On a result of James and Niven concerning unique factorization in congruence semigroups

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The theory of non-unique factorizations in integral domains and monoids is a very active area of current research (see both [1] and [4] to view recent trends in this work). To demonstrate the phenomena of non-unique factorizations, we consider a result from the classical setting on uniqueness of factorizations by James and Niven [11]. We proceed as follows: Let \( \mathbb{N} \) represent the natural numbers and suppose that \( M \subseteq \mathbb{N} \) is a multiplicative semigroup. \( M \) is called a congruence semigroup if there exists a natural number \( n \) such...

Im Hilbertschen Monoid \( 1+4\mathbb{N}_0 = \{1, 5, 9, 13, \ldots\} \) (\( \mathbb{N}_0 = \) natürliche Zahlen inklusive Null) ist die Zerlegung in irreduzible Faktoren nicht eindeutig: Es gilt zum Beispiel \( 441 = 9 \cdot 49 = 21 \cdot 21 \). Hilberts Monoid ist ein Beispiel einer Kongruenz-Halbgruppe. Ein klassisches Resultat von James und Niven besagt, dass in einer Kongruenz-Halbgruppe \( M \) genau dann der Fundamentalsatz der Arithmetik gilt, wenn \( M \) aus allen Zahlen besteht, die relativ prim zu einer festen Zahl \( n \in \mathbb{N} \) sind. Die Autoren der vorliegenden Arbeit untersuchen das andere Extrem, nämlich den Fall, wo \( M \) aus allen Zahlen besteht, die nicht relativ prim zu einer festen Zahl \( n \in \mathbb{N} \) sind. Sie zeigen, dass in diesem Fall wenigstens die Anzahl der Primfaktoren bei der Zerlegung einer Zahl eindeutig ist.
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that
\[ x \in M \text{ and } x \equiv y \pmod{n} \text{ for } y \in \mathbb{N} \text{ implies } y \in M. \]

If \( M \) is as above, then we call \( n \) a modulus of definition of \( M \). It follows directly from the definition that a congruence semigroup \( M \) of modulus \( n \) is completely determined by \( n \) and \( M \cap \{1, 2, \ldots, n\} \). In a congruence semigroup \( M \), we call an element \( x \) irreducible if \( x \) cannot be written in the form \( yz \) where \( y \) and \( z \) are nonunits of \( M \) (note that \( M \) possesses at most one unit, that being 1). The classic proof that all natural numbers can be factored as a product of primes can be easily modified to show that each nonunit of a congruence semigroup can be factored as a product of irreducible elements. In general, such a semigroup is called atomic. The interested reader can find more information on congruence semigroups in [8] and a review of basic algebra terminology in [10].

Examples of congruence semigroups can be found throughout the mathematical literature. In particular, Davenport [7, p. 21] uses the “Hilbert monoid”
\[ 1 + 4\mathbb{N}_0 = \{1, 5, 9, 13, 17, 21, \ldots\} \]
as an example of a multiplicative system where the Fundamental Theorem of Arithmetic fails. To be precise, in this system,
\[ 441 = 21 \cdot 21 = 9 \cdot 49 \]
and 9, 21 and 49 are all nonassociated irreducibles in \( 1 + 4\mathbb{N}_0 \).

Hence, it is reasonable to ask which congruence semigroups do satisfy the Fundamental Theorem of Arithmetic. This question was answered by James and Niven in [11], where they prove the following interesting result. We will require the following notation: if \( n \in \mathbb{N} \), then set
\[ A(n) = \{m \mid m \in \mathbb{N} \text{ and } \gcd(m, n) = 1\} \]
and \( B(n) = \mathbb{N} - A(n) \).

**Theorem (James and Niven [11]).** Let \( M \) be a congruence semigroup. \( M \) has unique factorization of elements into products of irreducible elements if and only if there exists a positive integer \( n \) with \( M \cap A(n) = A(n) \) and \( M \cap B(n) = \emptyset \). In other words, \( M \) has unique factorization if and only if \( M \) consists of all elements relatively prime to a fixed positive integer \( n \).

An alternate proof of this theorem due to Halter-Koch (which uses the divisor theory of a commutative cancellative monoid) can be found in [9]. As a byproduct of the theorem, we point out that the modulus for a congruence semigroup is not unique. Notice that letting \( n = 2 \) or 4 in the theorem produces the same semigroup. Hence, this \( M \) can be viewed with modulus of definition 2 or 4. While the modulus is not unique, it is obvious that each congruence semigroup has a unique minimal modulus.

We are struck by what happens in the other extreme suggested by the theorem (i.e., when \( M \) consists of all elements not relatively prime to a fixed positive integer \( n \)). It turns out that such an \( M \) also exhibits an interesting factorization property.
Proposition. Let \( n = p_1^{a_1} \cdots p_k^{a_k} \) be a positive integer where the \( p_i \)'s are distinct primes and the \( n_i \)'s positive integers. Set
\[
M = \{m \in \mathbb{N} \mid \gcd(m, n) \neq 1\}.
\]

\( M \) is a congruence semigroup with minimal modulus \( n' = p_1 \cdots p_k \) which satisfies the following factorization property: If \( x \in M \) and
\[
x = y_1 \cdots y_s = z_1 \cdots z_t
\]
where each \( y_i \) and \( z_j \) is irreducible in \( M \), then \( s = t \).

