Applications of Hofer’s geometry to Hamiltonian dynamics

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Abstract. We prove that for every subset $A$ of a tame symplectic manifold $(W, \omega)$ meeting a semi-positivity condition, the $\pi_1$-sensitive Hofer–Zehnder capacity of $A$ is not greater than four times the stable displacement energy of $A$,

$$c_{HZ}^i(A, W) \leq 4e(A \times S^1, W \times T^*S^1).$$

This estimate yields almost existence of periodic orbits near stably displaceable energy levels of time-independent Hamiltonian systems. Our main applications are:

- The Weinstein conjecture holds true for every stably displaceable hypersurface of contact type in $(W, \omega)$.
- The flow describing the motion of a charge on a closed Riemannian manifold subject to a non-vanishing magnetic field and a conservative force field has contractible periodic orbits at almost all sufficiently small energies.

The proof of the above energy-capacity inequality combines a curve shortening procedure in Hofer geometry with the following detection mechanism for periodic orbits: If the ray $\{\phi^t_F\}$, $t \geq 0$, of Hamiltonian diffeomorphisms generated by a compactly supported time-independent Hamiltonian stops to be a minimal geodesic in its homotopy class, then a non-constant contractible periodic orbit must appear.

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1. Introduction and results

On their search for periodic orbits of autonomous Hamiltonian systems, Hofer and Zehnder ([27], [28]) associated to every open subset $A$ of a symplectic manifold $(V, \omega)$ a number, the Hofer–Zehnder capacity $c_{HZ}(A) \in [0, \infty]$, in such a way that $c_{HZ}(A) < \infty$ implies almost existence of periodic orbits near any compact regular

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energy level of an autonomous Hamiltonian system on $A$. Showing that $c_{HZ}(A)$ is finite is, however, often a difficult problem. Our main result is that if a subset $A$ of a tame symplectic manifold meeting a suitable semi-positivity condition can be displaced from itself by a Hamiltonian isotopy in a stabilized sense, then the Hofer–Zehnder capacity of $A$ is indeed finite.

In order to set notations, we abbreviate $I = [0, 1]$ and consider an arbitrary smooth symplectic manifold $(V, \omega)$ without boundary. Denote by $\mathcal{H}_c(I \times V)$ the set of smooth functions $I \times V \to \mathbb{R}$ with compact support. The Hamiltonian vector field of $H \in \mathcal{H}_c(I \times V)$, defined by

$$\omega(X_{H_t}, \cdot) = -dH_t(\cdot),$$

generates a flow $h_t$. The time-1-maps $h$ form the group

$$\text{Ham}_c(V, \omega) := \{ h \mid H \in \mathcal{H}_c(I \times V) \}$$

defined as follows. We say that $F \in \mathcal{H}_c(V)$ is slow if all non-constant contractible periodic orbits of $f_t$ have period greater than 1. Following [27], [28] and [38], [53], [17] we define for each subset $A$ of $(V, \omega)$ the $\pi_1$-sensitive Hofer–Zehnder capacity

$$c_{HZ}^1(A, V, \omega) = \sup \{ \max F - \min F \mid F \in \mathcal{H}_c(\text{Int}(A)) \text{ is slow} \}. \quad (1)$$

We shall often suppress $\omega$ from the notation, and we shall write $c_{HZ}^1(V)$ instead of $c_{HZ}^1(V, V)$. The Hofer–Zehnder capacity $c_{HZ}(A)$ mentioned above is obtained by taking the supremum over the smaller class of functions $F \in \mathcal{H}_c(\text{Int}(A))$ for which all non-constant periodic orbits of $f_t$ have period $> 1$. Therefore, $c_{HZ}(A) \leq c_{HZ}^1(A, V)$.

The Hofer–Zehnder capacity we shall study is defined as follows. We say that $F \in \mathcal{H}_c(V)$ is slow if all non-constant contractible periodic orbits of $f_t$ have period greater than 1. Following [27], [28] and [38], [53], [17] we define for each subset $A$ of $(V, \omega)$ the $\pi_1$-sensitive Hofer–Zehnder capacity

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We shall compare the Hofer–Zehnder capacity $c_{HZ}^1(A, V)$ with the displacement energy defined in [21], [32]. The norm $\|H\|$ of $H \in \mathcal{H}_c(I \times V)$ is defined as

$$\|H\| = \int_0^1 \left( \max_{x \in V} H(t, x) - \min_{x \in V} H(t, x) \right) dt,$$

and the displacement energy $e(A, V) = e(A, V, \omega) \in [0, \infty]$ of a subset $A$ of $V$ is defined as

$$e(A, V) = \inf \{ \|H\| \mid H \in \mathcal{H}_c(I \times V), \ h(A) \cap A = \emptyset \}$$

if $A$ is compact and as

$$e(A, V) = \sup \{ e(K, V) \mid K \subset A \text{ is compact} \}$$
for a general subset $A$ of $V$. In fact, we shall compare $c_{HZ}^i(A, V)$ with the **stable displacement energy** defined as

$$e_s(A, V) := e(A \times S^1, V \times T^*S^1, \omega \oplus \omega_0)$$

where $\omega_0 = dp \wedge dq$ is the standard symplectic form on $T^*S^1$. We are able to do this for the following class of symplectic manifolds.

