Complex-dimensional Integral and Light-cone Singularities

by

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Abstract

The notion of a complex-dimensional integral is introduced in the complex n-dimensional Minkowski space. Its basic properties, such as Lorentz invariance, are investigated. Complex-dimensional invariant delta functions \( \delta_n(x; m^2) \), \( \delta^{(1)}_n(x; m^2) \), etc. are explicitly calculated in position space. It is proposed to define products of singular functions in the ordinary Minkowski space by analytically continuing the corresponding n-dimensional ones to \( n = 4 \). The light-cone singularities of \( \delta(x; m^2) \), \( \delta(x; m^2) \times \delta^{(1)}(x; m^2) \), and \( \delta^{(2)}(x; m^2) \) are shown to be unambiguously determined in this way.

Recently, in quantum field theory, much attention has been paid to complex-dimensional regularization [1]. The momentum-space Feynman integral is regularized by considering it in the complex n-dimensional space formally. The extension of the dimension 4 to the complex dimension \( n \) is easily done in the Feynman-parametric representation of the Feynman integral. The purpose of my talk is to formulate the theory of complex-dimensional integrals in the general framework and apply it to regularizing singular products in position space. Detailed accounts are presented in my papers [2, 3].

The complex n-dimensional Minkowski space \( M^n \) is a product of a one-dimensional Euclidean space \( R \) and a complex \( (n - 1) \)-dimensional space \( E^{n-1} \) such that the scalar product in \( M^n \) is defined by the difference between the product in \( R \) and the scalar product in \( E^{n-1} \). Here \( E^{n-1} \) is an abstract vector space equipped with a real-valued, symmetric scalar product. Except for the case in which \( n \) is a positive integer, however, \( E^{n-1} \) is not a topological space and therefore the number of linearly independent vectors in it is indefinite because it has no complete basis. It is assumed that any finite-dimensional subspace of \( E^{n-1} \) is a Euclidean space.
space. The notion of components \( p_1, p_2, \ldots \) of a vector \( p \in \mathbb{R}^{n-1} \) is meaningful only with reference to such a subspace. The index \( \mu \) of a vector \( p_\mu \in \mathbb{R}^n \) takes discrete values only when one works in a finite-dimensional subspace of \( \mathbb{R}^n \).

Let \( F(p_\mu) \) be a tempered distribution (or a Fourier hyperfunction) of scalar products \( p_\mu^s, p_\mu x^{(1)}, \ldots, p_\mu x^{(k)} \), where \( p_\mu \) is an integration vector and \( x_\mu^{(1)}, \ldots, x_\mu^{(k)} \) are constant vectors of \( \mathbb{R}^n \). Then I define the complex-dimensional integral of \( F(p_\mu) \) by

\[
\int d^n p F(p_\mu) = \frac{2\pi^{(n-k)/2}}{\Gamma((n-k-1)/2)} \int_{-\infty}^{+\infty} dp_0 \int_{-\infty}^{+\infty} dp_1 \ldots \int_{-\infty}^{+\infty} dp_k \int_{0}^{\infty} dp_\perp p_\perp ^{*-k-2} F(p_\mu; p_1, \ldots, p_k; p_\perp).
\]

Here \( p_1, \ldots, p_k \) are orthogonal coordinates in a generically \( k \)-dimensional subspace spanned by the spatial parts \( x^{(1)}, \ldots, x^{(k)} \) of \( x^{(1)}, \ldots, x^{(k)} \), and

\[
(2) \quad p_\perp ^2 = p_\perp ^2 - \sum_{j=1}^{k} p_j ^2.
\]

If \( x^{(1)}, \ldots, x^{(k)} \) happen to be linearly dependent, that is, for example, \( F \) is independent of \( p_\mu \), then setting \( p_\perp ^2 = p_\perp ^2 + p_\mu ^2 \), one can easily see that (1) reduces to the expression which is the same as (1) except that \( k \) is replaced by \( k - 1 \). Thus the definition (1) does not intrinsically depend on \( k \). From this fact it follows that (1) is invariant under a translation of the integration vector \( p_\mu \), as it should be. Of course, (1) reduces to the ordinary \( n \)-dimensional multiple integral when \( n \) is a positive integer.

The complex-dimensional integral defined by (1) is not manifestly Lorentz invariant, but its Lorentz invariance can be proved. More precisely, (1) can be shown to be a quantity depending only on scalar products formed from \( x^{(1)}, \ldots, x^{(k)} \). The proof is carried out by reducing the problem to that for the complex-dimensional Fourier transform\(^1\)

\[
(3) \quad \int d^n p e^{-ip_\mu \varphi(p_\mu)} = (2\pi)^{(n-1)/2} \int_{-\infty}^{+\infty} dp_0 \int_{0}^{\infty} d|p| |p|^{(n-1)/2} x^{-(n-3)/2} J_{(n-3)/2}(|p| x) e^{-ip_\mu x \varphi(p_\mu^2 - |p|^2)},
\]

where \( J_\alpha \) denotes a Bessel function.

