On some integral formulae containing Bessel functions

By

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So called Sounine's integral formula can be interpreted as follows [1]. Let $G_n$ be $n$-dimensional Euclidean motion group and $T_g$ be an irreducible unitary representation of class 1. The group $G_n$ contains the subgroup which is isomorphic to $G_{n-1}$. Being restricted to $G_{n-1}$, $T_g$ is decomposed into irreducible representations of $G_{n-1}$. Using this decomposition, we can express the zonal spherical function of $G_n$ by that of $G_{n-1}$. This expression is nothing but Sounine's first integral formula. Sounine's second integral formula can be proved in the same way.

On the other hand, the motion group of $n$-dimensional Lovachevsky space $G$ (the Lorentz group) contains, as a subgroup, $G_{n-1}$. As in the previous case, restricting the irreducible representation of $G$ to $G_{n-1}$ and decomposing it into irreducible factors, we can obtain an integral formula (6) involving Legendre and Bessel functions. In this sense, we may call the formula (6) an analogue to Sounine's formula.

If we consider imaginary Lovachevsky space, we obtain analogous formula (7), (8).

§ 1. Let $G$ be the $(n+2)$-dimensional Lorentz group, that is, the connected component of the orthogonal group of the quadratic form $-x_0^2+x_1^2+\cdots+x_{n+1}^2$.

We put,

$$K = \left\{ k = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} \in SO(n+1) \right\}$$
Then, $G = N_AK$ (Iwasawa decomposition)

Further,

$$M = \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix} ; \ m \in so(n)$$

is the normalizer of $N_-$ in $K$ and $M.N = N.M$ (semi-direct product) is isomorphic to $G_n$. We denote the Lie algebras corresponding to these subgroups by the letters, $u_-, a, \xi, m$, respectively.

The homogeneous space $X = G/K$ is $(n+1)$-dimensional Lovachevsky space. It is homeomorphic to $N/A$, so we can adopt the canonical coordinate of $A$ and $N_-$ as a global coordinate in $X$.

At first, we compute the invariant metric on $X$ (which is unique up to a constant factor) in terms of this coordinate,

Lemma. The invariant metric on $X$ is given by

$$ds^2 = dt^2 + e^{2t} \sum_{i=1}^{n} dx_i^2 \quad (1)$$

By the definition of the invariant metric, we must determine the positive definite quadratic form on the tangent space at $x(x \in X)$ $\varphi_x(Y)$, such that

$$\varphi_x(Y) = \varphi_{gx}(gY)$$
where we denote the transformation $x \rightarrow g \cdot x$ and its differential by the same letter.

For this, it is sufficient to construct the quadratic form on $\mathfrak{p}$, $\varphi(Y)$, invariant by $\text{Ad}_k$ ($\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ with respect to killing form, which can be identified with the tangent space at the origin $o$).

Then, $\varphi_s(Y) = \varphi(g^{-1}Y)$, where $g \cdot o = x$. The right hand side is independent of the choice of $g$.

In our case

$$
\varphi = \begin{pmatrix}
0 & y & \cdots & y_{n+1} \\
\vdots & \ddots & \ddots & \vdots \\
y & \cdots & 0 \\
y_{n+1}
\end{pmatrix}
$$

$y_h \in \mathbb{R}$

It is easy to see that $\varphi(Y) = \sum_{k=1}^{n+1} y_k^2$ (up to a constant factor).

Now, $\alpha + n_- \cong g/\mathfrak{k} \cong \mathfrak{p}$ (as vector spaces).

By this isomorphism, $\varphi(Y)$ can be regarded as a quadratic form on $\alpha + n_-$. That is,

$$
\varphi(Y) = t^2 + \sum_{k=1}^{n} y_k^2
$$

We identify all tangent space of $X$ with $\alpha + n_-$. Then the differential of the transformation $x \rightarrow gx$ is the linear transformation on $\alpha + n_-$ which is given by the Jacobian matrix.

$$
J_g = \frac{\partial (y', t')}{\partial (y, t)}|_{y = t = 0}
$$

where $(y', t')$ is the coordinate of $gx$.

