pi \) is the projection and \( i \)'s are the standard maps (cf. §6).

**Lemma 4.4.** The elements \( a_2' \in \text{Cotor}^{A_4}(\mathbb{Z}_2, \mathbb{Z}_2) \) and \( a_2 \in \text{Cotor}^{A_4} (\mathbb{Z}_2, \mathbb{Z}_2) \) are permanent cycles and \( i^*(a_2) = a_2' \).

**Proof.** Recall \( H^*(B\mathbb{Z}_2; \mathbb{Z}_2) \cong \mathbb{Z}_2[t] \). In the commutative diagram
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\[ \xymatrix{ BZ_2 \ar[dr] & \ar[d] \ar[r] & BSO(4n+2) \ar[d]^p \ar[l] \ar[r] & BPO(4n+2) \ar[l] \ar[d]^p \ar[r] & BSU(2n+1) \ar[dl] \ar[l] } \]

the elements \( a_2 \) and \( a'_2 \) represent the translation images of \( t \), and hence they are permanent cycles. For dimensional reason we have \( i^*(a_2) = a'_2 \).

Q.E.D.

The following relations among the elements in Theorem 3.7 and Proposition 3.9 are easily checked to be true:

(4.5.1) \( \pi^*(x_{4i}) = \bar{w}_{2i} + W_i \), where \( W_i \) is a sum of monomials containing elements of lower degree,

(4.5.2) \( i^*(\bar{w}_{2i}) = \bar{c}_i + \text{(decomp.)} \), (see § 6),

(4.5.3) \( \pi^*(\bar{c}_i) = \bar{c}_i + \text{(decomp.)} \),

(4.5.4) \( \pi^*(a_2) = \pi^*(a'_2) = 0 \).

Therefore

(4.6) \( i^*(x_{4i}) = \bar{c}_i^2 + \gamma_i \), where \( \gamma_i \) is a sum of monomials containing elements of lower degree.

Let \( E_r(1) \) be the Eilenberg-Moore spectral sequence with \( Z_2 \)-coefficient for \( PO(4n+2) \) and \( \{ E_r(2), d_r \} \) be the cartesian product of the Eilenberg-Moore spectral sequences of \( U(2n+1)/\Gamma_2 \) and \( SO(4n+2) \), i.e.,

\[ E_r(2) = \text{Cotor}^A(Z_2, Z_2) \times \text{Cotor}^A(Z_2, Z_2) \] and \( d_r = 0 \)

for all \( r \geq 2 \). Then the map \( i^* \times \pi^* \) induces a homomorphism between the spectral sequences:

\[ E_r(1) \longrightarrow E_r(2) \quad \text{for} \quad r \geq 2. \]

Lemma 4.7. \( i^* \times \pi^* : E_2(1) \rightarrow E_2(2) \) is injective.
Proof. Let \( f_1 \) be a sum of monomials containing \( a_2 \) and \( f_2 \) a sum of those not containing \( a_2 \). Suppose \((i^* \times \pi^*)(f_1 + f_2) = 0\) from which \( \pi^*(f_1 + f_2) = \pi^*(f_2) = 0 \) and hence \( f_2 = 0 \) by (4.5.1). Meanwhile \((i^* \times \pi^*) (f_1 + f_2) = 0\) implies \( i^*(f_1 + f_2) = 0 \), which implies \( i^*(f_1) = 0 \), and hence \( f_1 = 0 \) by (4.6). Thus \( i^* \times \pi^* \) is injective.

Q.E.D.

Thus we have shown

**Theorem 4.8.** The Eilenberg-Moore spectral sequence with \( \mathbb{Z}_2 \)-coefficient collapses for \( G = PO(4n+2) \).

In fact, Lemma 4.7 indicates that all differentials in \( E_r(1) \) are trivial. An immediate corollary is

**Theorem 4.9.** As a module

\[
H^*(BPO(4n+2); \mathbb{Z}_2) \cong \mathbb{Z}_2[a_2, \ x_{4l+4}, \ y'(I)]/R,
\]

where \( 1 \leq l \leq 2n, I \) runs over all sequences satisfying (3.3) and \( R \) is the ideal generated by \( a_2 y'(I), y'(I)^2 + \sum_{j=1}^r x_{4l+4} \ldots a_{2l+1} \ldots x_{4l+4} \) and \( y'(I)y'(J) + \sum I f_I y'(I)I \).

