Stability and Instability of Certain Foliations of 4-Manifolds by Closed Orientable Surfaces

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§0. Introduction

Let $\text{Fol}_q(M)$ denote the set of codimension $q$ $C^\infty$-foliations of a closed $m$-manifold $M$. $\text{Fol}_q(M)$ carries a natural weak $C^r$-topology ($0 \leq r \leq \infty$), which is described in [7]. We denote this space by $\text{Fol}_q(M)$. We say a foliation $F$ is $C^r$-stable if there exists a neighborhood $V$ of $F$ in $\text{Fol}_q(M)$ such that every foliation in $V$ has a compact leaf. We say $F$ is $C^r$-unstable if not. A foliation in a small neighborhood of $F$ in $\text{Fol}_q(M)$ is said to be a $C^r$-perturbation of $F$. It seems to be of interest to determine if $F$ is $C^r$-stable or not. Let $L$ be a compact leaf of $F$. Langevin-Rosenberg [8] showed, generalizing the Reeb stability theorem [11], that if $H^1(L; \mathbb{R}) = 0$, then $F$ is $C^1$-stable. Let $\pi_1(L) \to GL(q, \mathbb{R})$ be the action determined from the linear holonomy of $L$, where $q$ is the codimension of $F$. Then generalizing the results of Hirsch [7] and Thurston [16], Stowe [15] showed that if the cohomology group $H^1(\pi_1(L); \mathbb{R})$ is trivial, then $F$ is $C^1$-stable. On the other hand, let $F$ be the foliation of an orientable $S^1$-bundle over a closed surface $B$ by fibres. Seifert [13] showed that $F$ is $C^0$-stable if $\chi(B) = 0$, where $\chi(B)$ is the euler characteristic of $B$. The result was generalized by Fuller [6] to orientable circle bundles over arbitrary closed manifolds $B$ with $\chi(B) \neq 0$. Langevin-Rosenberg [9] considered a fibration $\pi: M \to B$ with fibre $L$ and showed that the foliation of $M$ by fibres is $C^0$-stable provided that 1) $\pi_1(L) \approx \mathbb{Z}$, 2) $B$ is a closed surface with $\chi(B) \neq 0$ and 3) $\pi_1(B)$ acts trivially on $\pi_1(L)$. The author [4] generalized the above result to compact codimension two foliations. Plante [10] classified all foliations of closed 3-manifolds by closed orientable surfaces into stable or unstable foliations. The author [5] classified all foliations of closed 3-manifolds by circles into stable or unstable foliations.

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We study here the case $M$ is a closed 4-manifold and $F$ is a foliation of $M$ by closed orientable surfaces. Our main results are as follows. See §1 for definitions.

Theorem A (Theorem 8). Let $F$ be a foliation with all leaves tori and only reflection leaves as singular leaves. Then we can regard a union of reflection leaves as $T^2 \times [0, 1]/h$, where $h$ is a diffeomorphism of $T^2$. If the induced automorphism $h_*: H_1(T^2; \mathbb{Z}) \to H_1(T^2; \mathbb{Z})$ is equal to $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, then $F$ is $C^1$-stable.

Theorem B (Theorem 17). Let $F$ be a foliation of $M$ without singular leaves. Then $F$ is $C^r$-unstable ($r \geq 0$) if one of the following is satisfied,

1. $M/F$ is homeomorphic to the 2-sphere and the genus of a generic leaf $\geq 2$,
2. $M/F$ is homeomorphic to the projective plane and the genus of a generic leaf $\geq 4$,
3. $M/F$ is neither homeomorphic to the 2-sphere nor the projective plane and the genus of a generic leaf $\geq 6$.

Theorem C (Theorem 25). Let $F$ be a foliation of $M$ with generic leaf of genus $g$ and $B=M/F$ be the leaf space. Suppose $F$ has $m$ rotation leaves with holonomy groups $\mathbb{Z}_{k_i} (i=1, 2, \ldots, m)$ and $m_j$ dihedral leaves with holonomy groups $\mathcal{D}_{i, j_k} (k=1, 2, \ldots, m_j)$ which correspond to points of $\partial_j B$ for each $j (1 \leq j \leq n')$. If $g \geq \max(3\max(k_i; 1 \leq i \leq m)+1, 8\max(l_{j,k}; 1 \leq j \leq n', 1 \leq k \leq m_j)+1, \varepsilon)$, then $F$ is $C^r$-unstable ($r \geq 0$), where $\varepsilon=0$ or 1 and $F$ has no reflection leaves if and only if $\varepsilon=0$.

The paper is organized as follows. In §1, we recall the (local) structure of compact codimension two foliations and prepare some definitions and notations. In §2, we discuss about foliations with all leaves tori and prove Theorem A. In §3, we discuss about foliations with generic leaf of genus $\geq 2$ and no singular leaves and prove Theorem B. In §4, we discuss about foliations with generic leaf of genus $\geq 2$ and singular leaves and prove Theorem C. All foliations we consider here are smooth of class $C^\infty$ and of codimension two.

§1. Compact Foliations and Singular Leaves

Let $M$ be a closed manifold and $F$ a compact foliation of codimension two. By the results of Epstein [2] and Edwards-Millett-Sullivan [1], we have a nice picture of the local behavior of $F$ as follows.