\(^1\) The right-hand side follows from the polar-coordinate form of (1) with \( k = 1 \).
The complex-dimensional invariant delta functions are defined by

\[ \delta_n(x; m^2) = -i (2\pi)^{-n+1} \int d^np \delta(p^2 - m^2) e^{-ipx}, \]

\[ \delta_n(x; m^2) = (2\pi)^{-n+1} \int d^np \delta(p^2 - m^2) e^{-ipx}, \]

etc. Their explicit expressions can be calculated by using (3). For example,

\[ \delta_n(x; m^2) = -\epsilon(x) \frac{\sqrt{x^2/m^{2-n}/2}}{2^{n+2/2}(n-2)/2} J_{(n-2)/2}(m\sqrt{x^2}) \theta(x^2). \]

It is easy to extend the definition of the complex-dimensional integral to the case in which the integrand is a Lorentz-covariant quantity \( G_{\rho\nu} \), which is defined by

\[ H \equiv y^\rho z^\nu G_{\rho\nu}, \]

where \( H \) is a Lorentz-invariant quantity and \( y^\rho, \cdots, z^\nu \) are artificially introduced constant vectors in \( M^n \). For example, consider \( G_{\rho\nu} = p_\rho p_\nu F(p^2, px) \). The complex-dimensional integral of \( y^\rho z^\nu G_{\rho\nu} \) is given by (1). Because of the Lorentz invariance of (1) and the proportionality in \( y^\rho \) and \( z^\nu \), I can write

\[ \int d^np (y^\rho p_\rho)(z^\nu p_\nu) F(p^2, px) = (y^\rho x^\rho)(z^\nu x^\nu) \Phi_1(x^2) + (y^\rho z^\nu) \Phi_2(x^2), \]

where \( \Phi_1 \) and \( \Phi_2 \) depend only on \( x^2 \). On introducing an abstract metric tensor \( g_{\mu\nu} \) of \( M^n \), I rewrite (8) as

\[ \int d^np p_\mu p_\nu F(p^2, px) = x_\mu x_\nu \Phi_1(x^2) + g_{\mu\nu} \Phi_2(x^2). \]

Then it can be proved that the formula

\[ g^{\mu\nu} \int d^np p_\mu p_\nu F(p_\rho) = \int d^np p^\rho F(p_\rho) \]

always holds if and only if one sets

\[ g_\rho^\rho = n. \]

The proof is carried out by showing that to prove (10) is equivalent

\[ \text{Necessity of (11) is well known and is shown easily.} \]
Finally, I mention the complex-dimensional regularization of singular products in position space. As is well known, the invariant delta functions in the ordinary Minkowski space exhibit light-cone singularities:

\begin{equation}
\mathcal{A}(x; m^2) = -\frac{\epsilon(x_o)}{2\pi} \left[ \delta(x^2) - \frac{m^2}{4} \theta(x^2) + \cdots \right],
\end{equation}

\begin{equation}
\mathcal{A}^{(1)}(x; m^2) = -\frac{1}{2\pi^2} \left[ \text{P} \frac{1}{x^2} - \frac{m^2}{4} \left( \log \frac{m^2|x^2|}{4} + 2\gamma - 1 \right) + \cdots \right],
\end{equation}

where \( \text{P} \) and \( \gamma \) denote Cauchy's principal value and Euler's constant, respectively. Therefore their products are not well defined. The complex-dimensional extensions \( \mathcal{A}_n \) and \( \mathcal{A}^{(1)}_n \) are, however, continuous on the light cone \( x^2 = 0 \) if \( \text{Re } n < 2 \). In that region, therefore, any product of \( \mathcal{A}_n \) and \( \mathcal{A}^{(1)}_n \) is always well defined. What I propose is to define singular products in the ordinary Minkowski space by analytically continuing in \( n \) the corresponding complex \( n \)-dimensional products to \( n = 4 \). After lengthy calculations, I have found that the products \( (\mathcal{A}_n)^2, \mathcal{A}_n \mathcal{A}^{(1)}_n, \) and \( (\mathcal{A}^{(1)}_n)^2 \) have no pole at \( n = 4 \). Accordingly, I obtain the regularized expressions for \( \mathcal{A}, \mathcal{A}^{(1)} \), and \( (\mathcal{A}^{(1)})^2 \) unambiguously [2]. They are consistent with another way of definitions

\begin{equation}
\mathcal{A}(x; m^2) \mathcal{A}^{(1)}(x; m^2) = 2\epsilon(x_o) \text{Im}[\mathcal{A}_f(x; m^2)]^2,
\end{equation}

\begin{equation}
[\mathcal{A}^{(1)}(x; m^2)]^2 - [\mathcal{A}(x; m^2)]^2 = 4 \text{Re}[\mathcal{A}_f(x; m^2)]^2
\end{equation}

where \( 2\epsilon = i\epsilon(x_o) \mathcal{A} + \mathcal{A}^{(1)} \) is a boundary value of an analytic function.

References