In particular, for $g = y \alpha_{t_0}$, $t' = t + t_0$, $y' = y_0 + e^{-t_0}y$

and

$$
J_g = \begin{pmatrix}
1 \\
e^{-t_0} \\
\vdots \\
e^{-t_0}
\end{pmatrix}
$$
Consequently, the form at \( x = (y, t) \) is given by
\[
\varphi_x(Y) = t^2 + e^{zt} \sum_{k=1}^{n} y_k^2,
\]
which proves (1).

**Corollary.** Let \( \Delta \) be the Laplace-Beltrami operator on \( X \). Then
\[
\Delta = \frac{\partial^2}{\partial t^2} + n \frac{\partial}{\partial t} + e^{zt} \Delta_0, \quad \Delta_0 = \sum_{k=1}^{n} \frac{\partial^2}{\partial y_k^2}
\]

§ 2. Irreducible unitary representations of class 1 of \( G \) are usually constructed in the function space on a maximal compact subgroup ([2], [3]). But for our purpose, the following realization is more convenient.

According to the decomposition,
\[
G = N_+ \begin{pmatrix} M & A \end{pmatrix}
\]

for \( g \in G \) and \( x \in N_- \), \( xg \) is uniquely expressed in the form
\[
xg = y_{\text{max}} \quad \text{(if \( xg \) is a "regular" element)}.
\]

We denote \( x', a \) by \( x_g, a_t(x, g) \) respectively. Then we have

**Lemma.** \( (x_g)_h = \frac{\frac{1}{2} g_{0,h}(1 + |x|^2) + \sum_{j=1}^{n} x_j g_{jk} + \frac{1}{2} 2g_{n+1,h}(|x|^2-1)}{\Delta(x, g)} \)

where
\[
ed^x = \Delta(x, g) = \frac{1}{2} (g_{0,0} - g_{0,n+1})(|x|^2 + 1) + \sum_{j=1}^{n} x_j (g_{j,0} - g_{j,n+1})
\]
\[
+ \frac{1}{2} (g_{n+1,0} - g_{n+1, n+1})(|x|^2 - 1).
\]

We prove this for \( n=2m \) (for \( n=2m+1 \), the proof is similar).

put \( \tilde{G} = \{ g = SgS^{-1}, \ g \in G, \ S = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \}
\)

we denote by \( \tilde{A}, \tilde{N}_+, \tilde{N}_- \) the subgroups corresponding to \( A, N_+, N_- \).
On some integral formulae containing Bessel functions

Then

\[ \tilde{A} = \begin{pmatrix} e^t & 0 \\ E_n & e^{-t} \end{pmatrix}, t \in \mathbb{R} \]

\[ \tilde{N}_- = \begin{pmatrix} 1 & 0 & 0 \\ z_1 & \ddots & z_m \\ \vdots & \ddots & \vdots \\ z_1 & \cdots & z_1 \\ |z|^2 & \tilde{z}_1 \cdots \tilde{z}_m & z_m \cdots z_1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ E_n \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} z_h = x_k + iy_{3m-k+1} \\ 1 \leq k \leq m \end{pmatrix} \]

\[ \tilde{N}_+ = \begin{pmatrix} 1 & w_1 \cdots w_m & w_m \cdots w_1 & |w|^2 \\ 0 & \ddots & \vdots & \ddots & \ddots \\ 0 & \cdots & 0 & w_m \\ 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_m \\ w_k = y_k + iy_{3m-k+1} \\ 1 \leq k \leq m \end{pmatrix} \]

From the equality \( \tilde{x}g = \tilde{y}m\tilde{y}^t \), we have

\[ e^t = \text{the (n+2)th component of } (|z|^2, \tilde{z}_1 \cdots \tilde{z}_m, z_m \cdots z_1, 1)^{g^t} \]

\[ e^t(|z'|^2, \tilde{z}_1' \cdots \tilde{z}_m', z_m' \cdots z_1', 1) = (|z|^2, \tilde{z}_1 \cdots \tilde{z}_m, z_m \cdots z_1, 1)^{g^t} \]

But \((|z'|^2, \tilde{z}_1' \cdots \tilde{z}_m', z_m' \cdots z_1', 1) = (1 + |x'|^2, 2x_1' \cdots 2x_n', \sqrt{x'^2 - 1})\]