**Definition** ([20], [56], [2]). A symplectic manifold $(W, \omega)$ is **tame** if $W$ admits an almost complex structure $J$ and a complete Riemannian metric $g$ such that

- $J$ is uniformly tame, i.e., there are positive constant $C_1$ and $C_2$ such that
  $$\omega(X, JX) \geq C_1 \|X\|^2 \quad \text{and} \quad |\omega(X, Y)| \leq C_2 \|X\| \|Y\|$$
  for all $X, Y \in TW$.
- The sectional curvature of $(W, g)$ is bounded from above and the injectivity radius of $(W, g)$ is bounded away from zero.

Examples of tame symplectic manifolds are closed symplectic manifolds, standard cotangent bundles $(T^*M, \omega_0)$ as well as twisted cotangent bundles $(T^*M, \omega_\sigma)$ over a closed base $M$, and symplectic manifolds which at infinity are isomorphic to the symplectization of a closed contact manifold. The class of tame symplectic manifolds is closed under taking products or coverings.

For technical reasons we also impose a semi-positivity condition on $(W, \omega)$. The first Chern class $c_1 \in H^2(W; \mathbb{Z})$ is defined as the first Chern class of the complex vector bundle $(TW, J)$, where $J$ is any almost complex structure such that $\omega(\cdot, J\cdot)$ is a Riemannian metric. Recall from [43], [23], [54], [44] that a $2n$-dimensional symplectic manifold $(W, \omega)$ is **strongly semi-positive** if for all $A \in \pi_2(W)$,

$$\omega(A) > 0, \quad c_1(A) \geq 2 - n \implies c_1(A) \geq 0.$$  

**Definition.** A $2n$-dimensional symplectic manifold $(W, \omega)$ is **stably strongly semi-positive** if for all $A \in \pi_2(W)$,

$$\omega(A) > 0, \quad c_1(A) \geq 1 - n \implies c_1(A) \geq 0.$$  

Equivalently, $(W, \omega)$ satisfies one of the following conditions.

(i) $\omega(A) = \lambda c_1(A)$ for every $A \in \pi_2(W)$ and some $\lambda \geq 0$;  

(ii) $c_1(A) = 0$ for every $A \in \pi_2(W)$;  

(iii) The minimal Chern number $N \geq 0$ defined by $c_1(\pi_2(W)) = N\mathbb{Z}$ is at least $n$.  

Since \((T^*S^1, \omega_0)\) is exact and has vanishing first Chern class, \((W, \omega)\) is stably strongly semi-positive if and only if \((W \times T^*S^1, \omega \oplus \omega_0)\) is strongly semi-positive. This assumption guarantees that the evaluation map used in the definition of the Gromov–Witten invariants relevant for our arguments is a pseudo-cycle. If one is willing to use Liu–Tian’s construction of the \(S^1\)-invariant virtual moduli cycle, this assumption can be dropped throughout the paper.

Our main result is the following energy-capacity inequality.

**Theorem 1.1.** Assume that \(A\) is a subset of a tame and stably strongly semi-positive symplectic manifold \((W, \omega)\). Then

\[
c^{\circ}_{HZ}(A, W) \leq 4e_S(A, W).
\]

We shall derive Theorem 1.1 from the following result by capitalizing on the fact that the definition of \(c^{\circ}_{HZ}\) involves only contractible periodic orbits and by using a stabilization trick found in Macarini’s work [41].

**Theorem 1.2.** Assume that \(A\) is a subset of a tame and strongly semi-positive symplectic manifold \((W, \omega)\). Then

\[
c^{\circ}_{HZ}(A, W) \leq 4e(A, W).
\]

Up to its slightly more restrictive hypothesis, Theorem 1.1 is stronger than Theorem 1.2. Indeed, it is elementary to see that \(e_S(A, V) \leq e(A, V)\) in general, and in the dynamically relevant Example 1.5 below we have \(e_S(A, V) < e(A, V) = \infty\).

The energy-capacity inequality

\[
c^{\circ}_{HZ}(A, V) \leq e(A, V)
\]

is known for every subset \(A\) of a weakly exact symplectic manifold \((V, \omega)\) which is closed or convex ([22], [53], [12], [16], [11]). For the open ball \(B^{2n}(r)\) of radius \(r\) in \((\mathbb{R}^{2n}, \omega_0)\) it holds that

\[
c^{\circ}_{HZ}(B^{2n}(r), \mathbb{R}^{2n}) = e\left(B^{2n}(r), \mathbb{R}^{2n}\right) = \pi r^2,
\]

see [28], and so (2) is sharp. It is conceivable that the factor 4 in Theorems 1.1 and 1.2 can be omitted.

Following Polterovich [50] we shall obtain Theorem 1.2 by combining an elementary curve shortening technique in Hofer’s geometry with the following detection mechanism for periodic orbits.

