An elementary approach to the location of the maximum
Stirling number(s) of the second kind

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Horst Wegner studied Mathematics at the University of Hamburg. After the diploma 1966 and a short activity in the industry, he promoted in 1970 over a problem with Stirling numbers second kind at the University of Cologne. Since 1973 he is as Academic Oberrat at the University Duisburg, first in the teacher training and since 1982 in the subject Stochastics.

1 The unimodality

The number of ways of partitioning a set of \( n \) elements into \( k \) nonempty subsets is usually denoted \( S(n, k) \). These numbers are called Stirling numbers of the second kind, which were so named by Niels Nielsen [5], who wrote in German “Stirlingsche Zahlen”.

Obviously \( S(n, 1) = S(n, n) = 1 \) and \( S(n, k) = 0 \) for all \( k > n \). Moreover it is useful to put \( S(0, 0) = 1 \) and \( S(0, k) = S(n, 0) = 0 \) for all \( n, k \in \mathbb{N} \). Elementary combinatorial arguments lead us to the two following recurrences

\[
S(n + 1, k) = kS(n, k) + S(n, k - 1), \quad n, k \in \mathbb{N}, \quad (1.1)
\]

\[
S(n + 1, k) = \sum_{i=0}^{n} \binom{n}{i} S(i, k - 1), \quad n, k \in \mathbb{N}. \quad (1.2)
\]
Using (1.1) and (1.2), it can be shown by mathematical induction on the value of $n$ that for fixed $n$ the numbers $S(n, k)$ are unimodal in the following sense (see [1], [6]).

**Theorem 1.1.** For all $n \in \mathbb{N}$ there exists a $K_n \in \mathbb{N}$, $1 \leq K_n \leq n$, such that

$$
\begin{align*}
S(n, k - 1) &< S(n, k) \quad \text{for} \quad 1 \leq k \leq K_n - 1, \\
S(n, k - 1) &\leq S(n, k) \quad \text{for} \quad k = K_n, \\
S(n, k - 1) &> S(n, k) \quad \text{for} \quad K_n + 1 \leq k \leq n + 1.
\end{align*}
$$

Furthermore $0 \leq K_{n+1} - K_n \leq 1$.

If we do without the inequality $K_{n+1} \leq K_n + 1$, Theorem 1.1 can be proved using only (1.1) (see [3]).

With regard to our further investigations it is useful to define

$$
K^*_{n} := \begin{cases} 
K_n - 1 & \text{if } S(n, K_n - 1) = S(n, K_n), \\
K_n & \text{if } S(n, K_n - 1) < S(n, K_n).
\end{cases}
$$

(1.3)

According to (1.3), the proof by induction of Theorem 1.1 shows that more precisely

$$
K_n \leq K^*_{n+1} \leq K_{n+1} \leq K_n + 1.
$$

(1.4)

Clearly $K^*_2 = K_2 - 1$. It is not known to the author whether there is another case such that $K^*_n = K_n - 1$. It seems to be an unsolved problem whether $S(n, k)$ always has a single maximum for $n \geq 3$. Some results concerning this problem and the value $K_n$ have been established (see e.g. [2], [3], [4], [7]).

The first aim of this paper is to obtain bounds for $K_n$, $K^*_n$, using quite elementary methods. The results, attained in this way, will be stated in the Theorems 3.2 and 3.5. Finally, in the last section, we will show how to determine exact values of $K_n$, $K^*_n$.