\((|z|^2, \tilde{z}_1 \cdots \tilde{z}_m, z_m \cdots z_1, 1)g = (1 + |x|^2, 2x_1 \cdots 2x_n, x^2 - 1)gS^{-1} \)

Therefore

\[ e^t = \frac{|x|^2 + 1}{2}(g_{00} - g_{0, n+1}) + \sum_{k=1}^n x_k(g_{k0} - g_{k, n+1}) - \frac{|x|^2 - 1}{2}(g_{m+1, 0} - g_{n+1, n+1}) \]

\[ = \Delta(x, g) \]

\[ x_k = \frac{|x|^2 + 1}{2}g_{kk} + \sum_{j=1}^n x_jg_{jk} + \frac{|x|^2 - 1}{2}g_{n+1, k}/\Delta(x, g) \quad (1 \leq k \leq n). \]

It is to be noticed that \( \frac{\partial(x \cdots x_n)}{\partial(x \cdots x_n)} = \Delta(x, g)^n \)

and \( \Delta(x, g, g_0) = \Delta(x, g_0)\Delta(x, g) \)
Using these, we can construct the unitary representation $\langle \mathcal{S}, U^p \rangle$ of $G$ in the following way.

$$\mathcal{S} = L^2(R^n)$$

$$U^p_g f(x) = \Delta(x, g)^{-\langle n/2 \rangle - i\rho} f(x_g) \quad (\rho \text{ is a real number parametrizing the representation.})$$

In particular,

for $g = y \in N$, $U^p_g f(x) = f(x + y)$ \hspace{1cm} (a)

for $g = a \in A$, $U^p_g f(x) = e^{i\langle n/2 + i\rho \rangle t} f(e^t x)$ \hspace{1cm} (b)

for $g = m \in M$, $U^p_g f(x) = f(x m)$ \hspace{1cm} (c)

From these, the irreducibility of $U^p_g$ can be easily seen. For, if $\tilde{U}^p_g$ is the Fourier transform of $U^p_g$ and $A$ is a bounded operator in $L^2(R^n)$ which commutes with $U^p_g (g \in G)$, then,

- by (a), $A$ is the multiplication operator by a bounded function $\varphi(x)$
- by (b), $\varphi(x)$ is independent of $|x|$
- by (c), $\varphi(x)$ is constant on sphere. if $n \geq 2$.

Therefore, $U^p_g$ is irreducible for $n \geq 2$. For $n = 1$ the above consideration is not sufficient for the irreducibility. We must prove that $\mathcal{S}_I = \{ f \in \mathcal{S}, f(x) = 0 \text{ for } x \geq 0 \}$ are not invariant subspaces.

If they are $\mathcal{S}_I$-invariant, the $\mathcal{S}_I$-invariant vector $f_0$ (see below) is contained in one of them. Then, being an even function, we have $f_0 \equiv 0$ which is a contradiction.

Next, we show the existence of $K$-invariant vector.

If $f \in \mathcal{S}$ is a $K$-invariant vector,

$$f(0) = (T_K f)(0) = \left\lfloor \frac{1 + g_{n+1, n+1}}{2} \right\rfloor^{-\langle n/2 \rangle - i\rho} f\left( \frac{g_{k, n+1}}{1 + g_{n+1, n+1}} \right)$$

We put $-g_{k, n+1}/1 + g_{n+1, n+1} = x_k$ (which take arbitrary real value) then

$$f(x) = c \frac{1}{1 + |x|^{\langle n/2 \rangle + i\rho}} \quad (c \text{ is a constant})$$

Conversely it is easy to see that such an $f$ is $K$-invariant (it is sufficient to verify this property for $g = 1$)

$$\begin{pmatrix}
\ddots & 0 \\
1 & 0 \\
0 & \cos \theta & \sin \theta \\
0 & -\sin \theta & \cos \theta
\end{pmatrix} = b_0$$
Therefore, $U^\alpha_k$ is of class 1 and the normalized $K$-invariant vector is

$$ f_\phi(x) = \left[ \frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)} \right]^{1/2} \frac{1}{(1 + |x|^2)^{n/2} + q^2} . $$