**Case II.** \( H^*(G(4n+2); \mathbb{Z}_2) \).

Consider the commutative diagram

\[
\begin{array}{ccc}
Sp(2n+1) & \xrightarrow{i} & SU(4n+2) \\
\downarrow{\pi} & & \downarrow{\pi} \\
PSp(2n+1) & \xrightarrow{i} & G(4n+2)
\end{array}
\]

where \( \pi \) is the projection and \( i \)'s are the standard maps.

**Lemma 4.4'.** The elements \( a_i \in \text{Cotor}^*_{B^*} (\mathbb{Z}_2, \mathbb{Z}_2) \) and \( a_i' \in \text{Cotor}^{B^*} (\mathbb{Z}_2, \mathbb{Z}_2) \) are permanent cycles and \( i^*(a_i) = a_i' \) for \( i = 2, 3 \).

Proof is similar to that of Lemma 4.4.

The following relations among the elements in Theorem 3.4 and Proposition 3.9 are easily checked to be true:
(4.10.1) \[ \pi^*(x^8_{i+8}) = \tilde{e}_{i+2}^2 + v_i, \]

(4.10.2) \[ \pi^*(\tilde{q}_i) = \tilde{q}_i + (\text{decomp.}), \]

where \( v_i \) is a sum of monomials containing elements of lower degree,

(4.10.3) \[ \pi^*(a_i) = \pi^*(a_i) = 0 \quad \text{for} \quad i = 2, 3, \]

(4.11) \[ i^*(\tilde{e}_2) = \tilde{q}_i + (\text{decomp.}). \]

**Lemma 4.7'.** Let \( f \in \text{Cotor}^B_*(\mathbb{Z}_2, \mathbb{Z}_2) \) such that \( \deg f \) is odd. Then \( i^*(f) = 0 \) iff \( f = 0 \).

**Proof.** \[ i^*(x^8_{i+8}) = \tilde{q}_{i+1}^2 + Q_i, \]

where \( Q_i \) is a sum of monomials containing elements of lower degree. So the elements \( i^*(x^8_{i+8}), 1 \leq l \leq 2n, i^*a_3 \) and \( i^*a_2 \) are algebraically independent. Q.E.D.

**Theorem 4.10.** The Eilenberg-Moore spectral sequence with \( \mathbb{Z}_2 \)-coefficient collapses for \( G = G(4n + 2) \).

**Proof.** Recall that \( a_2 \) and \( a_3 \) are permanent cycles. All generators of \( \text{Cotor}^B_*(\mathbb{Z}_2, \mathbb{Z}_2) \) except \( a_3 \) are of even degree. So \( d_*(\alpha) \) is of odd degree for \( \alpha \in \{ y(I), x^8_{i+8} \} \). By naturality \( i^*d_*(\alpha) = d_1i^*(\alpha) = 0 \). Hence by Lemma 4.7' \( d_*(\alpha) = 0 \). Thus all generators survive into \( E_{\infty} \). Q.E.D.

Immediate corollaries are

**Theorem 4.11.** As a module

\[ H^*(BG(4n + 2); \mathbb{Z}_2) \cong \mathbb{Z}_2[a_2, a_3, x^8_{i+8}, y(I)]/R, \]

where \( x^8_{i+8} = \{ b^2_{i+4} \} \) for \( 1 \leq l \leq 2n \) and \( I \) runs over all sequences satisfying (3.3) and \( R \) is the ideal generated by \( a_3y(I), y(I)^2 + \sum x^8_{i+8} \ldots a^2_{i+2} \ldots x^8_{i+8} \) and \( y(I)y(J) + \sum f_iy(I_i) \).

