The conics of Lucas’ configuration

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1 Introduction

Let us consider a figure formed by a triangle $ABC$ and its three inscribed squares $X_1X_2Y_3Z_4$, $Y_1Y_2Z_3X_4$, $Z_1Z_2X_3Y_4$, where the sides $X_1X_2$, $Y_1Y_2$, $Z_1Z_2$ are on the sides $AB$, $BC$, $CA$ of the triangle, and these three squares are homothetic to the external squares $BAB_1'B_2B_3'$, $CBC_1'C_2'C_3'$, respectively, from the vertices of $CAB$; see Fig. 1. We will call this figure “Lucas’ configuration”.

In fact, there are another three squares inscribed in the triangle $ABC$. These are the three squares $X_1'X_2'Z_3'Y_4'$, $Y_1'Y_2'X_3'Z_4'$, $Z_1'Z_2'Y_3'X_4'$, where the sides $X_1'X_2'$, $Y_1'Y_2'$, $Z_1'Z_2'$ are on the sides $A'B'$, $B'C'$, $C'A'$ of the triangle, and these three squares are homothetic to the internal squares $ABA''B''$, $BCB''C''$, $ACA''A''$, respectively, from the vertices of $CAB$. We will call this figure “Lucas’ internal configuration”; but the results and conditions are similar to Lucas’ configuration.

In [3], I. Panakis shows the relations found by Édouard Lucas between the circumcircles of the triangles $AX_4Z_3$, $BY_3X_4$, $CZ_4Y_3$ and the length of the sides of the triangle $ABC$. In [1], A.P. Hatzipolakis and P. Yiu show that these three circumcircles are mutually tangent to each other, and tangent to the circumcircle of $ABC$; see Fig. 1.

In this note we show that Lucas’ configuration has more geometric peculiarities. We find the following result:
2 Result

Theorem. Let $ABC$ be a triangle and let $X_1X_2Y_3Z_4, Y_1Y_2Z_3X_4, Z_1Z_2X_3Y_4$ be its three inscribed squares forming Lucas’ configuration. Then:

a) The vertices $X_1, X_2, Y_1, Y_2, Z_1, Z_2$ are on a conic.

b) The vertices $Y_3, Z_4, Z_3, X_4, X_3, Y_4$ are on an ellipse.

See Figs. 2, 3 and 4.

To prove the result we will concentrate our efforts on finding the equations of the conic.

We point out that, in the case of the three squares $X_1'X_2'Z_3'Y_4', Y_1'Y_2'X_3'Z_4', Z_1'Z_2'Y_3'X_4'$, which form the Lucas’s internal configuration, the result is the same, but the vertices $Z_3', Y_4', X_3', Z_4', Y_3', X_4'$ are on a conic which is not necessarily an ellipse.
Proof. To prove the result, let \(ABC\) be the triangle; we may assume that \(AB\) is the longest side, and we can consider a Cartesian system of coordinates such that

\[ A = (0, 0), \quad B = (1, 0), \quad C = (a, b) \quad \text{with} \quad a \in (0, 1], \quad b \in (0, 1]. \]

In this system, after a calculation we have:

\[ X_1 = \Gamma(a, 0), \quad X_2 = \Gamma(a + b, 0), \quad Y_3 = \Gamma(a + b, b), \quad Z_4 = \Gamma(a, b), \]
\[ Y_1 = \Delta(b + 1, -a + 1), \quad Y_2 = \Delta(a + b, -a + b + 1), \quad Z_3 = \Delta(a, b), \quad X_4 = \Delta(1, 0), \]
\[ Z_1 = \Lambda(a^2 + ab, ab + b^2), \quad Z_2 = \Lambda(a^2, ab), \quad X_3 = \Lambda(a^2 + b^2, 0), \]
\[ Y_4 = \Lambda(a^2 + b^2 + ab, b^2) \]

where \(\Gamma, \Delta, \Lambda\) have positive values:

\[ \Gamma = \frac{1}{b + 1}, \quad \Delta = \frac{b}{a^2 + b^2 - 2a + b + 1}, \quad \Lambda = \frac{1}{a^2 + b^2 + b}. \]

Then, with a long but straightforward calculation, we find that the points \(X_1, X_2, Y_1, Y_2, Z_1, Z_2\) verify the following equation

\[ Ax^2 + By^2 + Cxy + Dx + Ey + F = 0 \]

with

\[ A = b^2(b + 1)^2, \]
\[ B = -3a^4 - a^2b^2 + b^4 + 6a^3 + ab^2 + 2b^3 - 3a^2 + b^2, \]
\[ C = b(2a - 1)(2a^2 + b^2 - 2a), \]
\[ D = -b^2(b + 1)(2a + b), \]
\[ E = -b(2a + b)(a^2 + b^2 - ab - a + b), \]
\[ F = ab^2(a + b). \]
Also, with another long but straightforward calculation, we find that the points \( Y_3, Z_4, Z_3, X_4, X_3, Y_4 \) verify the following equation

\[
Ax^2 + By^2 + Cxy + Dx + Ey + F = 0
\]

with

\[
A = b^2(a^2 + b^2 + b)(a^2 + b^2 - 2a + b + 1),
\]

\[
B = a^6 + 2a^4b^2 + a^2b^4 - 3a^5 + b^5 + 3a^4b - 4a^3b^2 + 4a^2b^3 - ab^4 + 4a^4 + 2b^4
\]

\[
- 6a^3b + 5a^2b^2 - 4ab^3 - 3a^3 + 2b^3 + 3a^2b - 3ab^2 + a^2 + b^2,
\]

\[
C = -b(2a - 1)(a^2 + b^2 - a + b)^2,
\]

\[
D = -b^2(a^4 + b^4 + 2a^2b^2 - 2a^3 + 2a^2b - 2ab^2 + 2b^3 + a^2 + 2b^2),
\]

\[
E = b(a^5 + 2a^3b^2 + ab^4 - 3a^4 - 2b^4 + 2a^3b - 5a^2b^2 + 2ab^3 + 3a^3 - 2a^2b
\]

\[
+ 4ab^2 - 2b^3 - a^2 - b^2),
\]

\[
F = b^3(a^2 + b^2).
\]

Now that we have the previous equations, we can easily check that the first one corresponds to a conic which is not necessarily an ellipse, whereas the second one necessarily corresponds to an ellipse.

\[\square\]

**Remark.** If instead of considering the three inscribed squares we consider the three inscribed equilateral triangles each with a side parallel to a side of \( ABC \), then we can find similar results; see some of them in [2].

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**References**


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