$\varphi_\alpha(g) = (U^\alpha_k f_\alpha, f_\alpha)$ is by definition, the zonal spherical function of $G$. For

$$ g = k a \alpha (k, \alpha \in K) $$

$$ \varphi_\alpha(g) = \varphi_\alpha(t) = \frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)} \int_0^\infty \frac{e^{(n/2) + q^2 t}}{1 + e^{2t |x|^2}} \frac{1}{(1 + |x|^2)^{n/2} + q^2} dx $$

$$ = 2\pi^{n/2} \frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)} \int_0^\infty \frac{e^{(n/2) + q^2 t}}{1 + e^{2t |x|^2}} \frac{1}{(1 + |x|^2)^{n/2} + q^2} dx $$

$$ = 2\pi^{n/2} \frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)} \int_0^\infty \frac{1}{(1 + |x|^2)^{n/2} + q^2} d|t| $$

(Here, we put $y = 1 - r^2 / (1 + r^2)$)

$$ = \frac{1}{B\left(\frac{n}{2}, \frac{1}{2}\right)} \int_0^\pi \frac{\sin^{n-1} \theta}{(1 + \cos \theta) \sin \theta} d\theta $$

$$ = \frac{2^{(n/2)-1/2} \Gamma\left(\frac{n+1}{2}\right)}{\sin^{(n/2)+1/2} \sin \theta} \left( \frac{\gamma - \delta^2}{\delta + \gamma} \right) (\chi t) $$

(See also [2], [3])

§ 3. Let $\varphi(g)$ be a spherical function of $G$. As is known, $(g)$ is a function on $X$ and satisfies the differential equation

$$ \Delta \varphi = -\left( \rho^2 + \frac{n^2}{4} \right) \varphi $$

(4)

By (2), $\varphi(k, t) = \int_{\varphi(y, t) = \varphi(k, t)} dy$ satisfies

$$ \left[ \frac{d^2}{du^2} - \frac{n-1}{u} \frac{d}{du} - \frac{\rho^2 + n^2}{u} \right] \varphi = 0 $$

(Here, we put $u = e^t$)

put $\varphi(n) = u^m v(u)$, then $\frac{d^2 v}{du^2} + \frac{m}{u} \frac{dv}{du} - \left( \frac{\rho^2 + n^2}{u} \right) v = 0$

Let $f(x)$ be an even integrable function on $R^n$. Fourier transform of which does not vanish. We put $\varphi(g) = (U^\alpha_k f_\alpha, f)$, then
\[ \varphi(xa) = (U_a^*f_0, U_{-a}f) = U_a^*f_0f \]

and

\[ \widetilde{\varphi}(k, u) = T_a f_0 \cdot f \]

Therefore

\[ \widetilde{T_a f_0} = \frac{1}{\pi} \int_{\mathbb{R}^n} \frac{e^{i\langle x, k \rangle}}{(u^2 + |x|^2)^{\frac{n}{2}} + i\rho} \, dx = \frac{\varphi(k, u)}{f} \]

satisfies (4). Taking into account the behavior at infinity, we conclude \( \widetilde{T_a f_0}(k, u) = c(|k|)u^{n/2}K_{ip}(|k|, u) \) where \( c(t) \) is a homogeneous function of degree \( v = \frac{n}{2} \).

From

\[ \int_{\mathbb{R}^n} \frac{dx}{(1 + |x|^2)^{\frac{n}{2}} + \nu} = \frac{\pi^{n/2} \Gamma(\nu)}{\Gamma\left(\frac{n}{2} + \nu\right)} \]

and

\[ \lim_{\rho \to 0} z^2 K_v(z) = \frac{\Gamma(\nu)}{2^{\frac{\nu}{2}}} \]

we have

\[ c(t) = \frac{|t|^2 2^{1-v}\pi^{n/2}}{\Gamma\left(\nu + \frac{n}{2}\right)} \]