**Theorem 1.3.** Assume that \((W, \omega)\) is a tame and strongly semi-positive symplectic manifold, and that the autonomous Hamiltonian \(F \in \mathcal{H}_c(W)\) is slow. Then the path \(f_t, t \in [0, 1]\), is length minimizing in its homotopy class.
Here, the length of $f_i$ is defined as $||F||$. This result was discovered by Hofer [22] for $(\mathbb{R}^{2n}, \omega_0)$ and has been proved in [34] for weakly exact tame symplectic manifolds; it removes an additional assumption on $F$ in [9], [44] and verifies Conjecture 1.2 in [44] for tame strongly semi-positive symplectic manifolds.

Theorems 1.1 and 1.2 show that if $e_S(A, W)$ or $e(A, W)$ is finite, then so is $c^0_HZ(A, W)$, and the finiteness of $c^0_HZ(A, W)$ implies existence of contractible periodic orbits on almost every compact regular energy level of an autonomous Hamiltonian system on $A$. We thus want to understand which compact subsets of a symplectic manifold $V$ have finite (stable) displacement energy. Every compact subset of a symplectic manifold of the form $(V \times \mathbb{R}^2, \omega \oplus \omega_0)$ has finite displacement energy. Less obvious sufficient assumptions on $A$ alone are collected in the following proposition essentially due to Laudenbach [35] and to Polterovich [49] and Laudenbach–Sikorav [36]. Recall that a middle-dimensional submanifold $L$ of a symplectic manifold $(V, \omega)$ is called Lagrangian if $\omega|_L = 0$.

**Proposition 1.4.** Let $A$ be a compact subset of a $2n$-dimensional symplectic manifold $(V, \omega)$.

(i) If $A$ is contained in an embedded finite CW-complex $X$ of dimension $< n$, then $e_S(A, V) < \infty$.

(ii) If $A$ is contained in an $n$-dimensional closed submanifold $M$ which is not Lagrangian, then $e_S(A, V) = 0$.

(iii) If $A$ is strictly contained in a closed Lagrangian submanifold $L$, then $e_S(A, V) = 0$.

The example $S^1 \subset (T^*S^1, \omega_0)$ shows that neither the dimension assumption in (i) nor the assumption $\omega|_M \neq 0$ in (ii) nor the assumption $A \subset L$ in (iii) can be omitted.

The following example will play an important role in our applications.

**Example 1.5.** Let $\sigma$ be a non-vanishing closed 2-form on a closed manifold $M$ and let $\omega_\sigma = \omega_0 + \pi^*\sigma$ be the twisted symplectic form on its cotangent bundle $\pi : T^*M \to M$. Then $e_S(M, T^*M, \omega_\sigma) = 0$ by Proposition 1.4 (ii). Note that if the Euler characteristic $\chi(M)$ does not vanish, then $e(M, T^*M, \omega_\sigma) = \infty$.

Theorems 1.1 and 1.2 and Proposition 1.4, which are proved in the next section, have various applications to the existence problem of periodic orbits of time-independent Hamiltonian systems. Some of them are given in Section 3 below. Further such applications as well as an application to Lagrangian intersections can be found in [52].

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2. Proofs

2.1. Proof of Theorem 1.2. We follow Polterovich’s beautiful argument in [50, Section 9.A]. The proof consists of two steps.

Step 1. Curve shortening in Hofer’s geometry
Curve shortening in Hofer’s geometry was invented by Sikorav in [55] and further developed in [33, Proposition 2.2]. Here, we closely follow the proof of Theorem 8.3.A in [51], see also Theorem 3.3.A in [3].

We consider an arbitrary symplectic manifold \((V, \omega)\). Two Hamiltonians \(H, K \in \mathcal{H}_c(I \times V)\) are equivalent, \(H \sim K\), if \(h = k\) and the paths \(\{h_t\}, \{k_t\}, t \in [0, 1]\), are homotopic in \(\text{Ham}_c(V, \omega)\) with fixed end points. In other words, there exists a smooth family \(\{H^s\}, s \in [0, 1]\), in \(\mathcal{H}_c(I \times V)\) such that \(h_t^0 = h_t\) and \(h_t^1 = k_t\) for all \(t\) and \(h_t^s = h = k\) for all \(s\). The group of equivalence classes \(\mathcal{H}_c(I \times V)/\sim\) form the universal cover \(\tilde{\text{Ham}}_c(V, \omega)\) of \(\text{Ham}_c(V, \omega)\). We denote the lift of the Hofer norm to \(\tilde{\text{Ham}}_c(V, \omega)\) by

\[
\rho \left[ h_t \right] \equiv \rho[H] := \inf \left\{ \|K\| \mid K \sim H \right\}.
\]

Proposition 2.1. Consider a compact subset \(A\) of an arbitrary symplectic manifold \((V, \omega)\) such that \(e(A, V) < \infty\). If \(F: V \to \mathbb{R}\) is supported in \(A\) and \(\|F\| > 4e(A, V)\), then \(\rho[F] < \|F\|\).