Proposition 1 (Epstein [3]). There is a generic leaf $L_0$ with property that
there is an open dense saturated subset of $M$, where all leaves have trivial holo-

nomy and are diffeomorphic to $L_0$. Given a leaf $L$, we can describe a neighbor-

hood $U(L)$ of $L$, together with the foliation on the neighborhood as follows. There

is a finite subgroup $G(L)$ of $O(2)$ such that $G(L)$ acts freely on $L_0$ on the right and $L_0/G(L)\approx L$. Let $D^2$ be the unit disk. We foliate $L_0\times D^2$ with leaves of the form $L_0\times\{pt\}$. This foliation is preserved by the diagonal action of $G(L)$, de-

fined by $g(x, y)=(x'g, g\cdot y)$ for $g\in G(L)$, $x\in L_0$ and $y\in D^2$, where $G(L)$ acts linearly on $D^2$. So we have a foliation induced on $U=L_0\times D^2$. The leaf cor-

responding to $y=0$ is $L_0/G(L)$. Then there is a $C^\infty$-imbedding $\varphi: U\to M$ with

$\varphi(U)=U(L)$, which preserves leaves and $\varphi(L_0/G(L))=L$.

**Remark 2.** $U(L)$ can be considered to be the total space of a normal
disk bundle of $L$ in $M$ with structure group $G(L)$.

Since $G(L)$ is a finite subgroup of $O(2)$, $G(L)$ is isomorphic to a rotation
group $Z_k (k>1)$, a dihedral group $D_l (l>1)$ which consists of $l$ rotations and $l$
reflections or a group $D$ consisting of only one reflection, which is called a
reflection group.

**Definition 3.** A leaf $L$ is singular if $G(L)$ is not trivial. The order of

$G(L)$ is called the order of holonomy of $L$. We say such an $L$ is a rotation
leaf, a reflection leaf or a dihedral leaf if $G(L)$ is isomorphic to $Z_k (k>1)$, $D$
or $D_l (l>1)$ respectively.

Let $B=M/F$ be the leaf space. $B$ is a compact $V$-manifold of dimension

two and is also a topological manifold. The quotient map $\pi: M\to B$ is a

$V$-bundle (see Satake [12] for definitions). Since $M$ is compact, there are

finitely many rotation leaves and dihedral leaves in $F$. Dihedral and reflection
leaves correspond to the boundary points of $B$. Let $n=\sum n_j$ be the num-
ber of boundary components of $B$. We let $L_i (i=1, 2, \cdots, m)$ be all rotation
leaves with holonomy groups $Z_{k_i}$ and $L_{i,k} (j=1, 2, \cdots, n'; k=1, 2, \cdots, m_j)$
al all dihedral leaves with holonomy groups $D_{l_{i,k}}$ respectively such that $L_{i,k}$
$k=1, 2, \cdots, m_j$ correspond to points in the $j$-th boundary $\partial jB$ of $B (1\leq j\leq n')$.

All points in other boundaries $\partial \alpha_{i+j} B (1\leq j\leq n')$ of $B$ correspond to reflection
leaves. Choose saturated neighborhoods $U(L_i)$ as in Proposition 1 to be
disjoint and take saturated neighborhoods $U'(L_i)$ such that $U'(L_i)\subset U(L_i)$, where $U'$ denotes the interior of $U$. Let $V_i =\pi(U(L_i))$ and $V'_i =\pi(U'(L_i))$.

Let $B_0 = B - \bigcup_{i=1}^m V_i$, $B_1 = B_0 - \partial B$ and $M_1 = \pi^{-1}(B_1)$. The restricted map $\pi: M_1 \to B_1$ is a fibre bundle with generic leaf $L$ as fibre. We now assume that
$M$ is a closed 4-manifold and $F$ is a foliation of $M$ by closed orientable surfaces of genus $\geq 2$. Then the bundle $\pi: M_{1} \to B_{1}$ is represented as follows. Let $B_{0}$ be a compact surface of genus $h$ with $m+n$ boundaries. First we consider the case $B$ is orientable. Take simple closed curves $c_{i}$ $(i=1, 2, \cdots, 2h)$ and arcs $d_{j}$ $(j=1, 2, \cdots, m+n)$ on $B_{0}$ such that 1) $c_{i}$ and $c_{i+1}$ intersect at points $p_{i}$ $(i=1, 2, \cdots, 2h-1)$, 2) $d_{j}$ $(j=1, 2, \cdots, m)$ join points $p_{2k+j-1}$ of $2k$ and $\partial V'_{j}$ respectively and $d_{m+j}$ $(j=1, \cdots, n)$ join points $p_{2k+m+j-1}$ of $2k$ and points $q_{j}$ of $\partial jB$, where $q_{j}=\pi(L_{j,1})$ for $j=1, \cdots, n'$, and 4) cutting off $B_{0}$ along $c_{i}$ $(i=1, \cdots, 2h)$ and $d_{j}$ $(j=1, \cdots, m+n)$ yields a compact manifold $B_{2}$ which is homeomorphic to a disk (see Fig. 1). $c_{1}$ is separated to an arc $c_{1}'$ by $c_{2}, c_{1}$ $(i=2, \cdots, 2h-1)$ are separated to two arcs $c_{i+1}, c_{i+2}$ by $c_{i+1}$ and $c_{i+2}$, and $c_{2h}$ is separated to $m+n+1$ arcs $c_{2h+j}$ $(j=1, \cdots, m+n+1)$ by $c_{2h-1}$ and $d_{j}$ $(j=1, \cdots, m+n)$. Cutting off $B_{1}$ along $c_{i}$ and $d_{j}$ yields a subset $B_{3}$ of $B_{2}$ whose interior is homeomorphic to an open disk. Then we obtain $M_{1}$ by making the following identifications in $L \times B_{3}$ as follows;