Therefore

\[ \int_{\mathbb{R}^n} \frac{e^{i\langle x, k \rangle}}{(u^2 + |x|^2)^{\frac{n}{2}} + \nu} \, dx = 2\pi^{n/2} \left(\frac{|k|}{2u}\right)^\nu \frac{1}{\Gamma\left(\nu + \frac{n}{2}\right)} K_v\left(|k|, u\right) \quad (5) \]

As is seen from §2 (a) and (c), if we restrict \( U_g^* \) to the subgroup \( G_n = MN_+ \), we obtain the quasi-regular representation \( V_g \) of \( G_n \) (the regular representation of \( R^1 \), for \( n=1 \)). Fourier transform of which are decomposed as follows.

\[ \tilde{V}_g = d_n \int_0^\infty V_g R^{-1} dR, \quad d_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \]

where \( (V_g, \tilde{\mathfrak{g}}_R) \) is the irreducible unitary representation of class 1 of \( G \) [1]. This means that to \( f \in \tilde{\mathfrak{g}} \), there corresponds \( f_R \in \tilde{\mathfrak{g}}_R \), such that

\[ \|f\|^2 = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \int_0^\infty \langle f_R, f_R \rangle R^{-1} dR \]

\[ (V_g f)_R = V_g^R f_R \]

\( \langle f_R, f_R \rangle \) is the inner product in \( \tilde{\mathfrak{g}}_R \).
We denote this correspondence by \( f = d_n \int dr R^{n-1} dR \). Then by (5)
\[
\hat{f} = \int_0^\infty c(R) K_{ip}(R) \varphi_0 R^{n-1} dR
\]
\[
T_{at} f_0 = \int_0^\infty c(R) a^{n/2} K_{ip}(aR) \varphi_0 R^{n-1} dR \cdot
\]
(here, \( \varphi_0 \) is the normalized \( M \)-invariant vector in \( S_R \) and \( a = e^t \))

Therefore, for \( g = a_{i_1} x a_{t_2} \)
\[
\varphi_0(g) = \langle T_x T_{at_2} f_0, T_{at_1} f_0 \rangle = \frac{1}{(2\pi)^n} \langle T_x T_{at_2} \hat{f}_0, T_{at_1} \hat{f}_0 \rangle
\]
\[
= \frac{1}{(2\pi)^n} \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^\infty c(R)^2 (ab)^{n/2} K_{ip}(aR) K_{ip}(bR) \langle T_x^2 \varphi_0, \varphi_0 \rangle R^{n-1} dR
\]

As is known [1], \( \langle T_x^2 \varphi_0, \varphi_0 \rangle = 2^{(n/2)-1} \Gamma \left( \frac{n}{2} \right) (|x| R)^{-(n/2)} J_{(n/2)-1}(|x| R) \) (the zonal spherical function of \( G \))

On the other hand, \( \varphi_0(g) \) is given by (5) for \( g = ka_{j_1} k' \).

By an easy computation, from \( a_{i_1} x a_{t_2} = ka_{j_1} k' \), we obtain
\[
\chi t = \frac{r^2 + a^2 + b^2}{2ab} \quad (a = e^t, \ b = e^{t_2})
\]

(compare the \((1, 1)\)-component of the matrices on both sides.)

thus we have proved the following formula

\[
(6) \quad \int_0^\infty t^{n/2} K_{ip}(as) K_{ip}(bs) J_{(n/2)-1}(us) ds
\]
\[
= \pi^{1/2} u^{(n/2)-1} \Gamma \left( \frac{n}{2} + i p \right) \Gamma \left( \frac{n}{2} - i p \right) \frac{\mathfrak{H}_1^{(1/2)}(t)}{\sin^{(n-1)/2} \tan} \quad \text{for } n \geq 2 ,
\]

and
\[
\int_0^\infty 4 \sqrt{ab} K_{ip}(as) K_{ip}(bs) \cos ut \, dt
\]
\[
= \pi \Gamma \left( \frac{1}{2} + i p \right) \Gamma \left( \frac{1}{2} - i p \right) \mathfrak{H}_{-1/2}^{(1/2)+i p}(cht) \quad \text{for } n = 1 .
\]

\( \S 4 \). Instead of \( K = SO(n+1) \), we consider the subgroup
\( H = \left\{ h = \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \in G \right\} \) which is isomorphic to \((n+1)\)-dimensional Lorentz group.