**Theorem 4.12.** As a module

\[ H^*(BPU(4n + 2); \mathbb{Z}_2) \cong \mathbb{Z}_2[a_2, a_3, x^8_{i+8}, y(I)]/R \]
§5. Some Generators in $H^*(BG(4n+2); \mathbb{Z}_2)$ and $H^*(BPO(4n+2); \mathbb{Z}_2)$

Let $G$ be a compact, connected Lie group and $H$ its closed subgroup. Let $EG$ and $EH$ be the total spaces of the universal $G$- and $H$-bundles respectively. Then the following diagram is commutative:

$$
\begin{array}{ccc}
H & \longrightarrow & EH \\
\downarrow & & \downarrow \\
G & \longrightarrow & EG
\end{array}
\quad
\begin{array}{ccc}
& \longrightarrow & BH \\
\downarrow & & \downarrow \\
& \longrightarrow & BG
\end{array}
$$

Then naturality of the transgression implies

**Lemma 5.1.** Let $k$ be a commutative field.

1. If $x \in H^*(G/H; k)$ is transgressive with respect to the bottom fiber-
ing, then $p^*(x) \in H^*(G; k)$ is universally transgressive.

2. If $x \in H^*(G; k)$ is universally transgressive, so is $j^*(x) \in H^*(H; k)$.

3. Suppose $H^i(G/H; k) = 0$ for $i < n$. Let $x \in H^i(G; k)$ and $i < n - 1$. If $j^*(x)$ is universally transgressive, so is $x$.

Recall the following:

(5.2) $G(2) = SO(3)$,

(5.3) $H^*(SO(3); \mathbb{Z}_2) = \mathbb{Z}_2[a]/(a^4)$ where $a$ is universally transgressive.

Now we prove

**Proposition 5.4.** The elements $a$ and $x_{4j+1}$ of $H^*(G(4n+2); \mathbb{Z}_2)$, $1 \leq j \leq 2n$, are all universally transgressive.

**Proof.** Proof is induction on $n$. The case $n=0$ is clear from (5.2) and (5.3). Suppose as the inductive hypothesis that the elements $a$ and $x_{4j+1}, 1 \leq j \leq 2n-1$, are universally transgressive in $H^*(G(4n-2);$
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It follows from Proposition 1.5 and (2), (3) of Lemma 5.1 that the elements \( a \) and \( x_{4j+1}, 1 \leq j \leq 2n-1 \), are universally transgressive. Clearly the element \( \bar{x}_{8n-3} \) is transgressive with respect to the fibering \( C(4n-2, 2n+1) \to BG(4n-2) \to BG((4n-2)(2n+1)) \), and hence so is \( \bar{x}_{8n+1} \), since \( \bar{x}_{8n+1} = Sq^4 x_{8n-3} \) by Proposition 1.14. Thus by (1) of Lemma 5.1 the elements \( x_{8n-3} \) and \( x_{8n+1} \) are universally transgressive. Q.E.D.

It follows from (2.2) and (2.3) that \( H^*(BG(2); Z_2) \cong H^*(BSO(3); Z_2) \cong \mathbb{Z}[a_2, a_3] \), where \( a_2 = \tau(a) \) and \( a_3 = \tau(a^2) \) with \( \deg a_i = i \). As \( \Delta_{2n+1}^* : H^i(BG(4n+2); Z_2) \to H^i(BG(2); Z_2) \) is isomorphic for \( i \leq 4 \) by (1) of Theorem 1.7, we denote by \( a_2 = \tau(a) \) and \( a_3 = \tau(a^2) \) the generators of \( H^i(BG(4n+2); Z_2) \cong \mathbb{Z}_2 \) for \( i = 2, 3 \).

**Lemma 5.5.** \( Sq^1 a_3 = 0 \) and \( Sq^2 a_3 = a_2 a_3 \) in \( H^*(BG(4n+2); Z_2) \).

**Proof.** We obtain the above formula by virtue of the Wu formula, since \( a_i \) is the inverse image of \( \Delta_{2n+1}^* \) of the \( i \)-th Stiefel-Whitney class. Q.E.D.

**Proposition 5.6.** There exist elements \( a_{4j+2}, 1 \leq j \leq 2n, \) in \( H^*(BG(4n+2); Z_2) \) such that

1. \( \deg a_{4j+2} = 4j + 2 \),
2. \( a_{4j+2} = \tau(x_{4j+1}) \mod (\text{decomp.}) \),
3. \( a_3 a_{4j+2} = 0 \).