$$(x, y) \sim (\varphi_{i}(y)(x), y) \text{ for } x \in L, y \in c_{i}' \text{, } (x, y) \sim (\varphi_{i}(y)(x), y) \text{ for } x \in L, y \in c_{i+j} \text{ (i=2, } \cdots, 2h-1; j=1, 2), (x, y) \sim (\varphi_{2h+j}(y)(x), y) \text{ for } x \in L, y \in c_{2h,j} \text{ (j=1, } \cdots, m+n+1) \text{ and } (x, y) \sim (\psi_{d}(y)(x), y) \text{ for } x \in L, y \in d_{k} \text{ (k=1, } \cdots, m+n),$$

where $\varphi_{i}: c_{1} \times L \to L, \varphi_{i+j}: c_{i+j} \times L \to L, \varphi_{2h,i}: c_{2h,i} \times L \to L$ and $\varphi_{d}: d_{k} \times L \to L$ are smooth maps such that $\varphi_{i}(y), \varphi_{i+j}(y), \varphi_{2h,i}(y)$ and $\psi_{d}(y)$ for $y \in c_{1}', c_{i}, c_{2h,i}$ and $d_{k}$ are diffeomorphisms of $L$ respectively. We may assume that $\varphi_{i}(y), \varphi_{i+j}(y), \varphi_{2h,i}(y)$ and $\psi_{d}(y)$ are constant diffeomorphisms on neighborhoods of the boundaries.
Next we consider the non-orientable case. Let $B_0$ be a closed surface obtained by pasting disks to $B_0$ along the boundaries. $B_0$ is homeomorphic to $\sum_{h_i} \# P^2$ or $\sum_{h_i} \# K^2$ according to $h=2h_1+1$ or $2h_1+2$, where $\sum_{h_i}$ is an orientable surface of genus $h_i$, $P^2$ is the projective plane and $K^2$ is the Klein bottle. We can identify $B_0$ with $\sum_{h_i} \# P^2 - \bigcup_{i=1}^{m+n} D_i$ or $\sum_{h_i} \# K^2 - \bigcup_{i=1}^{m+n} D_i$.

**Case of $h=2h_1+1$.** Take simple closed curves $c_i (i=1, 3, 4, \ldots, h+1)$ and arcs $d_j (j=1, 2, \ldots, m+n)$ on $B_0$ such that 1) $c_i$ and $c_{i+1}$ intersect at points $p_i (i=3, 4, \ldots, h)$, 2) $c_1$ generates the fundamental group of $P^2$, 3) $c_2$ joins a point $p_1$ of $c_1$ and a point $p_2$ of $c_3$, 4) $d_j (j=1, \ldots, m)$ join points $p_{h+j}$ of $c_{h+1}$ and points $q_j$ of $\partial_i B$, where $q_j = \pi (L_{j,1})$ for $j=1, \ldots, n^2$ and 5) cutting off $B_0$ along $c_i (i=1, \ldots, h+1)$ and $d_j (j=1, \ldots, m+n)$ yields a compact topological manifold $B_2$ which is homeomorphic to a disk.

**Case of $h=2h_1+2$.** Take simple closed curves $c_1, c_2$ on $B_0$ instead of $c_1, c_2$ in the case of $h=2h_1+1$ as follows: their homotopy classes $\{c_1\}$ and $\{c_2\}$ are generators of the fundamental group of $K^2$ with the relation $\{c_1\} \cdot \{c_2\} = \{c_2\}^{-1} \cdot \{c_1\}$ and $c_1$ and $c_2$ intersect at $p_1$. Take other simple closed curves $c_i (i=3, 4, \ldots, h+1)$ and arcs $d_j (j=1, 2, \ldots, m+n)$ on $B_0$ in the same way as in the case of $h=2h_1+1$. Note that $c_2$ and $c_3$ intersect at a point $p_2$. $c_1$ is separated to an arc $c_1'$ by $c_2$, $c_1 (i=3, 4, \ldots, 2h_1-1)$ are separated to two arcs $c_{i,1}$, $c_{i,2}$ by $c_{i-1}$ and $c_{i+1}$, and $c_{2k}$ is separated to $m+n$ arcs $c_{2k,j} (j=1, 2, \ldots, m+n)$ by $c_{2k-1}$ and $d_j (j=1, 2, \ldots, m+n)$. In the case of $h=2h_1+2$, $c_2$ is separated to two arcs $c_{2,1}$, $c_{2,2}$ by $c_1$ and $c_2$. The rest is similar to the orientable case.

**Definition 4.** Let $Y$ be a subset of a manifold $X$. We say a diffeomorphism $f: X \to X$ satisfies the property $P(Y,r)$ if 1) $f$ is sufficiently $C^r$-close to 1$_X$, 2) $f$ is equal to 1$_X$ on the outside of $Y$ and 3) $f$ has no fixed points in $Y$.

We fix the following notations.