The the unitary representation $(\mathcal{G}, U^\varphi)$ is of “class 1” with respect to this subgroup. $H$-invariant vectors are (up to constant factors) $f_0(x) = (1 - |x|^2)^{-(n/2)+ip}$ and $f_1(x) = (1 - |x|^2)^{-(n/2)+ip} = U^\varphi_1 f_0(x)$ (for $\varepsilon$, see below)

where, by definition, $F(x)_+ = \begin{cases} F(x) & \text{if } F(x) > 0 \\ 0 & \text{otherwise} \end{cases}$

$$
\varphi^{(1)}_\varphi(g) = (U^\varphi_0 f_0, f_0) \quad \text{and} \quad \varphi^{(2)}_\varphi(g) = (U^\varphi_1 f_0, f_1)
$$

are “zonal spherical functions”

1) for $g = a_t$ ($t \geq 0$)

$$
\varphi^{(1)}_\varphi(g^{-1}) = \varphi^{(1)}_\varphi(-t) = \int \frac{1}{(1 - e^{-2t} |x|^2)^{n/2} + p} dx
$$

$$
= \alpha_n \int_0^1 \frac{e^{-\frac{y^2}{2}}}{(1 - e^{-2t} |x|^2)^{n/2} + p} \frac{y^{n-1}}{1 - e^{-2t} |x|^2} dy
$$

$$
= \frac{\alpha_n \pi^{n/2}}{2^{n-1} \Gamma\left(\frac{n}{2}\right)} \int_0^\infty \frac{s^{n-1}}{(1 + y^2)^{n/2} + p} ds \quad (y = ch s)
$$

$$
\varphi^{(1)}_\varphi(t) = \varphi^{(1)}_\varphi(-t)
$$

2) for $g = b_0$

$$
\varphi^{(1)}_\varphi(g) = \varphi^{(1)}_\varphi(\theta) = \int \left\{ (1 - |x|^2) \cos \theta + 2x_n \sin \theta \right\}^{-(n/2)+ip} dx
$$

$$
\times \left\{ \frac{\cos \theta}{1 - |x|^2} + \frac{2x_n \sin \theta}{1 - |x|^2} \right\}^{-(n/2)+ip} \frac{dx}{(1 - |x|^2)^n}
$$

$$
= \frac{1}{2^n} \int_{R^n} (\cos \theta + x_n \sin \theta)^{-(n/2)+ip} (1 + |x|^2)^{-(1/2)} dz
$$

$$
(z_k = 2x_k / 1 - |x|^2, 1 \leq k \leq n)
$$

$$
= \frac{1}{2^n} \sin^{n-1} \theta \int_{s < 0} (1 + |u|^2 - 2u_n \cos \theta) u_n^{-(n/2)+ip} du
$$

$$
(u_n = \cos \theta + x_n \sin \theta, u_k = z_k \sin \theta \quad 1 \leq k \leq n-1)
$$

$$
= \frac{\pi^{-(n/2)} \Gamma\left(1 - \frac{n}{2}\right)}{2^n \sin^{n-1} \theta} \int_0^\infty u^{-(n/2)+ip} (u^2 - 2u \cos \theta + 1)^{-(n/2)-1} du
$$
3) for \( g = a_t \) \((t > 0)\)

\[
\varphi_p^{(2)}(g) = \varphi_{-p}^{(2)}(-t) = \int_{|x|<\infty} \frac{e^{i(n/2)+ipx}}{1-e^{-2t|x|^2}(n/2)+ip(|x|^2-1)} \frac{1}{1-x} dx
\]

\[
= \frac{\pi n}{2^{\nu_2}} \int_{c}^{\infty} \frac{(y^2-1)^{(n/2)-1}}{\cosh t} dy \quad \left( y = \frac{r^2+1}{r^2-1} \right)
\]

\[
= \frac{\pi n}{2^{\nu_2}} \int_{c}^{\infty} y^{(n/2)-1} (y^2+2y \cosh t + 1)^{(n/2)-1} dy
\]

\[
\varphi_p^{(2)}(t) = 0
\]

So far as \( f_0, f_1 \) do not belong to the Hibert space \( \mathcal{H} \), these integral are, in general, divergent.