Proof. Since the product of two numbers not relatively prime to \( n \) is again not relatively prime to \( n', M \) is closed under multiplication and is a multiplicative semigroup. It follows directly from the hypothesis of the proposition and elementary number theory that \( M \) is a congruence semigroup of modulus \( n \). We show that \( M \) also has modulus \( n' = p_1 \cdots p_k \).

Setting
\[
M' = \{m \in \mathbb{N} \mid \gcd(m, n') \neq 1\}
\]
we obtain, as above, that \( M' \) is a congruence monoid of modulus \( n' \). For \( m \in \mathbb{N} \) it follows that \( \gcd(m, n) \neq 1 \) if and only if \( \gcd(m, n') \neq 1 \). Hence \( M = M' \) and \( n' \) is a modulus of definition for \( M \). We argue that this is the minimal modulus. Suppose \( M \) is defined by some modulus \( d < n' \). Then there exists an \( i \) such that \( p_i \nmid d \). Now, by definition \( p_i \in M \), but note that \( p_i^{\varphi(d)} \equiv 1 \pmod{d} \) (where \( \varphi \) represents the Euler \( \varphi \)-function), and hence \( 1 \in M \), a contradiction.

We now show that \( M \) satisfies \((*)\). By the definition of \( M \), if \( x \in M \), then \( x = p_1^{a_1} \cdots p_k^{a_k} \) where the \( a_i \)'s are nonnegative integers (with at least one nonzero) and \( w \in \mathbb{N} \) with \( \gcd(w, n') = 1 \). Define a function \( f : M \to \mathbb{N} \) by
\[
f(x) = \sum_{i=1}^{k} a_i.
\]
It is easy to verify that for \( x \) and \( y \in M \) we have \( f(xy) = f(x) + f(y) \).

Claim: \( x \in M \) is irreducible in \( M \) if and only if \( f(x) = 1 \).

Proof of Claim: \((\Rightarrow)\) Suppose that \( x \in M \) and \( f(x) > 1 \). Write \( x = pqk \) where \( p \) and \( q \) are not necessarily distinct primes which divide \( n' \) and \( k \in \mathbb{N} \). By definition, \( p, q \) and \( qk \) are in \( M \). Hence \( pqk = (p)(qk) \) and thus is not irreducible in \( M \).

\((\Leftarrow)\) Suppose \( x = pk \) where \( p \) is a prime divisor of \( n' \) and \( \gcd(k, n') = 1 \). If \( x = yz \) where \( y \) and \( z \in M \), then \( 1 = f(x) = f(yz) = f(y) + f(z) \), which implies that either \( f(y) \) or \( f(z) = 0 \), a contradiction.

Now, suppose that \( x \in M \) and
\[
x = y_1 \cdots y_s = z_1 \cdots z_t
\]
where each \( y_i \) and \( z_j \) is irreducible in \( M \). Then \( f(x) = \sum_{i=1}^{s} f(y_i) = \sum_{i=1}^{t} f(z_i) \). Since each \( f(y_i) = 1 = f(z_i) \), we have that \( f(x) = s = t \) and the result follows. \( \square \)
We close with some comments concerning the proposition.

(1) An atomic semigroup (or monoid) which satisfies (\(\ast\)) is called half-factorial. More information on the half-factorial property can be found in [5].

(2) The James and Niven result indicates that the set of odd integers, when viewed as a semigroup, has unique factorization. On the other hand, the proposition indicates that the set of even integers is half-factorial. As the proof indicates, a non-unique factorization in the set of even integers is given by

\[ 6 \cdot 10 = 2 \cdot 30, \]

where 2, 6, 10 and 30 are all irreducible as even integers.

(3) Not all half-factorial congruence semigroups are of the form \(M\) in the proposition. The Hilbert Monoid, \(H = 1 + 4\mathbb{N}_0\) is also half-factorial. To see this, notice that \(x \in H\) is irreducible if and only if

(i) \(x\) is prime in \(\mathbb{N}\), or

(ii) \(x = q_1q_2\) where \(q_1\) and \(q_2\) are not necessarily distinct primes in \(\mathbb{N}\) which are congruent to 3 (mod 4).

Hence, if \(x \in H\) is of the form

\[ x = p_1 \cdots p_s q_1 \cdots q_t, \]

where each \(p_i\) is a prime congruent to 1 (mod 4) and each \(q_j\) is a prime congruent to 3 (mod 4), then any irreducible factorization of \(x\) in \(H\) has length \(s + \frac{t}{2}\) (note that \(t\) will necessarily be even). Half-factorial congruence semigroups which are also arithmetic sequences have been characterized in [3, Theorem 2.6].

(4) To exhibit a congruence semigroup which is neither factorial nor half-factorial, let \(M = 1 + 5\mathbb{N}_0\). In \(M\) we have

\[ 81 \cdot 2401 = 21 \cdot 21 \cdot 21 \cdot 21 \]

and each of 81, 2401 and 21 are irreducible in \(M\). A good general reference on monoids which do not satisfy the unique factorization property is [6].

(5) The function \(f\) in the proof of the proposition is known as a semi-length function on \(M\). The reader can find more information on semi-length functions in [2].

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