**Proof.** Proof is induction on \( n \). The case \( n = 0 \) is clear from (2.2) and (2.3). Suppose that the assertion is true for \( BG(4n-2) \). By Theorem 1.7 the homomorphism \( \Delta_{2n+1}^* \) is injective for \( \deg \leq 8n-4e+1 \) with \( e = \pm 1 \). Put \( a_i = \Delta_{2n-1}^* \circ \Delta_{2n-1}^* (a_i) \) for \( 1 \leq 8n-6 \). Then \( a_i \) satisfies the properties (1), (2), (3) by the inductive hypothesis. For the transgression \( \tau \) of the fibering

\[
C(4n-2, 2n+1) \longrightarrow BG(4n-2) \longrightarrow BG((4n-2)(2n+1))
\]

we put \( a_{8n-2} = \Delta_{8n-3}^* (\bar{x}_{8n-3}) \). The element \( x_{8n-3} \in H^*(G(4n+2); Z_2) \) is not universally transgressive, since it is not primitive by Proposition
1.9'. So the corresponding element $\tilde{x}_{8n-1}$ of $H^*(C(4n-2, 2n+1); \mathbb{Z}_2)$ is not transgressive in the fibering (5.7). That is, in the cohomology Serre spectral sequence $\{E_r^{*,*}, d_r\}$ with $\mathbb{Z}_2$-coefficient of (5.7) we have $d_3(1 \otimes \tilde{x}_{8n-1}) = a_3 \otimes \tilde{x}_{8n-3}$, from which we get $a_3 \tau(\tilde{x}_{8n-3}) = 0$. Applying $\Delta_{2n-1}^*$ we obtain $a_3 a_{8n-2} = 0$. Thus the element $a_{8n-2}$ satisfies (1), (2), (3). Next, we put

$$a_{8n+2} = \Delta_{2n-1}^* \tau(\tilde{x}_{8n+1}) + a_2 Sq^2 a_{8n-2} + a_3 Sq^1 a_{8n-2}.$$ 

Then

$$a_3 a_{8n+2} = a_3 (Sq^* a_{8n-2} + a_2 Sq^2 a_{8n-2} + a_3 Sq^1 a_{8n-2})$$

$$= Sq^4 (a_3 a_{8n-2})$$

$$= 0.$$ 

So the element $a_{8n+2}$ satisfies (1), (2), (3). Q.E.D. 

Quite similarly one can prove

**Proposition 5.8.** There exist elements $a_2, a_{2j+1}, 1 \leq j \leq 2n$, in $H^*(BPO(4n+2); \mathbb{Z}_2)$ such that

(1) \hspace{1cm} \text{deg } a_2 = 2, \quad \text{deg } a_{2j+1} = 2j+1,

(2) \hspace{1cm} a_2 = \tau(y), \quad a_{2j+1} = \tau(z_{2j}), \quad 1 \leq j \leq 2n,

(3) \hspace{1cm} a_2 a_{2j+1} = 0.

**Remark 5.9.** The elements $a_i$ in Theorems 4.9, 4.11 and 4.12 are thus the transgression images of some generators in $H^*(G(4n+2); \mathbb{Z}_2)$, $H^*(PU(4n+2); \mathbb{Z}_2)$ or $H^*(PO(4n+2); \mathbb{Z}_2)$. The relations among them are given in Propositions 5.6 and 5.8.

§ 6. Exterior Power Representations

To begin with we recall the definition of the exterior power representation (p. 90 of [14]).

Let $G$ be a group and $k$ a commutative field. Denote by $GL(n, k)$ the general linear group. Let $A=(a_{ij}): G \rightarrow GL(n, k)$ be a matrix rep-
representation. For a pair of sequences of \( r \) integers \( I = (i_1, \ldots, i_r) \) and \( J = (j_1, \ldots, j_r) \) such that

\[
(*) \quad 1 \leq i_1 < \cdots < i_r \leq n,
\]

\[
1 \leq j_1 < \cdots < j_r \leq n,
\]

we define

\[
a_{IJ}(x) = \det \begin{pmatrix}
   a_{i_1j_1}(x) & \cdots & a_{i_rj_1}(x) \\
   \vdots & \ddots & \vdots \\
   a_{i_1j_r}(x) & \cdots & a_{i_rj_r}(x)
\end{pmatrix}
\text{ for } x \in G.
\]