**Notation 5.** $B$ is the leaf space, and $B_0$, $B_1$ and $B_2$ are subsets of $B$ as is stated above. $g$ is the genus of a generic leaf $L$, and $\alpha_i$, $\beta_i (1 \leq i \leq g)$ are simple closed curves on $L$ such that $\langle \alpha_i, \alpha_j \rangle = 0$, $\langle \beta_i, \beta_j \rangle = 0$ and $\langle \alpha_i, \beta_j \rangle = \delta_{ij}$ for any $i, j$, where $\langle \ , \ \rangle$ denotes the algebraic intersection number of 1-cycles in $L$. We denote by $[\alpha_i]$ and $[\beta_i]$ the homology classes of $\alpha_i$ and $\beta_i$. Then $[\alpha_i], [\beta_i] (1 \leq i \leq g)$ form a canonical symplectic basis for $H_1(L; \mathbb{Z})$. We denote by $\{\alpha_i\}$, $\{\beta_i\}$ the homotopy classes of $\alpha_i$, $\beta_i$. Then $\{\alpha_i\}$, $\{\beta_i\} (1 \leq i \leq g)$
§ 2. Stability of a Foliation with All Leaves Tori

In this section we study about perturbations of foliations with all leaves tori and singular leaves.

**Proposition 6.** If a foliation with all leaves tori has a rotation or dihedral leaf, then the foliation is $C^1$-stable.

**Proof.** This follows from Theorem 1.1 of Hirsch [7] since a certain linear holonomy of such a leaf has not 1 as an eigenvalue.

We consider here the case a foliation $F$ has only reflection leaves as singular leaves. Each connected component of the union $R(F)$ of reflection leaves of $F$ is diffeomorphic to $T^2 \times [0, 1]/h$, where $(x, 0)$ and $(y, 1)$ are identified by a diffeomorphism $h$ of $T^2$. We denote a connected component of $R(F)$ by the same letter. Let Möb be the Möbius band obtained in the product $S^1 \times (-1, 1)$ with coordinate $(\theta, u)$, $\theta \in S^1 = \mathbb{R}/\mathbb{Z}$, $u \in (-1, 1)$ by identifying $(\theta, u)$ and $(\theta + 1/2, -u)$. The foliation on $S^1 \times (-1, 1)$ with leaves of form $S^1 \times \{pt\}$ induces a foliation $F_1$ on Möb. So we define a foliation $F_2$ on Möb $\times S^1 \times [0, 1]$ with leaves of form $L \times S^1 \times \{pt\}$, $L \in F_1$. Let $U$ be a saturated tubular neighborhood of $R(F)$ in $M$. Then $(U, F)$ is diffeomorphic to $\text{(Möb} \times S^1 \times [0, 1], F_2)/H$, where $H : \text{Möb} \times S^1 \rightarrow \text{Möb} \times S^1$ is a foliation preserving diffeomorphism extended from $h$ and a point $p$ in Möb $\times S^1$ is assumed to be fixed by $H$. We take generators $\alpha$ and $\beta$ of $\pi_1(\text{Möb} \times S^1, p)$ corresponding to generators of $\pi_1(\text{Möb})$ and $\pi_1(S^1)$ respectively. Let $h_* : H_1(T^2) \rightarrow H_1(T^2)$ be the automorphism.

**Lemma 7.** $h_* = \begin{pmatrix} 2k+1 & l \\ 2m & 2n+1 \end{pmatrix}$, where $k, l, m, n \in \mathbb{Z}$ and $(2k+1)(2n+1) - 2ml = \pm 1$.

**Proof.** The holonomy along $\alpha$ is non-trivial and of order two and the holonomy along $\beta$ is trivial. So the holonomy along $h_*(\beta) = 2m\alpha + n'\beta$ $(m, n' \in \mathbb{Z})$. Since $h_*$ belongs to $GL(2, \mathbb{Z})$, diagonal components are odd numbers.

We consider the special case $k = -1$ and $n = -1$.

**Theorem 8.** Let $F$ and $U$ be as above. Suppose $h_* = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Then
every sufficiently small $C^1$-perturbation of $F$ has a compact leaf in $U$. Hence $F$ is $C^1$-stable.

Proof. We may assume that $U = \text{Mob} \times S^1 \times [0, 1]/H$ with coordinate $(\theta, u, \varphi, t), (\theta, u) \in \text{Mob}, \varphi \in S^1 = R/Z$ and $t \in [0, 1]$, and $p = (0, 0, 0) \in \text{Mob} \times S^1$ and a segment $E = \{ (0, u, 0) ; -1 < u < 1 \} \subset \text{Mob} \times S^1$ are left invariant by $H$.

The set $E \times [0, 1]/H$ can be considered to be the set $E \times S^1$, if necessary, by taking an appropriate double covering $\tilde{U}$ of $U$. For, let $F'$ be a small $C^1$-perturbation of $F$. Then the foliation $\tilde{F}'$ induced on $\tilde{U}$ is also a small $C^1$-perturbation of the foliation $\tilde{F}$ induced on $\tilde{U}$. If $\tilde{F}'$ has a compact leaf in $\tilde{U}$, then $F'$ has a compact leaf in $U$.