Proof. Choose a path \(\{h_t\}, t \in [0, 1]\), in \(\text{Ham}_c(V, \omega)\) such that \(h(A) \cap A = \emptyset\) and

\[
\rho \left[ h_t \right] < \frac{1}{4} \|F\|. \tag{3}
\]

For \(t \in [0, 1]\) we decompose the path \(f_t\) as

\[
f_t = \left( f_{t/2} \circ h_t \circ f_{t/2} \circ h_t^{-1} \right) \circ \left( h_t \circ f_{t/2}^{-1} \circ h_t^{-1} \circ f_{t/2} \right) \equiv b_t \circ a_t.
\]

As we shall see below,

\[
\rho \left[ a_t \right] < \frac{1}{2} \|F\| \quad \text{and} \quad \rho \left[ b_t \right] \leq \frac{1}{2} \|F\|. \tag{4}
\]
Since \( \{ b_t \circ a_t \} \) is equivalent to the juxtaposition of \( \{ a_t \} \) and \( \{ b_t \circ a_1 \} \) and since \( \rho \) satisfies the triangle inequality, the estimates (4) imply Proposition 2.1. In order to prove the first estimate in (4), notice that the paths \( \{ f_{t/2}^{-1} \circ h_t^{-1} \circ f_{t/2} \} \) and \( \{ f_{t/2}^{-1} \circ h_t^{-1} \circ f_{t/2} \} \) are equivalent and that

\[
\rho \left[ f_{t/2}^{-1} \circ h_t^{-1} \circ f_{t/2} \right] = \rho \left[ h_t^{-1} \right] = \rho \left[ a_t \right].
\]

Together with the triangle inequality and the estimate (3) we can estimate

\[
\rho \left[ a_t \right] = \rho \left[ h_t \circ f_{t/2}^{-1} \circ h_t^{-1} \circ f_{t/2} \right] \\
\leq \rho \left[ h_t \right] + \rho \left[ f_{t/2}^{-1} \circ h_t^{-1} \circ f_{t/2} \right] \\
= 2 \rho \left[ h_t \right] \\
< \frac{1}{2} \| F \|.
\]

To prove the second estimate in (4), notice that the path \( \{ b_t \} = \{ f_{t/2} \circ h_t \circ f_{t/2} \circ h^{-1} \} \) is equivalent to the path \( \{ f_{t/2} \circ h \circ f_{t/2} \circ h^{-1} \} \) generated by the Hamiltonian

\[
K(t, x) = \frac{1}{2} F(x) + \frac{1}{2} F(h^{-1} f_{t/2} x), \quad t \in [0, 1].
\]

Since \( F \) is autonomous, \( F = F \circ f_{t/2} \), and since \( h \) displaces \( \text{supp} \ F \subset A \), so does \( h^{-1} \). Therefore,

\[
\| K_t \| = \frac{1}{2} \| F + F \circ h^{-1} \circ f_{t/2}^{-1} \| \\
= \frac{1}{2} \| F \circ f_{t/2} + F \circ h^{-1} \| \\
= \frac{1}{2} \| F + F \circ h^{-1} \| \\
= \frac{1}{2} \| F \|,
\]

and so \( \rho \left[ b_t \right] \leq \frac{1}{2} \| F \| \). The proof of Proposition 2.1 is complete. \( \square \)

**Step 2. The cut point has a non-constant contractible periodic orbit**

Consider an arbitrary symplectic manifold \( (V, \omega) \). We recall from the introduction that \( F \in H_c(V) \) is slow if all non-constant contractible periodic orbits of \( f_t \) have period \( > 1 \). We say that \( F \in H_c(V) \) is flat if all non-constant periodic orbits of the linearized flow of \( F \) at its critical points have period \( > 1 \).

**Lemma 2.2.** Assume that \( (W, \omega) \) is a tame strongly semi-positive symplectic manifold, and that the autonomous Hamiltonian \( F \in H_c(W) \) is slow and flat. Then the path \( f_t, t \in [0, 1] \), is length minimizing in its homotopy class.

**Proof.** If \( W \) is closed, this result is proved in [9], [44], see also [34]. If \( (W, \omega) \) is not closed but tame, then the compactness theorems in [20], [56] hold, and so the arguments in [44] establishing compactness of the relevant Floer moduli space go through. \( \square \)
Following a suggestion by Viktor Ginzburg, we derive Theorem 1.3 from Lemma 2.2 by elementary means:

**Proof of Theorem 1.3.** Let $F \in \mathcal{H}_c(W)$ be slow. Arguing by contradiction, we assume that $\rho[F] < \|F\|$. Choose $\varepsilon > 0$ so small that

$$\rho[F] + 2\varepsilon < \|F\|.$$ 

Since $F$ is smooth and compactly supported and by Sard’s theorem, the set $C$ of critical values of $F$ is compact and has zero Lebesgue measure. If $F(W) = [a, b]$, we thus find finitely many intervals $[a_i, b_i] \subset [a, b] \setminus C$ such that $\sum (b_i - a_i) \geq (b - a) - \varepsilon$. Choose a smooth function $r: [a, b] \to \mathbb{R}$ such that $r(a) = a$ and such that $0 \leq r'(t) \leq 1$ for all $t$ and $r'(t) = 1$ if $t \in \bigcup_i [a_i, b_i]$ and $r'(t) = 0$ if $t \in C$.