**Definition 6.1.** Let \( 1 \leq r \leq n \). We define a representation \( A^{(r)}(x) : G \to GL((p), k) \) by

\[
A^{(r)}(x) = \begin{pmatrix}
   & & & & \\
   & & & & \\
   & & & & J \\
   & & & & \\
   \vdots & \ddots & \vdots & \ddots & \vdots \\
   & & & & \\
   a_{i_1j_r}(x) & \cdots & a_{i_rj_r}(x) & &
\end{pmatrix}
\]

where \( I \) and \( J \) run over all sequences satisfying \((*)\). We call \( A^{(r)} \) the exterior power representation of degree \( r \) of \( G \).

If \( G \) is a topological group and \( k = \mathbb{R} \) or \( \mathbb{C} \) and if \( A : G \to GL(n, k) \) is continuous, so is \( A^{(r)} \), namely, \( A^{(r)} \) is a representation of \( G \).

When \( G \) is a compact group and \( k = \mathbb{C} \) (resp. \( \mathbb{R} \)), we may suppose

\[
A^{(r)} : G \to U((p)) \quad \text{(resp. } A^{(r)} : G \to O((p))\text{)}
\]

by making use of the \( G \)-invariant Hermitian (resp. Riemannian) metric (see [2]).

**Proposition 6.2.** Let \( G \) be a subgroup of \( GL(n, k) \). Let \( A : G \to GL(n, k) \) be an inclusion. For \( G \ni x = \left( \begin{smallmatrix} -1 & \cdots & 0 \\ 0 & \ddots & -1 \\ \vdots & \ddots & 0 \\ 0 & \cdots & -1 \end{smallmatrix} \right) \), we have \( A^{(r)}(x) = \left( \begin{smallmatrix} 1 & \cdots & 0 \\ 0 & \ddots & 1 \end{smallmatrix} \right) \in GL((p), k) \) if \( r \) is even.

**Proof.** By definition

\[
a_{IJ}(x) = \begin{cases}
   (-1)^r & \text{if } I = J \\
   0 & \text{if } I \neq J.
\end{cases}
\]

Q.E.D.
In the below we regard the identity map \( \lambda: G = U(n) \to U(n) \) (or the inclusion \( \lambda: SU(n) \to U(n) \)) as an \( n \)-dimensional complex representation.

**Corollary 6.3.** Let \( n \) be even. Then there exists a map \( \lambda^{(2)} \) such that the right diagram commutes:

\[
\begin{array}{c}
SU(n) \xrightarrow{\lambda^{(2)}} U(\tfrac{n}{2}) \\
\pi \downarrow \quad \downarrow \lambda^{(2)} \\
G(n) \quad 
\end{array}
\]

Let \( t_k \) be a generator of \( H^2(BT^n; \mathbb{Z}) \) corresponding to the torus

\[
T^1 = \left\{ \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} : 0 \leq \theta < 2\pi \right\} \subset T_n \subset U(n).
\]

Then according to Borel-Hirzebruch (p. 492 of [6]) the total Chern class \( c(\lambda^{(2)}) \) of the second exterior power representation \( \lambda^{(2)} \) is given by

\[
(6.4) \quad c(\lambda^{(2)}) = \prod_{1 \leq i < j \leq n} (1 + t_i + t_j) \in H^*(BU(n); \mathbb{Z}).
\]

**Remark 6.5.** \( t_1 + \cdots + t_n = 0 \) if \( G = SU(n) \).

Let \( \alpha_i, 1 \leq i \leq n, \) be indeterminates with \( \deg \alpha_i = 1 \). Express

\[
\prod_{1 \leq i < j \leq n} (1 + \alpha_i + \alpha_j) = \beta_1 + \cdots + \beta_n + \text{(higher terms)},
\]

where \( \beta_k \) is a homogeneous term of degree \( k \). Denoting by \( \sigma_k \) the \( k \)-th elementary symmetric function, we have \( \beta_k = a_k \sigma_k(\alpha_1, \ldots, \alpha_n) + \text{(decomp.)} \) for some integer \( a_k \).