Let $\alpha$ and $\beta$ be loops in $L_{px;q}$ with base point $p \times \{ 0 \} = (0, 0, 0)$, representing generators of $\pi_1(L_{px;q}) \approx Z \oplus Z$ such that the holonomy along $\alpha$ (resp. $\beta$) is non-trivial (resp. trivial). Let $\alpha(t)$ and $\beta(t)$ be translations of $\alpha$ and $\beta$ along the curve $p \times \{ t \}, t \in [0, 1]$. Then we can define perturbed holonomy maps $H(F', \alpha(t)), H(F', \beta(t)) : E_\delta \times \{ t \} = \{ (0, u, 0) ; -\delta < u < \delta \} \times \{ t \} \to E \times S^1$ for each $t$ and some $\delta > 0$, which are imbeddings (cf. Hirsch [7], Langevin-Rosenberg [9] and Fukui [4]). Note that 1) $H(F', \alpha(t_0))$ and $H(F', \beta(t_0))$ are extended to maps $H(F', \alpha_{t_0})$ and $H(F', \beta_{t_0}) : E_\delta \times (t_0 - r, t_0 + r) \to E \times S^1$ for some small $r$, which are local diffeomorphisms, 2) the extended maps $H(F', \alpha_{t_0})$ and $H(F', \beta_{t_0})$ coincide on the intersections of their domains respectively if $t_0$ and $t_1$ are close, 3) $H(F', \alpha(t))$ and $H(F', \beta(t))$ are $C^1$-close to the map $R(u, t) = (-u, t)$ and $id(u, t) = (u, t)$ respectively because $F$ and $F'$ are $C^1$-close and 4) $H(F', \alpha(1)) = H(F', -\alpha(0))$ and $H(F', \beta(1)) = H(F', -\beta(0))$ may not coincide with $H(F', \alpha(0))$ and $H(F', \beta(0))$ respectively.

We put $\tilde{S}^1 = R/2Z$ and let $\pi : \tilde{S}^1 \to S^1$ be the double covering map defined by $\pi(\tilde{t}) = \tilde{t} (\text{mod } 1), \tilde{t} \in \tilde{S}^1$. Then there exist the maps $H_{\alpha}(F')$ and $H_{\beta}(F') : E_\delta \times \tilde{S}^1 \to E \times \tilde{S}^1$ extended from $H(F', \alpha(t))$ and $H(F', \beta(t))$ (cf. [4], [9]) respectively, such that the following diagram commutes;

\[
\begin{array}{ccc}
E_\delta \times \tilde{S}^1 & \xrightarrow{H_{\alpha}(F')} \ & E \times \tilde{S}^1 \\
\uparrow i & \ & \downarrow 1 \times \pi \\
E_\delta \times \{ t \} & \xrightarrow{H(F', \alpha(t)) \ (\text{resp. } H(F', \beta(t)))} & E \times S^1
\end{array}
\]

where $i((0, u, 0, t)) = (0, u, 0, t)$ and $(1 \times \pi) (0, u, 0, \tilde{t}) = (0, u, 0, \pi(\tilde{t}))$. We put $H_{\alpha}(F')(u, \tilde{t}) = (f_{\delta}(u, \tilde{t}), f_{\delta}(u, \tilde{t}))$ using the coordinate $(u, \tilde{t})$ of $E_\delta \times \tilde{S}^1$. Then there exists a unique $u(\tilde{t})$ for each $\tilde{t} \in \tilde{S}^1$ such that $u(\tilde{t}) = f_{\delta}(u(\tilde{t}), \tilde{t})$. The set $\tilde{I} = \{ (u(\tilde{t}), \tilde{t}); \tilde{t} \in \tilde{S}^1 \}$ is a loop in $E \times \tilde{S}^1$, so $(1 \times \pi) (\tilde{I})$ is a loop in $E \times S^1$, which
rotates twice around $S^1$. Therefore there exists a $t_0 (t_0 \in S^1)$ with $u(t_0) = u(t_0 + 1)$. We may assume $t_0 = 0$. So the set $\{(u(t), t); t \in S^1\}$ in $E \times S^1$ is a loop $\ell$, which may have a corner at $(u(0), 0)$. By the same argument as in the proof of Theorem 4 of Fukui [5], there exists a point $q = (u(t_1), t_1) \in \ell$ such that $H(F', \alpha(t_1))(q) = q$, that is, $q$ is a fixed point of $H_a(F')$.

We consider the behavior of $H^m_a(F')(q) (n \in \mathbb{Z})$ for a fixed point $q$ of $H_a(F')$.

**Lemma 9.** $H^m_a(F')(q)$ is a fixed point of $H_a(F')$ and belongs to $\tilde{l}$.

**Proof.** $H_a(F') \circ H^m_a(F') = H(F', \alpha \cdot \beta) = H(F', \beta \cdot \alpha) = H^m_a(F') \circ H_a(F')$. Thus $H_a(F')(H^m_a(F')(q)) = H_a(F')(H_a(F')(q)) = H^m_a(F')(q)$. Hence $H^m_a(F')(q)$ is a fixed point of $H_a(F')$. Since we have a unique $u(\tilde{t})$ with $u(\tilde{t}) = f(u(\tilde{t}), \tilde{t})$ for each $\tilde{t} \in S^1$, $H_a(F')(q)$ lies in $\tilde{l}$.

For the simplicity, we put $\varphi' = H_a(F')$. We denote by $\text{Fix}(H_a)$ the fixed point set of $H_a(F')$. Note that $\varphi'(\text{Fix}(H_a)) = \text{Fix}(H_a)$ and $\varphi'(\text{Fix}(H_a)) \subseteq \tilde{l}$ by Lemma 9.

