A short note on the Erdös-Debrunner inequality

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Introduction

Let $ABC$ be an arbitrary triangle and $D, E, F$ arbitrary points on sides $BC, CA, AB$, resp., all three being different from the vertices of $ABC$.

Then, triangle $ABC$ is divided into four smaller triangles, a central one $DEF$, and three corner ones $AEF, BDF, CED$.

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Let $F_1, F_2, F_3$ be the areas of the three corner triangles and $F_0$ be the one of the central triangle. Then the Erdős-Debrunner inequality says

$$F_0 \geq \min(F_1, F_2, F_3),$$

where equality occurs if and only if $D, E, F$ are the midpoints of the respective sides. See [1, p. 81] for an extensive list of references concerning this inequality; furthermore, the appropriate chapters in [3] and [4] report a host of results (and some of their proofs) related to two triangles, one inscribed in the other.

Speaking in the language of power-means, inequality (1) reads

$$F_0 \geq M_{-\infty}(F_1, F_2, F_3),$$

where the $p$-th power-mean of three positive real numbers $x, y, z$ is defined by

$$M_p(x, y, z) = \begin{cases} 
\left( \frac{x^p + y^p + z^p}{3} \right)^{1/p} & p \neq 0, \\
\sqrt[3]{xyz} & p = 0.
\end{cases}$$

Then, $M_p(x, y, z)$ is (weakly) increasing as $p$ increases, and

$$M_{-\infty}(x, y, z) = \lim_{p \to -\infty} M_p(x, y, z) = \min(x, y, z).$$

Therefore, it is natural to ask whether or not there do exist inequalities of the type

$$F_0 \geq M_p(F_1, F_2, F_3),$$

where $p > -\infty$.

Subsequently, we will show that this is indeed so and we will give a bound for the maximum value $p_{\text{max}}$ of $p$. Thereby, we will also falsify a result stated and “proven” in [2]. At the end of this note, we shall state two conjectures for further research.

**Bounds for $p_{\text{max}}$**

Before stating the announced result, we are going to introduce the method of proof frequently applied in situations as the present one.
Let $BC, CA, AB$ be divided by $D, E, F$ in ratios $t : (1 - t), u : (1 - u), v : (1 - v)$, resp., where $0 < t, u, v < 1$. Then, we have

$$F_1 = (1 - u) \cdot v \cdot F_\Delta, \quad F_2 = (1 - v) \cdot t \cdot F_\Delta, \quad F_3 = (1 - t) \cdot u \cdot F_\Delta,$$

where $F_\Delta$ denotes the area of triangle $ABC$. For this, note for instance for $F_1$: $AF = v \cdot AB$, and $AE = (1 - u) \cdot AC$. Therefore, $F_0 = F_\Delta - F_1 - F_2 - F_3$ becomes

$$F_0 = (t \cdot u \cdot v + (1 - t) \cdot (1 - u) \cdot (1 - v)) \cdot F_\Delta.$$

Furthermore,

$$\frac{F_0}{F_1} = \frac{1 - t - u - v + tu + tv + uv}{(1 - u)v} = \frac{1 - t}{v} + \frac{t}{1 - u} - 1.$$

Since we get similar expressions for $F_0/F_2$ and $F_0/F_3$, we introduce the notation

$$x = \frac{t}{1 - u}, \quad y = \frac{u}{1 - v}, \quad z = \frac{v}{1 - t},$$

yielding

$$\frac{F_0}{F_1} = \frac{1}{z} + x - 1, \quad \frac{F_0}{F_2} = \frac{1}{x} + y - 1, \quad \frac{F_0}{F_3} = \frac{1}{y} + z - 1.$$

We now show that $p$ has to be negative for inequality (2) to hold in general. Indeed, let $p = 0$. Then, for (2) the inequality $F_0/F_1 \cdot F_0/F_2 \cdot F_0/F_3 \geq 1$ had to be valid. But $t = 1/2$, $u = 1/3$ and $v = 2/3$ lead to the contradiction $8/9 \geq 1$.

Therefore, we let $p = -q$, where $q > 0$, and thus, obtain for (2) the equivalent inequality

$$F_1^{-q} + F_2^{-q} + F_3^{-q} \geq 3 \cdot F_0^{-q},$$

i.e.,

$$\left(\frac{F_0}{F_1}\right)^q + \left(\frac{F_0}{F_2}\right)^q + \left(\frac{F_0}{F_3}\right)^q \geq 3,$$

hence,

$$\left(\frac{1}{z} + x - 1\right)^q + \left(\frac{1}{x} + y - 1\right)^q + \left(\frac{1}{y} + z - 1\right)^q \geq 3,$$

where of course $x, y, z > 0$ have to satisfy

$$\frac{1}{z} + x - 1 \geq 0, \quad \frac{1}{x} + y - 1 \geq 0, \quad \frac{1}{y} + z - 1 \geq 0.$$

We are now in the position to state and prove the following

**Theorem.** The quantity $p_{\text{max}}$ in inequality (2) satisfies

$$-1 \leq p_{\text{max}} \leq -\frac{\ln(3/2)}{\ln(2)}.$$
Proof. In order to prove this assertion, we have to show that the minimal value $q_{\text{min}}$ such that inequality (3) holds true in general, fulfils $\ln\left(\frac{3}{2}\right)/\ln(2) \leq q_{\text{min}} \leq 1$.

i) Case $q_{\text{min}} \leq 1$: Indeed, inequality (3) becomes for $q = 1$

$$x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} \geq 6.$$ 

But this inequality follows from $t + 1/t \geq 2$, whenever $t > 0$.

ii) Case $q_{\text{min}} \geq \ln\left(\frac{3}{2}\right)/\ln(2)$: We let $t = 1/2$, and $v = 1 - u$ ($0 < u < 1$). Then, we find

$$\frac{F_0}{F_1} = \frac{u}{1-u}, \quad \frac{F_0}{F_2} = \frac{F_0}{F_3} = 2(1-u),$$

whence inequality (3) reads

$$\left(\frac{u}{1-u}\right)^q + 2 \cdot (2(1-u))^q \geq 3$$

with $0 < u < 1$. Since the expression on the left-hand side of this inequality is continuous as $u \to 0$, we arrive at $2 \cdot 2^q \geq 3$, which completes the proof of the theorem. $\Box$

Remark. In [2] it is "shown" by an erroneous argument that $p_{\text{max}}$ equals $-1/3$ contradicting the inequality $p_{\text{max}} \leq -\ln(3/2)/\ln(2) = -0.58\ldots$

Two conjectures

At the end of this note we state two conjectures. (The second of them is very likely to be settled by non elementary means only.)

Conjecture 1. Let $x$, $y$ and $z$ be positive real numbers such that $1/z + x - 1 \geq 0$, $1/x + y - 1 \geq 0$ and $1/y + z - 1 \geq 0$. Then, for any $q > 0$ the minimum of the left-hand expression in (3) is attained at $x$, $y$ and $z$ satisfying $x \cdot y \cdot z = 1$.

Conjecture 2. In the above theorem, the equality $p_{\text{max}} = -\ln(3/2)/\ln(2)$ holds true.

References


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