**Lemma 6.6.** If \( n \) is odd, \( a_i \) is odd for \( 2 \leq i \leq n \).

(A proof will be given at the end of the section.)

Let \( i: Sp(n) \to SU(2n) \) be the usual inclusion map defined by

\[
q_{ij} = \alpha_{ij} + j\beta_{ij} \mapsto c_{ij} = \begin{pmatrix} \alpha_{ij} & -\tilde{\beta}_{ij} \\ \beta_{ij} & \tilde{\alpha}_{ij} \end{pmatrix},
\]
where $\alpha_{i,j}, \beta_{i,j} \in \mathbb{C}$.

Let $s_i$ be a generator of $H^2(BT^n; \mathbb{Z})$ corresponding to the torus

$$T^1 = \left\{ \begin{pmatrix} 1 & \ldots & e^{i\theta} & 0 \\ \ldots & e^{i\theta} & \ldots \\ 0 & \ldots & 1 \end{pmatrix} \in Sp(n) \right\} \subset T^n \subset Sp(n).$$

Then

$$i^*(t_{2i-1}) = s_i \quad \text{and} \quad i^*(t_{2i}) = -s_i. \quad (6.7)$$

Consider the composite of the maps

$$BSp(n) \xrightarrow{i} BSU(2n) \xrightarrow{j^{(2)}} BU((\frac{1}{2} n)).$$

**Proposition 6.8.** The mod 2 reduction of $i^*c(\lambda^{(2)})$ is given by

$$i^*c(\lambda^{(2)}) = \prod_{1 \leq i < j \leq n} (1 + s_i^4 + s_j^4) \in H^*(BSp(n); \mathbb{Z}_2).$$

**Proof.**

$$i^*c(\lambda^{(2)}) = i^*\left( \prod_{1 \leq i < j \leq n} (1 + t_i + t_j) \right) \quad \text{by (6.4)}$$

$$= \prod_{1 \leq i < j \leq n} (1 + s_i + s_j)^4 \quad \text{by (6.7)}$$

$$= \prod_{1 \leq i < j \leq n} (1 + s_i^4 + s_j^4). \quad \text{Q.E.D.}$$

Next we consider the commutative diagram:

$$\begin{array}{ccc}
BSp(2n+1) & \xrightarrow{i} & BSU(4n+2) \xrightarrow{j^{(2)}} BU((\frac{4n+2}{2})) \\
\downarrow{\pi} & & \downarrow{\pi} \\
BPSp(2n+1) & \xrightarrow{i} & BG(4n+2) \xrightarrow{\lambda^{(2)}} \\
\end{array} \quad (6.9)$$

For the mod 2 reduction of the Chern class $c_{4i} \in H^{8i}(BU((\frac{4n+2}{2})); \mathbb{Z}_2)$ we put

$$x_{8i} = \lambda^{(2)*}(c_{4i}) \in H^{8i}(BG(4n+2); \mathbb{Z}_2), \quad 2 \leq i \leq 2n+1.$$ 

Then by the commutativity of the diagram (6.9)

$$i^*\pi^*\Sigma x_{8i} = i^*\pi^*\lambda^{(2)*}(\Sigma c_{4i}).$$
Apply Lemma 6.6 and we obtain

\[ i^*\pi^* x_{8i} = \sigma_i(s_1^4, \ldots, s_{2n+1}^4) + (\text{decomp.}). \]

Denoting by \( q_i \) the mod 2 reduction of the \( i \)-th symplectic Pontrjagin class, we have

\[ i^*\pi^* x_{8i} = q_i^2 + P, \]

where \( P \) is a sum of monomials containing \( q_j \) (\( j < i \)).