The function $G = r \circ F$ belongs to $\mathcal{H}_c(W)$ and is both slow and flat. Moreover,

$$\max G = r(b) \geq r(a) + (b - a) - \varepsilon = \max F - \varepsilon.$$ 

Since the path $\{gt \circ f_t^{-1}\}$ is generated by $G - F = r \circ F - F$ and since $\|r \circ F - F\| = \max F - \max G \leq \varepsilon$, we have $\rho[gt \circ f_t^{-1}] \leq \varepsilon$. Therefore,

$$\rho[G] = \rho[gt \circ f_t^{-1} \circ f_t] \leq \rho[gt \circ f_t^{-1}] + \rho[F] \leq \varepsilon + \rho[F] < \|F\| - \varepsilon \leq \|G\|.$$ 

We have constructed a slow and flat $G \in \mathcal{H}_c(W)$ with $\rho[G] < \|G\|$, in contradiction to Lemma 2.2.

We would like to point out that the proof of Lemma 2.2 is the only place where we use a semi-positivity assumption on $(W, \omega)$. As explained in [44] the $S^1$-invariant virtual moduli cycle can be used to establish Lemma 2.2 for arbitrary tame symplectic manifolds. The above argument then yields Theorem 1.3 and hence Conjecture 1.2 in [44] for all tame symplectic manifolds.

**End of the proof of Theorem 1.2.** We can assume that $e(A, W) < \infty$, and in view of the definitions of the capacity $c^e_{\mathcal{H}}$ and the displacement energy $e$ we can assume that $A$ is compact. Let $F \in \mathcal{H}_c(\text{Int} A)$ be such that $\max F - \min F = \|F\| > 4e(A, W)$. According to Proposition 2.1 we have $\rho[F] < \|F\|$, and so Theorem 1.3 shows that $F$ is not slow. Therefore, $c^e_{\mathcal{H}}(A, W) \leq 4e(A, W)$. 

2.2. Proof of Theorem 1.1. We shall derive Theorem 1.1 from Theorem 1.2 by a stabilization argument. Let \( G(q, p) = \frac{1}{2}p^2 \) be the Hamiltonian generating the geodesic flow on \( T^* S^1 \), and abbreviate \( G^\varepsilon = \{(q, p) \mid G(q, p) \leq \varepsilon \} \).

**Lemma 2.3.** For any subset \( A \) of a symplectic manifold \((V, \omega)\) and any \( \varepsilon > 0 \),

\[
e_{HZ}^c (A, V) \leq e_{HZ}^c (A \times G^\varepsilon, V \times T^* S^1).
\]

**Proof.** We can assume that \( \text{Int} A \neq \emptyset \). Let \( F \in \mathcal{H}_c (\text{Int} A) \) be slow. We choose a smooth function \( a : \mathbb{R} \to [0, 1] \) such that

\[
a(t) = 1 \text{ if } t \leq \frac{1}{3} \varepsilon \quad \text{and} \quad a(t) = 0 \text{ if } t \geq \frac{2}{3} \varepsilon.
\]

The function \( F_S : V \times T^* S^1 \to \mathbb{R} \) given by \( (v, w) \mapsto F(v)a(G(w)) \) belongs to \( \mathcal{H}_c (\text{Int} (A \times G^\varepsilon)) \). In order to see that \( F_S \) is slow, assume that \( x(t) \) is a contractible periodic orbit of its Hamiltonian flow. Then \( x(t) = (x_1(t), x_2(t)) \subset V \times T^* S^1 \), where both \( x_1(t) \) and \( x_2(t) \) are contractible periodic orbits. Denoting the Hamiltonian vector fields of \( F \) and \( G \) by \( X_F \) and \( X_G \), we find

\[
\dot{x}_1(t) = a(G(x_2(t)))X_F (x_1(t)),
\]

\[
\dot{x}_2(t) = F(x_1(t))a'(G(x_2(t)))X_G (x_2(t)).
\]

Therefore, the orbits \( x_1(t) \) and \( x_2(t) \) are, up to reparametrization, orbits of \( X_F \) and \( X_G \). Since \( F \) and \( G \) are autonomous, we conclude that the functions \( a(G(x_2(t))) \) and \( F(x_1(t))a'(G(x_2(t))) \) are constant. Since \( |a(G(x_2))| \in [0, 1] \) and \( F \) is slow, the orbit \( x_1(t) \) is constant or has period \( > 1 \), and since all contractible periodic orbits of the flow of \( G \) are constant, the orbit \( x_2(t) \) is constant. We have constructed for every slow \( F \in \mathcal{H}_c (\text{Int} A) \) a slow \( F_S \in \mathcal{H}_c (\text{Int} (A \times G^\varepsilon)) \) with max \( F = \max F_S \). Lemma 2.3 thus follows. \( \square \)