Let $\overline{\varphi}: E \times S^1 \rightarrow \tilde{l}$ be the map defined by $\overline{\varphi}(u, \tilde{t}) = (u(\tilde{t}), \tilde{t})$ and $\varphi = \overline{\varphi} \circ \varphi' | \tilde{l}: \tilde{l} \rightarrow \tilde{l}$. Then we easily see that $\varphi$ is a diffeomorphism of $\tilde{l}$ and $\varphi$ and $\varphi'$ coincide on $\text{Fix}(H_a)$.

**Proposition 10.** There exists a point $p$ in $\tilde{l}$ such that $p$ is a fixed point of $H_a(F')$ and a periodic point of $H_a(F')$.

We prove Proposition 10 as follows. We suppose that $\varphi$ has no periodic points on $\text{Fix}(H_a)$. We introduce on $\tilde{l}$ a fixed orientation. If $a$ and $b$ are different points of $\tilde{l}$, the $\widehat{ab}$ denotes the oriented simple arc connecting $a$ with $b$, and the formula $a < c < b$ means that the point $c$ lies on the arc $\widehat{ab}$. Since $\varphi$ is $C^1$-close to the identity, $\varphi$ preserves the orientation.

Then we can prove Lemma 11 and use it to prove Lemma 12 similarly as in the proofs of Lemmas 1 and 2 of Siegel [14].

**Lemma 11.** Let $q$ be a point in $\text{Fix}(H_a)$. For a non-zero integer $m$ be given, then there exists an integer $h$ such that $q < \varphi^h(q) < \varphi^m(q)$.

We suppose that a point $p \in \text{Fix}(H_a)$ is not ergodic, that is, the orbit set $O(p) = \{\varphi^n(p); n \in \mathbb{Z}\}$ is not dense in $\tilde{l}$.

$\tilde{l} - O(p)$ is an open and non-empty set. Choose in $\tilde{l} - O(p)$ an open arc $\widehat{ab}$ whose end points belong to $O(p)$. The end points $a_n, b_n$ of all images arcs $\widehat{a_n b_n} = \varphi^n(\widehat{ab}) (n \in \mathbb{Z})$ lie in $O(p)$ and the inner points of these arcs lie in $\tilde{l} - O(p)$,
hence $\alpha_n b_n (n \in \mathbb{Z})$ are disjoint.

**Lemma 12.** Let $a_n, b_n (n \in \mathbb{Z})$ be as above. For an arbitrarily large natural number $N$, there exists an integer $m > N$ such that either the $m$-arcs $a_{m-k} b_{m-k}$ or $a_{m-1} b_{m-1}$ $(k=1, 2, \ldots, m)$ are disjoint.

By the similar argument as in Siegel [14], Lemma 12 leads us to a contradiction. Hence $O(p)$ is dense in $\mathcal{L}$. This implies $\text{Fix}(H_\alpha) = \mathcal{L}$. We put $\varphi'(u, \bar{t}) = (g_1(u, \bar{t}), g_2(u, \bar{t}))$ for $(u, \bar{t}) \in E_0 \times S^1$. From $(*)$, $H(F', \beta(t)) (u, t) = (g_1(u, t), g_2(u, t))$. Since we suppose that $\varphi'$ has no periodic points on $\text{Fix}(H_\alpha)$, the point $(u(0), 0)$ is not a fixed point of $\varphi'$ and if $g_2(u(0), 0) > 0$, then $g_2(u(1), 1) > 1$. On the other hand, $g_2(u(1), 1) < 1$ because $H(F', \beta(1)) = H(F', -\beta(0))$. This is a contradiction. Hence we have Proposition 10.

The following proposition is proved by the standard argument (cf. Langevin-Rosenberg [8]).

**Proposition 13.** If $p$ in $\mathcal{L}$ is a fixed point of $H_\alpha(F')$ and a periodic point of $H_\beta(F')$, then $L'_p$ is compact, where $L'_p$ is a leaf of $F'$ through $p$.

We complete the proof of Theorem 8 by Propositions 10 and 13.

**Remark 14.** Theorem 8 holds for $C^0$-foliations.

**Theorem 15.** Let $F$ and $U$ be as above. Suppose $h_* = \begin{pmatrix} \pm 1 & 1 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 2m & \pm 1 \end{pmatrix}$ $(l, m \in \mathbb{Z})$. Then there is a foliation $F'$ such that $F'$ is $C^r$-close to $F$ and $F'$ has no compact leaves in $U$.