But, for \( n = 1 \), they converge almost everywhere and can be expressed by special functions:

\[
\varphi_p^{(1)}(t) = \int_{0}^{\infty} (\cosh t + \sinh t) \left( 1 + e^{-t} \right) ds = Q_{(1/2)}(\cosh t)
\]

\[
\varphi_p^{(1)}(\theta) = \frac{1}{2} \int_{0}^{\infty} u^{-(1/2)-ip} (1 + u^2 - 2u \cos \theta)^{1/2} du
\]

\[
= \frac{1}{2} \Gamma\left( \frac{1}{2} + i\rho \right) \Gamma\left( \frac{1}{2} - i\rho \right) P_{-(1/2)-ip}(-\cos \theta)
\]

for \( t < 0 \)

\[
\varphi_p^{(2)}(t) = \int_{0}^{\infty} y^{-(1/2)-ip} (1 + y^2 + 2y \cosh t)^{1/2} dy
\]

\[
= \Gamma\left( \frac{1}{2} + i\rho \right) \Gamma\left( \frac{1}{2} - i\rho \right) 3_{-(1/2)+ip}^{(1/2)-ip}(\cosh t)
\]

The following lemma can be proved easily.

**Lemma.**

(1) if \( r < |a-b| \), \( a_{inter1} x_{inter1} = h a, h' \) \((h, h' \in \mathcal{H})\)

and \( \cosh t = \frac{a^2 + b^2 - r^2}{2ab} \) \((a = e^{t_1}, b = e^{t_2}, r = |x|)\)

\( t \geq 0 \) according as \( b \geq a \)

(2) if \( |a-b| < r < a+b \), \( a_{inter1} x_{inter1} = h h' \)

and \( \cos \theta = \frac{a^2 + b^2 - r^2}{2ab} \)

(3) if \( r > a+b \), \( a_{inter1} x_{inter1} = h e a, h' \), \( e = \begin{pmatrix} 1 & \cdots & 1 \\ -1 \end{pmatrix} \)
From the above results, we obtain the following.

\begin{align*}
(7) \quad \sqrt{ab} \int_0^\infty J_\nu(au) J_{\nu'}(bu) \cos ut \, du &= \frac{\text{ch} \pi \rho}{\pi} Q_{-\frac{1}{2}+i\rho} \frac{a^2 + b^2 - t^2}{2ab} \\
&= P_{-\frac{1}{2}+i\rho} \left( \frac{t^2 - a^2 - b^2}{2ab} \right) \\
& \quad \text{for } |a-b| < t < a+b \\
& = 0 \quad \text{for } t > a+b
\end{align*}

\begin{align*}
(8) \quad \sqrt{ab} \int_0^\infty Y_\nu(au) Y_{-\nu'}(bu) \cos ut \, du &= \frac{\text{ch} \pi \rho}{\pi} Q_{-\frac{1}{2}+i\rho} \frac{(a^2 + b^2 - t^2)}{2ab} \\
&= P_{-\frac{1}{2}+i\rho} \left( \frac{t^2 - a^2 - b^2}{2ab} \right) \\
& \quad \text{for } |a-b| < t < a+b \\
& = 2 \Im P_{-\frac{1}{2}+i\rho} \left( \frac{t^2 - a^2 - b^2}{2ab} \right) \\
& \quad \text{for } t > a+b
\end{align*}

The proof is the same as in §3. Instead of (5) we use the following.

\begin{align*}
\int_{|t|<a} \frac{e^{it\mu}}{(a^2 - t^2)^{1/2} \gamma} \, dt &= \frac{1}{2} \pi^{1/2} \Gamma \left( \frac{1}{2} - \nu \right) \left( \frac{u}{2a} \right)^\gamma J_\nu(au) \\
\int_{|t|>a} \frac{e^{it\mu}}{(t^2 - a^2)^{1/2} \gamma} \, dt &= -\frac{1}{2} \pi^{1/2} \Gamma \left( \frac{1}{2} - \nu \right) \left( \frac{u}{2a} \right)^\gamma Y_\nu(au)
\end{align*}

LITERATURE