In order to prove Theorem 1.1 we need to show that for every compact subset \( A \) of \( W \),

\[
e_{HZ}^c (A, W) \leq 4e (A \times S^1, W \times T^* S^1).
\]

We can assume that \( e(A \times S^1, W \times T^* S^1) \) is finite. Fix \( \delta > 0 \), and choose \( H \in \mathcal{H}_c (I \times W \times T^* S^1) \) such that \( h \) displaces \( A \times S^1 \) and

\[
\|H\| \leq e (A \times S^1, W \times T^* S^1) + \delta.
\]

We then find \( \varepsilon > 0 \) such that \( h \) displaces \( A \times G^\varepsilon \). It follows that

\[
e (A \times G^\varepsilon, W \times T^* S^1) \leq \|H\| \leq e (A \times S^1, W \times T^* S^1) + \delta.
\]
Since both \((W, \omega)\) and \((T^*S^1, \omega_0)\) are tame, so is their product, and since \((W, \omega)\) is stably strongly semi-positive, \((W \times T^*S^1, \omega \oplus \omega_0)\) is strongly semi-positive. Together with Lemma 2.3 and Theorem 1.2 we can thus estimate
\[
c^h_{HZ}(A, W) \leq c^h_{HZ}(A \times G^\varepsilon, W \times T^*S^1) \\
\leq 4e(A \times G^\varepsilon, W \times T^*S^1) \\
\leq 4e(A \times S^1, W \times T^*S^1) + 4\delta.
\]
Since \(\delta > 0\) was arbitrary, inequality (5) follows, and so Theorem 1.1 is proved. 

2.3. **Proof of Proposition 1.4.** (i) By assumption, the set \(A \times S^1\) is contained in the finite CW-complex \(X \times S^1\) of dimension \(\leq n + 1\) in the \((2n + 2)\)-dimensional symplectic manifold \((V \times T^*S^1, \omega \oplus \omega_0)\). Since \(X \times S^1\) can be displaced from itself in \(V \times T^*S^1\) by a smooth isotopy, a result of Laudenbach [35] implies that \(X \times S^1\) can be displaced from itself in \((V \times T^*S^1, \omega \oplus \omega_0)\) by a Hamiltonian isotopy. It follows that \(e_S(A, V) \leq e_S(X, V) < \infty\).

(ii) Consider the closed submanifold \(M \times S^1\) of \(V \times T^*S^1\). Since \(\omega|_M \neq 0\) we have \(\omega \oplus \omega_0|_{M \times S^1} \neq 0\). Moreover, the Euler characteristic of \(M \times S^1\) vanishes. A result of Polterovich [49] and Laudenbach–Sikorav [36] thus implies that \(e(M \times S^1, V \times T^*S^1) = 0\), and so \(e_S(A, V) = 0\).

(iii) The proof of the case \(n = 1\) is elementary and omitted. So assume that \(n \geq 2\). Since \(A\) is compact, \(L \setminus A\) is open. Using the Lagrangian Neighbourhood Theorem we easily find a closed submanifold \(L'\) of \(V\) which is not Lagrangian and such that \(A \subset L'\). By assertion (ii) we have \(e_S(L', V) = 0\), and so \(e_S(A, V) = 0\).

3. **Applications**

Throughout this section, \((V, \omega)\) denotes an arbitrary symplectic manifold, while \((W, \omega)\) denotes a tame and stably strongly semi-positive symplectic manifold. We say that a compact subset \(A\) of \((V, \omega)\) is **displaceable** if there exists \(h \in \text{Ham}_c(V, \omega)\) such that \(h(A) \cap A = \emptyset\), and we say that \(A\) is **stably displaceable** if \(A \times S^1\) is displaceable in \((V \times T^*S^1, \omega \oplus \omega_0)\). Thus \(A \subset V\) is (stably) displaceable if and only if \(e(A, V) < \infty\) (resp. \(e_S(A, V) < \infty\)). Note that if \(A\) is (stably) displaceable, then an entire neighbourhood of \(A\) is (stably) displaceable.

3.1. **Almost existence of closed characteristics and the Weinstein conjecture.**

A **hypersurface** \(S\) in a symplectic manifold \((V, \omega)\) is a smooth compact connected orientable codimension 1 submanifold of \(V\) without boundary. A closed characteristic on \(S\) is an embedded circle in \(S\) all of whose tangent lines belong to the distinguished
line bundle

\[ \mathcal{L}_S = \{ (x, \xi) \in TS \mid \omega(\xi, \eta) = 0 \text{ for all } \eta \in T_x S \}. \]

Examples show that \( \mathcal{L}_S \) might not carry any closed characteristic, see [15], [17]. We therefore follow [26] and consider parametrized neighbourhoods of \( S \). Since \( S \) is orientable, there exists an open neighbourhood \( I \) of 0 and a smooth diffeomorphism \( \vartheta : S \times I \to U \subset V \) such that \( \vartheta(x, 0) = x \) for \( x \in S \). We call \( \vartheta \) a thickening of \( S \), and we abbreviate \( S_\varepsilon = \vartheta(S \times \{ \varepsilon \}) \). Denote by \( \mathcal{P}_\varepsilon(S_\varepsilon) \) the set of closed characteristics on \( S_\varepsilon \) which are contractible in \( V \). The refinement of the Hofer–Zehnder argument [28, Sections 4.1 and 4.2] in [42] shows

**Proposition 3.1.** For any thickening \( \vartheta : S \times I \to U \subset V \) of a hypersurface \( S \) in \((V, \omega)\) with \( c_{HZ}^0(U, V) < \infty \) it holds that \( \mathcal{P}^0(S_\varepsilon) \neq \emptyset \) for almost all \( \varepsilon \in I \).