On the other hand, since \( i^* : H^m(BSU(4n + 2); \mathbb{Z}_2) \rightarrow H^m(BSp(2n + 1); \mathbb{Z}_2) \) is trivial for \( m \equiv 0 \pmod{4} \), we have

\[ i^*\pi^*(a_2) = i^*\pi^*(a_3) = i^*\pi^*(a_{4j+2}) = 0, \quad \text{and hence} \]

\[ i^*\pi^*(y(I)) = 0. \]

Thus we have shown

**Theorem 6.10.** There exist non-decomposable elements \( x_{8i+8} \in H^{8i+8}(BG(4n + 2); \mathbb{Z}_2), 1 \leq i \leq 2n, \) such that \( i^*\pi^*(x_{8i+8}) = q_{i+1}^2 + P, \) where \( P \) is a sum of monomials containing \( q_j \) (\( j < i+1 \)).

Now we turn to the orthogonal case.

Let \( \lambda : SO(n) \rightarrow O(n) \) be the natural inclusion and regard it as a real representation. As before we consider its exterior power representation \( \lambda^{(2)} : SO(n) \rightarrow O(n). \) The total Stiefel-Whitney class is then given as

\[ w(\lambda^{(2)}) = \prod_{1 \leq i < j \leq 2n} (1 + t_i + t_j), \]

where \( t_i \) is a generator of \( H^1(B\mathbb{Z}_2^n; \mathbb{Z}_2) \) corresponding to

\[ \mathbb{Z}_2 = \begin{bmatrix} 1 & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}; \quad e = \pm 1 \subset (\mathbb{Z}_2)^n \subset O(n). \]
Remark. \( t_1 + \cdots + t_n = 0. \)

Let \( i: U(n) \to SO(2n) \) be the inclusion defined by the correspondence \( b + c \sqrt{-1} \mapsto \left( \begin{array}{c} b \\ c \\ -b \end{array} \right) \). Let \( s_i \) be a generator of \( H^1(B(Z_2^n); Z_2) \) corresponding to

\[
Z_2 = \left\{ \left( \begin{array}{cccc}
1 & & & \\
& \ddots & & \\
& & 1 & \\
& & & 1
\end{array} \right); \; \varepsilon = \pm 1 \right\} \subset (Z_2)^n \subset U(n).
\]

Then

\[ (6.11) \quad i^*(t_{2i-1}) = i^*(t_{2i}) = s_i. \]

Let \( w_i \) be the Stiefel-Whitney class. Then

\[ i^*(w_{2i-1}) = 0, \]

\[ i^*(w_{2i}) = c_i, \] the mod 2 reduction of the \( i \)-th Chern class.

Consider the following commutative diagram

\[
\begin{array}{ccc}
BU(2n+1) & \xrightarrow{i} & BSO(4n+2) \\
\downarrow{\pi} & & \downarrow{\pi} \\
B(U(2n+1)/\Gamma_2) & \xrightarrow{i} & BPO(4n+2)
\end{array}
\]

where \( \pi \) is the natural projection and \( \bar{\lambda}^{(2)} \) the one induced from \( \lambda^{(2)} \). Then

\[ i^* \pi^* \bar{\lambda}^{(2)*} \left( \sum_{i=0}^{l} w_i \right) = i^* \left( w\left( \lambda^{(2)} \right) \right) \quad \text{with} \quad l = (4n+2), \]

where \( w\left( \lambda^{(2)} \right) = \prod_{1 \leq i < j \leq 4n+2} (1 + t_i + t_j). \)

So by Lemma 4.6 we have

\[ i^* \pi^* \bar{\lambda}^{(2)*} \left( \sum_{i=0}^{l} w_i \right) = \prod_{1 \leq i < j \leq 2n+1} (1 + s_i^* + s_j^*). \]
Thus by a similar argument to the unitary case we have

**Theorem 6.12.** There exist non-decomposable elements \( x_{4j+4} \in H^4 j+4(BPO(4n+2); Z_2) \), \( 1 \leq j \leq 2n \), such that \( i^* \pi^* x_{4j+4} = c_{j+1}^2 + P \), where \( P \) is a sum of monomials containing \( c_k \) \((k < j + 1)\).