2 **Preparatory remarks**

Let $n, k \in \mathbb{N}$ and let $X, Y$ be sets with $|X| = n$, $|Y| = k$. It is evident that $k!S(n, k)$ is the number of surjective functions from $X$ to $Y$. Thus, by simple combinatorial considerations, we obtain for the number of all functions from $X$ to $Y$ the following formula:

$$
k^n = \sum_{i=0}^{k-1} \binom{k}{k-i}(k-i)!S(n, k-i).
$$

Hence

$$
k^n = \sum_{i=0}^{k-1} \frac{1}{i!}S(n, k-i).
$$

(2.1)

Furthermore, using the principle of inclusion and exclusion, the number of non-surjective functions from $X$ to $Y$ is

$$
k^n - k!S(n, k) = \sum_{i=1}^{k-1} (-1)^{i-1} \binom{k}{k-i}(k-i)^n
$$
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(see [8], Section 4), and this implies

$$S(n, k) = \sum_{i=0}^{k-1} \frac{(-1)^i}{i!} \frac{(k-i)^n}{(k-i)!}. \quad (2.2)$$

By the formulas (2.1), (2.2) (clearly, they are well-known), we obtain

$$kn! - \frac{(k-1)^n}{(k-1)!} = \sum_{i=0}^{k-1} \frac{1}{i!} (S(n, k-i) - S(n, k-i-1)) \quad (2.3)$$

and

$$S(n, k) - S(n, k-1) = \sum_{i=0}^{k-1} \frac{(-1)^i}{i!} \frac{(k-i)^n}{(k-i)!} - \frac{(k-i-1)^n}{(k-i-1)!}. \quad (2.4)$$

The last two formulas will lead us to the desired results in Theorem 3.2 and Theorem 3.5.

3 Bounds for $K_n$, $K^*_n$

First we want to establish an upper bound for $K_n$, which is already given in [7]. Previously we are beginning with a result, which is evident.

**Lemma 3.1.** Let $n \in \mathbb{N}$, $n \geq 2$. Then there is a unique $s_n \in (2, \infty)$ such that

$$x \left(1 - \frac{1}{x}\right)^n \begin{cases} < 1 & \text{for } 2 \leq x < s_n, \\ = 1 & \text{for } x = s_n, \\ > 1 & \text{for } x > s_n. \end{cases}$$

**Theorem 3.2.** Let $n \in \mathbb{N}$, $n \geq 2$, and let $K_n$ be given by Theorem 1.1. Furthermore let $s_n$ be the unique root of $x(1 - \frac{1}{x})^n = 1$ in the interval $(2, \infty)$ (see Lemma 3.1). Then $K_n \leq [s_n].$

**Proof.** Let $k = K_n$. Then, by (2.3),

$$\frac{k^n}{k!} \left(1 - k \left(1 - \frac{1}{k}\right)^n\right) = \frac{k^n}{k!} - \frac{(k-1)^n}{(k-1)!} \geq 0.$$ 

Hence $k(1 - \frac{1}{k})^n \leq 1$, and then Lemma 3.1 implies $K_n = k \leq s_n$. Hence $K_n \leq [s_n]$. \qed

Before giving a lower bound for $K^*_n$ we establish two further lemmas.

**Lemma 3.3.** Let $n \in \mathbb{N}$. Then the function

$$f(x) := \frac{1}{x} \left(1 + \frac{1}{x}\right)^{n-1} + x \left(1 - \frac{1}{x}\right)^n$$

is strictly convex on the interval $[1, \infty)$.

**Proof.** It is easy to show that $f''(x) > 0$ for all $x \geq 1$. \qed
Lemma 3.4. Let $n \in \mathbb{N}$, $n \geq 3$, and let $s_n$ be given by Lemma 3.1. Then there is a unique $r_n \in (2, s_n)$ such that

$$
\frac{1}{x-1} \left(1 + \frac{1}{x-1} \right)^{n-1} + (x-1) \left(1 - \frac{1}{x-1} \right)^n \begin{cases} 
> 2 & \text{for } 2 \leq x < r_n, \\
= 2 & \text{for } x = r_n, \\
< 2 & \text{for } r_n < x \leq s_n.
\end{cases}
$$

Proof. For abbreviation we put

$$
g(x) := \frac{1}{x-1} \left(1 + \frac{1}{x-1} \right)^{n-1} + (x-1) \left(1 - \frac{1}{x-1} \right)^n.
$$

Obviously

(i) $g(2) > 2$.