Together with Theorem 1.2 we obtain

**Corollary 3.2.** Assume that \( S \) is a stably displaceable hypersurface in \((W, \omega)\). Then for any stably displaceable thickening \( \vartheta : S \times I \to U \subset W \) it holds that \( \mathcal{P}^0(S_\varepsilon) \neq \emptyset \) for almost all \( \varepsilon \in I \).

In [61], Zehnder constructed a symplectic form on the 4-torus \( T^4 = (\mathbb{R}/\mathbb{Z})^4 \) such that none of the hypersurfaces \( \{ x_4 = \text{const} \} \) carries a closed characteristic. The assumption in Corollary 3.2 that \( S \) is stably displaceable thus cannot be omitted.

A hypersurface \( S \) in a symplectic manifold \((V, \omega)\) is called of contact type if there exists a Liouville vector field \( X \) (i.e., \( \mathcal{L}_X \omega = d\iota_X \omega = \omega \)) which is defined in a neighbourhood of \( S \) and is everywhere transverse to \( S \). Weinstein conjectured in [60] that every hypersurface \( S \) of contact type with \( H^1(S; \mathbb{R}) = 0 \) carries a closed characteristic.

**Corollary 3.3.** Assume that \( S \) is a stably displaceable hypersurface of contact type in \((W, \omega)\). Then \( \mathcal{P}^0(S) \neq \emptyset \). In particular, the Weinstein conjecture holds true for \( S \).

The Weinstein conjecture has been proved for various classes of hypersurfaces of contact type in various classes of symplectic manifolds ([57], [26], [24], [10], [25], [29], [40], [58], [38], [59], [4], [37], [39], [46]). Corollary 3.3 generalizes or complements the results in [57], [26], [10], [59], [37], where the ambient symplectic manifold is of the form \((V \times \mathbb{R}^2, \omega \oplus \omega_0)\). Under the additional assumption that \((W, \omega)\) is weakly exact and convex, Corollary 3.3 has been proved in [12].
3.2. **Periodic orbits of autonomous Hamiltonian systems.** We consider a smooth proper Hamiltonian $F$ on $(V, \omega)$ which attains its minimum at 0. We abbreviate the sublevel set $F^{-1}([0, r])$ by $F^r$, and define $d_1(F) \in [0, \infty]$ by

$$d_1(F) = \sup \{r \in \mathbb{R} \mid F^r \text{ is stably displaceable}\}.$$ 

Thus $d_1(F) > 0$ if and only if $F^{-1}(0)$ is stably displaceable. Denote by $\mathcal{P}^o(F^{-1}(r))$ the set of non-constant periodic orbits on $F^{-1}(r)$ which are contractible in $V$. Since the set of critical values of $F$ is closed and, by Sard’s theorem, of Lebesgue measure zero, Corollary 3.2 yields

**Corollary 3.4.** Consider a proper Hamiltonian $F$ on $(W, \omega)$ with minimum 0, and assume that $d_1(F) > 0$. Then $\mathcal{P}^o(F^{-1}(r)) \neq \emptyset$ for almost all $r \in [0, d_1(F)]$.

**Discussion. 1.** Recall that Corollary 3.4 becomes relevant in conjunction with Proposition 1.4 applied to $A = F^{-1}(0)$.

2. According to [17], every symplectic manifold $(V, \omega)$ of dimension $2n \geq 4$ admits a proper $C^2$-smooth Hamiltonian $F$ with minimum 0 and $d_1(F) > 0$ such that for a sequence $r_k \to 0$ of regular values the levels $F^{-1}(r_k)$ carry no periodic orbit, and if $2n \geq 6$, then $F$ can be chosen $C^\infty$-smooth.

3. Consider a tame symplectic manifold $(W^{2n}, \omega)$ for which $\pi_2(W)$ and $c_1$ vanish on $\pi_2(W)$, and assume that the proper function $F : W \to \mathbb{R}$ attains its minimum 0 along a closed symplectic submanifold $M^{2k}$ of $(W, \omega)$. It has been shown in [17, Corollary 2.16] that $\mathcal{P}^o(F^{-1}(r)) \neq \emptyset$ for almost all $r \in [0, b(F)]$, where

$$b(F) = \sup \{r \in \mathbb{R} \mid F^r \subset B(M, F)\} \in [0, \infty]$$

and $B(M, F)$ is “the $F$-maximal symplectic ball neighbourhood of $M$ in $(W, \omega)$”, see [17, Section 4.1] for details. For $k \in \{0, 1, \ldots, [n/2]\}$, this result is covered by Proposition 1.4 and Corollary 3.4 with $d_1(F) > 0$ instead of $b(F)$. It would be interesting to compare these two constants.