First we consider the case \( G = G(4n+2) \). The projection \( \pi: SU(4n + 2) \to G(4n + 2) \) induces \( \pi^*: \operatorname{Cotor}^{B_4}(Z_2, Z_2) \to \operatorname{Cotor}^{B_4}(Z_2, Z_2) \) on the \( E_2 \)-level of the Eilenberg-Moore spectral sequence. By naturality we have

\[
\pi^* x_{8i+8} = \pi^* b_{4i+4}^2 = \pi^* b_{4i+4}^2 \quad \text{for} \quad 1 \leq i \leq 2n,
\]

which survives in the \( E_\infty(SU(4n+2))-\)term, since \( E_2(SU(4n+2)) \cong E_\infty(SU(4n+2)) \cong \mathcal{O} H^*(BSU(4n+2); Z_2) \) by Proposition 4.3. On the other hand, since \( q_{i+1} = i^* c_{2i+2} \), it follows from Theorem 6.10 that for \( \pi^*: H^*(BG(4n+2); Z_2) \to H^*(BSU(4n+2); Z_2) \) we have

\[
\pi^* x_{8i+8} = c_{2i+2}^2 + P'. \quad 1 \leq i \leq 2n,
\]

where \( P' \) is a sum of monomials containing \( c_j \) \((j < i + 1)\).

Thus we obtain

**Theorem 6.13.** The element \( x_{8i+8} \in \operatorname{Cotor}^{B_4}(Z_2, Z_2) \) survives in the \( E_\infty(G(4n+2))-\)term and represents \( x_{8i+8} \in H^*(BG(4n+2); Z_2) \).

Similarly,

**Theorem 6.13'.** The element \( x_{4j+4} \in \operatorname{Cotor}^{A_4}(Z_2, Z_2) \) survives in the \( E_\infty(PO(4n+2))-\)term and represents \( x_{4j+4} \in H^*(BPO(4n+2); Z_2) \).

**Proof of Lemma 6.6.** Let \( m \) be an odd integer. We regard the identity map \( \lambda: U(m) \to U(m) \) as an \( m \)-dimensional complex representation as before. Let \( t_k \) be a generator of \( H^2(BT^m; Z) \) corresponding to the torus

\[
T^1 = \begin{pmatrix}
1 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
\cdots & \cdots & 1 \\
1 & \cdots & 1
\end{pmatrix}
\quad 0 \leq \theta < 2\pi \subset T^m \subset U(m).
\]
Then by (6.4) the total Chern class of the exterior representation of degree 2 of $\lambda$ is given by

$$c(\lambda^{(2)}) = \prod_{1 \leq i < j \leq m} (1 + t_i + t_j) \in H^*(BU(m); \mathbb{Z}).$$

We will show that the integer $a_k$ is odd by taking $\alpha = t_i$ and $\beta_i = c_i(\lambda^{(2)})$, the $i$-th Chern class of $\lambda^{(2)}$.

Let $\Phi^k$ be the Adams operation on representations and $ch_q$ the Chern character. Denote by $\lambda^2$ the tensor product $\lambda \otimes \lambda$.

**Lemma 6.14.**

1. $ch_q \Phi^2(\lambda) = 2^q ch_q(\lambda)$.
2. $\Phi^2(\lambda) = \lambda^2 - 2\lambda^{(2)}$.
3. $ch_q(\lambda^2) = 2mch_q(\lambda) + (\text{decomp.})$.
4. Let $m \geq 3$. For $\eta = \lambda$ or $\lambda^{(2)}$

$$ch_q(\eta) = \frac{(-1)^q}{(q-1)!} c_q(\eta) + (\text{decomp.}).$$

**Proof.** (1), (2), (3) follow directly from the definition (also see [1]). (4) follows from the Newton formula. Q.E.D.

By this lemma we have

$$ch_i(\lambda^{(2)}) = \frac{1}{2} \{ ch_i(\lambda^2) - ch_i(\Phi^2(\lambda)) \}$$

$$= \frac{1}{2} \{ 2(n - 2^{i-1})ch_i(\lambda) \} + (\text{decomp.})$$

$$= (n - 2^{i-1})ch_i(\lambda) + (\text{decomp.}).$$

Now by (4) we obtain

$$c_i(\lambda^{(2)}) = (n - 2^{i-1})c_i(\lambda) + (\text{decomp.})$$

$$= (n - 2^{i-1})\sigma_i(t_1, \ldots, t_n) + (\text{decomp.}),$$

where $(n - 2^{i-1})$ is odd if $i \geq 2$. Q.E.D.
References