Now let $x = s_n$. Then Lemma 3.1 implies

$$
1 = x \left(1 - \frac{1}{x} \right)^n > (x-1) \left(1 - \frac{1}{x-1} \right)^n \quad \text{and}
$$
$$
1 = \frac{1}{x} \left(1 - \frac{1}{x} \right)^{-n} = \frac{1}{x-1} \left(1 + \frac{1}{x-1} \right)^{n-1}.
$$

These two relations imply

(ii) $g(s_n) < 2$.

With the function $f$ given in Lemma 3.3 we have $g(x) = f(x-1)$, and then it follows from Lemma 3.3 that $g$ is strictly convex on the interval $[2, \infty)$. With regard to (i), (ii), this implies that there is a unique $r_n \in (2, s_n)$ such that

$$
g(x) \begin{cases} 
> 2 & \text{for } 2 \leq x < r_n, \\
= 2 & \text{for } x = r_n, \\
< 2 & \text{for } r_n < x \leq s_n.
\end{cases} \quad \Box
$$

Now we are prepared to establish a lower bound for $K^*_n$.

Theorem 3.5. Let $n \in \mathbb{N}$, $n \geq 3$, and let $K^*_n$ be given by Theorem 1.1 and (1.3). Furthermore let $r_n$ be the unique root of

$$
\frac{1}{x-1} \left(1 + \frac{1}{x-1} \right)^{n-1} + (x-1) \left(1 - \frac{1}{x-1} \right)^n = 2
$$

in the interval $(2, s_n)$ (see Lemma 3.4), where $s_n$ is given by Lemma 3.1. Then $[r_n] \leq K^*_n$.

Proof. Our aim is to apply (2.4).

Let $k = [r_n]$, hence $k \geq 2$. Then, for $i = 0, 1, \ldots, k-1$ we put

$$
\Delta_i := \frac{(k-i)^n}{(k-i)!} - \frac{(k-i-1)^n}{(k-i-1)!} = \frac{(k-i)^n}{(k-i)!} \left(1 - (k-i) \left(1 - \frac{1}{k-i} \right)^n \right).
$$
Since \(1 \leq k - i \leq r_n < s_n\), it follows from Lemma 3.1 that
\[
\Delta_i > 0 \quad \text{for } i = 0, 1, \ldots, k - 1.
\] (3.1)

Moreover, for \(i = 0, 1, \ldots, k - 2\) one has
\[
\begin{align*}
\Delta_i - \Delta_{i+1} &= \frac{1}{(k-i)!} ((k-i)^{n-1} - 2(k-i-1)^n + (k-i-1)(k-i-2)^n) \\
&= \frac{(k-i-1)^n}{(k-i-1)!} \left( \frac{1}{k-i-1} \left(1 + \frac{1}{k-i-1}\right)^{n-1} + (k-i-1) \left(1 - \frac{1}{k-i-1}\right)^n - 2 \right).
\end{align*}
\]

Since \(2 \leq k - i \leq r_n\), it follows then from Lemma 3.4 that
\[
\Delta_i - \Delta_{i+1} \geq 0 \quad \text{for } i = 0, 1, \ldots, k - 2.
\] (3.2)

In particular, we obtain for \(i = k - 2\)
\[
\Delta_{k-2} - \Delta_{k-1} = 2^{n-1} - 2 \geq 2^2 - 2 > 0.
\] (3.3)

With respect to (3.1), we obtain from (3.2), (3.3)
\[
\frac{\Delta_i}{i!} \geq \frac{\Delta_{i+1}}{(i+1)!} \quad \text{for } i = 0, 1, \ldots, k - 2
\] (3.4)

and in particular
\[
\frac{\Delta_{k-2}}{(k-2)!} > \frac{\Delta_{k-1}}{(k-1)!}
\] (3.5)

According to (3.1), (3.4), (3.5), it follows from (2.4) that
\[
S(n, k) - S(n, k - 1) = \sum_{i=0}^{k-1} (-1)^i \frac{\Delta_i}{i!} > 0.
\]

Hence \(K^*_n \geq k = [r_n]\).