3.3. **Closed trajectories of a charge in a magnetic field and a potential.** Consider a closed Riemannian manifold $(M, g)$ of dimension at least 2, and endow the cotangent bundle $T^*M$ with the standard symplectic form $\omega_0 = \sum_i dp_i \wedge dq_i$. We fix a closed 2-form $\sigma$ on $M$ and define the twisted symplectic form $\omega_\sigma$ on $\pi : T^*M \to M$ by $\omega_\sigma = \omega_0 + \pi^*\sigma$. We also fix a function $V$ on $M$ with minimum 0. The flow of the Hamiltonian system

$$F_V : (T^*M, \omega_\sigma) \to \mathbb{R}, \quad F_V(q, p) = \frac{1}{2} |p|^2 + V(q),$$

describes (for example) the motion of a unit charge on $(M, g)$ subject to the magnetic field $\sigma$ and the potential $V$, cf. [45], [31], [14]. As before we denote by $\mathcal{P}^o(F_V^{-1}(r))$ the set of periodic orbits on the level $F_V^{-1}(r)$ which are contractible in $T^*M$ and hence project to contractible closed trajectories on $M$. 
Corollary 3.5. Consider a closed Riemannian manifold \((M, g)\) endowed with a closed 2-form \(\sigma\) which does not vanish identically, and let \(V\) be a potential on \(M\) with minimum 0. Then \(d_1(F_V) > 0\) and \(\mathcal{P}^\circ(F_V^{-1}(r)) \neq \emptyset\) for almost all \(r \in \]0, d_1(F_V)\].

Proof. It is shown in [5] that for any closed 2-form \(\sigma\) on a closed manifold \(M\) the symplectic manifold \((T^*M, \omega_\sigma)\) is tame. Since the kernel of the differential of the projection \(\pi : T^*M \to M\) defines a Lagrangian distribution in the tangent bundle of \((T^*M, \omega_\sigma)\), the first Chern class vanishes, so that \((T^*M, \omega_\sigma)\) is stably strongly semi-positive. Moreover, \(F_V\) is proper, has minimum 0, and \(F_V^{-1}(0) \subset M\); and since \(\sigma\) does not vanish, \(M\) is not Lagrangian. Proposition 1.4 (ii) thus yields \(d_1(F_V) > 0\), and so Corollary 3.5 follows from Corollary 3.4. \(\square\)

Specializing to the case \(V = 0\), we set \(d_1(g, \sigma) = d_1(F_0)\) and denote the sphere bundle \(F_0^{-1}(r)\) by \(E_r\).

Corollary 3.6. Consider a closed Riemannian manifold \((M, g)\) endowed with a closed 2-form \(\sigma\) which does not vanish identically. Then \(d_1(g, \sigma) > 0\) and \(\mathcal{P}^\circ(E_r) \neq \emptyset\) for almost all \(r \in \]0, d_1(g, \sigma)\].

Discussion. 1. There has been much recent progress in the existence problem for periodic orbits of a charge in a magnetic field ([45], [31], [1], [13], [24], [14], [38], [50], [18], [30], [7], [19], [5], [17], [41], [8], [6], [12], [47]). Corollary 3.6 solves the almost existence problem at small energies. Under additional assumptions on \(M\), \(g\) or \(\sigma\), stronger results are known. We refer to [14], [52], [47] for the state of the art.

2. If \(\sigma\) is exact, \(d_1(g, \sigma) \leq \frac{1}{2} \max_{x \in M} |\alpha(x)|^2\) for all \(\alpha\) with \(d\alpha = \sigma\), see [12]. If \(\sigma\) is non-exact, \(d_1(g, \sigma)\) can be infinite; examples with infinite \(d_1(g, \sigma)\) are non-exact closed 2-forms \(\sigma\) on tori, see [18], [52].

3. One cannot expect that \(\mathcal{P}^\circ(E_r) \neq \emptyset\) for almost all \(r > 0\) in general. Indeed, let \(M\) be a closed oriented surface of genus 2, and let \(g\) and \(\sigma\) either be a Riemannian metric of constant curvature \(-1\) and its area form or the Riemannian metric and the exact 2-form constructed in [48]. Then \(\mathcal{P}^\circ(E_r) = \emptyset\) for all \(r \geq \frac{1}{2}\), see [14, Example 3.7] and [48].

4. Assume that \(M\) is neither a 2-sphere nor an orientable surface of genus \(\geq 2\). If \(\sigma\) is non-exact, then none of the hypersurfaces \(E_r\) in \((T^*M, \omega_\sigma)\) is of contact type, see e.g. [52]. Therefore, Corollary 3.6 does not follow from existence results of closed characteristics on contact type hypersurfaces.

References


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