**Proof.** We prove the case $h_* = \begin{pmatrix} \pm 1 & 1 \\ 0 & 1 \end{pmatrix}$. It is proved similarly for the case $h_* = \begin{pmatrix} 1 & 0 \\ 2m & \pm 1 \end{pmatrix}$. We consider the product $\text{Möb} \times S^1 \times \mathbb{R}$ with coordinate $(\theta, u, \varphi, t), (\theta, u) \in \text{Möb}, \varphi \in S^1$ and $t \in \mathbb{R}$, and define a foliation $G$ on $\text{Möb} \times S^1 \times \mathbb{R}$ to be the set of leaves whose tangent spaces are spanned by $\frac{\partial}{\partial \varphi}$ and $\frac{\partial}{\partial \theta}$. Let $\hat{H} : \text{Möb} \times S^1 \times \mathbb{R} \rightarrow \text{Möb} \times S^1 \times \mathbb{R}$ be a diffeomorphism defined by $\hat{H}(\theta, u, \varphi, t) = (\pm \theta + l \cdot \varphi, eu, \varphi, t+1), \epsilon = \pm 1$. Then $(U, F)$ is diffeomorphic to $(\text{Möb} \times S^1 \times \mathbb{R}, G)/\hat{H}$. Now we define a new foliation $G'$ on $\text{Möb} \times S^1 \times \mathbb{R}$ to be the set of leaves whose tangent spaces are spanned by $\frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial \varphi} + \lambda \frac{\partial}{\partial t}$, where $\lambda$ is a small irrational number. Then $\hat{H}$ preserves $G'$ because $\hat{H}_*(\frac{\partial}{\partial \theta}) = \pm \frac{\partial}{\partial \theta}$
and \( \hat{H}(\frac{\partial}{\partial \varphi} + \lambda \frac{\partial}{\partial t}) \frac{\partial}{\partial \varphi} + \lambda \frac{\partial}{\partial t} \). Hence we can define a foliation \( F_1 = \hat{G'}/\hat{H} \) on \( \text{Möb} \times S^3 \times \mathbb{R}/\hat{H} \). We can easily extend \( F_1 \) to \( F' \) on \( M \) such that \( F \) and \( F' \) are \( C^r \)-close \((r \geq 0)\). It is easy to see that \( F' \) has no compact leaves in \( U \).

§3. Instability of Foliations without Singular Leaves

In this section we consider the case \( F \) has no singular leaves, that is, \( \pi: M \to B \) is a fibre bundle. To begin with we show the following which is true for arbitrary closed manifolds \( B \).

**Proposition 16.** If the bundle \( \pi: M \to B \) is trivial and \( g \geq 2 \), then \( F \) is \( C^r \)-unstable \((r \geq 0)\).

**Proof.** Take diffeomorphisms \( f_1 \) and \( f_2 \) of \( B \) such that \( f_1 \) and \( f_2 \) are sufficiently \( C^r \)-close to \( 1_B \) and the periodic point sets of \( f_1 \) and \( f_2 \) are disjoint. Then we define a homomorphism \( \Phi: \pi_1(L) \to \text{Diff}(B) \) by \( \Phi(\{\alpha_i\}) = f_i \) \((i = 1, 2)\), \( \Phi(\{\alpha_i\}) = 1_B \) \((i = 3, 4, \ldots, g)\) and \( \Phi(\{\beta_i\}) = 1_B \) \((i = 1, 2, \ldots, g)\). This defines a foliation \( F' \) of \( M = L \times B \) whose leaves are transverse to the fibres of another fibre bundle \( \pi': M \to L \) with fibre \( B \). From the properties of \( f_i \), we see that \( F' \) is sufficiently \( C^r \)-close to \( F \) and has no compact leaves.

**Theorem 17.**

1. If \( B \) is homeomorphic to the 2-sphere \( S^2 \) and \( g \geq 2 \), then \( F \) is \( C^r \)-unstable \((r \geq 0)\).
2. If \( B \) is homeomorphic to the projective plane \( P^2 \) and \( g \geq 4 \), then \( F \) is \( C^r \)-unstable \((r \geq 0)\).
3. If \( B \) is neither homeomorphic to \( S^2 \) nor \( P^2 \) and \( g \geq 6 \), then \( F \) is \( C^r \)-unstable \((r \geq 0)\).

**Proof of (1).** In this case, it is an immediate consequence of Proposition 16 because that any bundle over \( S^2 \) with fibre \( L \) of genus \( \geq 2 \) is trivial.

**Proof of (2).** Any bundle over \( P^2 \) with fibre \( L \) of genus \( \geq 2 \) is obtained by making the identifications in \( L \times D^2 \) as follows: \((x, y) \sim (\varphi(x), -y)\) for \( x \in L \), \( y \in D^2 \), where \( \varphi: L \to L \) is a diffeomorphism with \( \varphi^2 = 1_L \).

**Step 1.** We perturb \( F \) on \( \pi^{-1}(D^2) \approx L \times \hat{D}^2 \) as follows. Let \( G: D^2 \to D^2 \) be a diffeomorphism satisfying the property \( P(D^2, r) \). Let \( Q \) be an open tubular neighborhood of \( \alpha_i \) in \( L \) such that its closure \( \bar{Q} \) is homeomorphic to \( \alpha_i \times [0, 1] \) with coordinate \((s, t)\), \( s \in \alpha_i \), \( 0 \leq t \leq 1 \). We start with the foliation of \((L-Q) \times \hat{D}^2 \) having leaves of form \((L-Q) \times \{y\}\), \( y \in \hat{D}^2 \) and make the identi-
fications \((s, 0, y) \sim (s, 1, G(y))\) to obtain a foliation of \(L \times \mathcal{D}^2\) having no compact leaves. Replacing \(F\) on \(\pi^{-1}(\mathcal{D}^2)\) by this foliation yields a new foliation \(F'\) which has no compact leaves on \(\pi^{-1}(\mathcal{D}^2)\).

**Step 2.** Take a point \(p\) of \(\partial \mathcal{D}^2/\sim\) and a conic tubular neighborhood \(A\) of \(\partial \mathcal{D}^2/\sim - p\) in \(\mathcal{D}^2/\sim\) as in Fig. 2.