This section shall end with some examples illustrating the results of Theorems 3.2 and 3.5:

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<th>50</th>
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<th>250</th>
<th>500</th>
<th>1000</th>
<th>2500</th>
<th>5000</th>
<th>10000</th>
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<td>10</td>
<td>17</td>
<td>29</td>
<td>61</td>
<td>107</td>
<td>190</td>
<td>415</td>
<td>755</td>
<td>1383</td>
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<tr>
<td>([r_n])</td>
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<td>16</td>
<td>27</td>
<td>58</td>
<td>103</td>
<td>185</td>
<td>407</td>
<td>745</td>
<td>1370</td>
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</tbody>
</table>
4 Exact values of $K_n$

In many cases the exact values of $K_n$, $K^*_n$ can be determined by a procedure first published by Comtet and Fiolet [4], which shall be described here.

We know that the partial sums of (2.2) successively overcount and undercount the number $S(n, k)$ (see [8]). Thus, for $l = 0, 1, \ldots, k - 1$ the expressions

$$\sum_{i=0}^{l} \frac{(-1)^i}{i!} \frac{(k-i)^n}{(k-i)!} - \sum_{i=0}^{l-1} \frac{(-1)^i}{i!} \frac{(k-1-i)^n}{(k-1-i)!}$$

$$= \sum_{i=0}^{l} \frac{(-1)^i}{i!} \frac{i+1}{(k-i)!} \frac{(k-i)^n}{(k-i)!} = \frac{k^n}{k!} \sum_{i=0}^{l} (-1)^i (i+1) \binom{k}{i} \left(1 - \frac{i}{k}\right)^n$$

are successively upper and lower bounds for the difference $S(n, k) - S(n, k-1)$. Putting for abbreviation

$$v(n, k, l) := \sum_{i=0}^{l} (-1)^i (i+1) \binom{k}{i} \left(1 - \frac{i}{k}\right)^n,$$

we have for $l = 0, 1, \ldots, k - 1$ the relations

$$S(n, k) - S(n, k-1) \begin{cases} \leq \frac{k^n}{k!} v(n, k, l), & \text{if } l \text{ is even}, \\ \geq \frac{k^n}{k!} v(n, k, l), & \text{if } l \text{ is odd}. \end{cases} \quad (4.1)$$

Using (4.1), Theorem 1.1, (1.3), we obtain

$$\begin{cases} l \text{ even} & v(n, k, l) < 0 \Rightarrow K_n \leq k - 1, \\ l \text{ odd} & v(n, k, l) > 0 \Rightarrow K^*_n \geq k. \end{cases} \quad (4.2)$$

Now let us apply (4.2) to our last example of Section 3, which gives us

$$1370 \leq K^*_{10000} \leq K_{10000} \leq 1383.$$

As a first step we check $v(10000, k, l)$ for $k = 1383$. Then $l = 6$ is the first even number such that $v(10000, 1383, l)$ is negative, namely

$$v(10000, 1383, 6) = -0.000510 \ldots$$

Thus, by (4.2), $K_{10000} \leq 1382$. Already a second step shows us that

$$v(10000, 1382, 7) = 0.000314 \ldots > 0$$

and therefore (4.2) implies $K^*_{10000} \geq 1382$.

Hence $K_{10000} = K^*_{10000} = 1382.$
This example shows us that the upper bound $[s_n]$ from Theorem 3.2 is much sharper than the lower bound from Theorem 3.5. This fact will be emphasized by many other examples. (According to (1.4), (4.2), it is quite easy to determine $K_{n+1}$, if the value of $K_n$ is given.)

In the overwhelming majority of cases we obtain $K_n = [s_n] - 1$ and only in a few cases $K_n = [s_n]$. So we can fall into temptation to suppose that $K_n \geq [s_n] - 1$ for all $n \geq 3$. But no elementary proof of this inequality can be offered.

References


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