A short proof of the formula of Faà di Bruno

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While Leibniz’ formula $(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$ for the higher-order derivatives of the product of two functions is common mathematical knowledge, its analogue for the composition of two functions is much less well known.

**Formula of Faà di Bruno.** If $f$ and $g$ possess derivatives up to order $n$, then

$$(f \circ g)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{k_1!k_2!\cdots k_n!} \left( f^{(k_1)} \circ g \right) \left( g' \right)^{k_1} \frac{g''}{2!}^{k_2} \cdots \left( g^{(n)} \right)^{k_n}.$$

The formula is due to Francesco Faà di Bruno (see [1]) who lived from 1825 to 1888 and enjoys the rare (at least for mathematicians) distinction of being a Saint of the Catholic church (canonization in 1988 by Pope John Paul II). A proof using basic umbral calculus...
was given by Steven Roman in [3] where also references to other approaches can be found; a derivation using Hirzebruch’s m-sequences is given in [4]. In this paper we present a completely elementary (and extremely short) proof which requires almost no prerequisites and allows the formula of Faa di Bruno to be incorporated into undergraduate calculus courses. (Some uses of the formula are given in [2].)

**Proof.** A trivial induction shows that there are polynomials $P_{n,k}$ (where $n$ is the number of variables of $P_{n,k}$) such that

$$(f \circ g)^{(n)} = \sum_{k=0}^{n} (f^{(k)} \circ g) \cdot P_{n,k}(g', g'', \ldots, g^{(n)}) \quad (\ast)$$

for all $f$ and $g$. In fact, the induction shows that these polynomials are recursively given by $P_{0,0}(x) = 1$ and $P_{n+1,k}(x_1, \ldots, x_n, x_{n+1}) = x_1 \cdot P_{n,k-1}(x_1, \ldots, x_n) + \sum_{i=k}^{n} x_i+1 \cdot (\partial_i P_{n,k})(x_1, \ldots, x_n)$, if we interpret $P_{n,0}$ and $P_{n,n+1}$ as zero, but this is irrelevant for our argument. What is important to realize from (\ast) is that $(f \circ g)^{(n)}(x_0)$ depends only on the values $g^{(k)}(x_0)$ and $f^{(k)}(g(x_0))$ where $0 \leq k \leq n$; hence to establish the validity of the formula at any given point $x_0$, we may replace the given functions $f$ and $g$ with any functions $F$ and $G$ which have the same derivatives up to order $n$ as $f$ and $g$ at $g(x_0)$ and $x_0$, respectively. Hence, it suffices to prove the formula of Faa di Bruno for polynomials! Assuming $x_0 = 0$ and $g(x_0) = 0$ without loss of generality, we may thus write $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ and $g(x) = b_1 x + b_2 x^2 + \cdots + b_n x^n$ where $a_k = f^{(k)}(0)/k!$ and $b_k = g^{(k)}(0)/k!$ for all $k$. In this case the formula to be proved reduces to the claim that the coefficient of $x^n$ in the expansion of $f(g(x))$ is

$$\sum_{k=0}^{n} \sum_{k_1 + \cdots + k_n = k} \frac{k!}{k_1! \cdots k_n!} a_kb_1^{k_1} b_2^{k_2} \cdots b_n^{k_n}.$$ 

But this is trivial! In fact, applying the multinomial formula

$$(X_1 + \cdots + X_n)^k = \sum_{k_1 + \cdots + k_n = k} \frac{k!}{k_1! \cdots k_n!} X_1^{k_1} X_2^{k_2} \cdots X_n^{k_n}$$

with $X_k := b_k x^k$, we find

$$f(g(x)) = \sum_{k=0}^{n} a_k (b_1 x + b_2 x^2 + \cdots + b_n x^n)^k = \sum_{k=0}^{n} a_k \sum_{k_1 + \cdots + k_n = k} \frac{k!}{k_1! \cdots k_n!} b_1^{k_1} b_2^{k_2} \cdots b_n^{k_n} x^{k_1+2k_2+\cdots+nk_n}.$$
References


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