![Figure 2](https://example.com/figure2.png)

We want to perturb \(F'\) on \(\pi^{-1}(A) \simeq L \times A\) to obtain a foliation \(F''\) such that 1) \(F''\) is sufficiently \(C^r\)-close to \(F\), 2) \(F''\) has no compact leaves on \(\pi^{-1}(\mathcal{P}^2-p)\) and 3) \(L_p = \pi^{-1}(p)\) is the only compact leaf of \(F''\). Since \(g \geq 4\), there exists a simple closed curve \(\delta\) on \(L\) in \(\pi^{-1}(\partial \mathcal{D}^2/\sim)\) such that \([\delta] \neq 0\) in \(H_1(L; \mathbb{Z})\) and \(\langle \delta, \alpha_i \rangle = (\delta, \varphi(\alpha_i)) = 0\) (see Plante [10]). In fact \(\langle \alpha_i, \varphi(\alpha_i) \rangle \beta_2 - \langle \beta_2, \varphi(\alpha_i) \rangle \alpha_2\) is homologous to a multiple of a simple closed curve. Let \(\tau\) be an arbitrary closed curve on \(L\). Then the holonomy of \(L\) in \(F'\) along \(\tau\) is trivial if and only if \(\langle \tau, \alpha_i \rangle = (\tau, \varphi(\alpha_i)) = 0\). Hence the holonomy of \(L\) in \(F'\) along \(\delta\) is trivial. Take a tubular neighborhood \(Q_\delta\) of \(\delta\) in \(L\) such that its closure \(\overline{Q}_\delta\times A\) is homeomorphic to \(\delta \times [0, 1] \times \{y\}, y \in A\). Thus we start with the foliation of \(\delta \times [0, 1] \times A - \delta \times (1/3, 2/3) \times A\) having leaves of form \(\delta \times [0, 1/3] \cup [2/3, 1] \times \{y\}, y \in A\) and make the identifications \((s, 1/3, y) \sim (s, 2/3, H(y))\) to obtain a foliation of \(\overline{Q}_\delta \times A\). Replacing \(F'\) on \(\overline{Q}_\delta \times A\) by this foliation, we obtain a required foliation \(F''\).

**Step 3.** Finally we perturb \(F''\) on a neighborhood of the compact leaf \(L_p\). Since \(g \geq 4\), there is a simple closed curve \(\eta\) on \(L_p\) such that \([\eta] \neq 0\) in
$H_1(L_p; \mathbb{Z})$ and (1) $\langle \eta, \alpha_i \rangle = \langle \eta, \delta \rangle = 0$, (2) $\langle \eta, \varphi(\alpha_i) \rangle = \langle \eta, \varphi(\delta) \rangle = 0$. For, put 
$$\eta = \sum_{i=1}^{k} m_i \alpha_i + \sum_{i=1}^{k} n_i \beta_i \ (m_i, n_i \in \mathbb{Z}).$$
Then the equations (1) are satisfied. So we consider the equations (2). We can solve (2) over integers since $g \geq 4$. Let $\eta_0$ be a non-trivial general solution of (2). Then we can choose a simple closed curve $\eta$ such that $\eta_0$ is homologous to a multiple of $\eta$ because $\eta = \sum_{i=1}^{k} m_i \alpha_i + \sum_{i=1}^{k} n_i \beta_i$ is realizable by a simple closed curve on $L_p$ if $m_3, \cdots, m_g, n_3, \cdots, n_k$ are relatively prime. The holonomy of $L_p$ in $F''$ along $\eta$ is trivial. Hence the rest of the proof is done similarly as in Step 2.

**Proof of (3).** We prove the case $B$ is orientable. It is proved similarly for the non-orientable case. Note that $B_1$ is a closed surface in this case.

**Lemma 18.** There exists a foliation $F_1$ of $M$ such that 1) $F_1$ is sufficiently $C^\infty$-close to $F$, 2) $F_1$ has no compact leaves on $\pi^{-1}(B-\{p_1, \cdots, p_{2h-1}\})$ and 3) $L_{p_i} = \pi^{-1}(p_i) \ (i = 1, 2, \cdots, 2h-1)$ are the compact leaves of $F_1$.

**Proof.** First we perturb $F$ on $\pi^{-1}(B) = L \times \tilde{B}^2$ as in Step 1 in the proof of (2) using $\alpha_1$ on $L$ and a diffeomorphism of $B$ satisfying the property $P(B_2, r)$. We let $F'$ be the resulting foliation. Next take conic tubular neighborhoods $A_i$, $A_{ij} \ (i=2, 3, \cdots, 2h-1; j = 1, 2)$ and $A_{2h,1}$ of $c_i$, $c_{ij}$ and $c_{2h,1}$ such that they are disjoint and the closures of $A_i$ and $A_{2j}$ have the only point $p_i$ in common ($j = 1, 2$) and the closures of $A_i$ and $A_{2j} \ (i = 2, \cdots, 2h-1)$ have the points $p_{i-1}$ and $p_i$ in common and the closures of $A_{2h-1,i}$ and $A_{2h,1}$ have the only point $p_{2h-1}$ in common ($j = 1, 2$) (see Fig. 3).

![Diagram](image-url)

Figure 3

There exist simple closed curves $\delta_i$, $\delta_{ij} \ (i = 2, \cdots, 2h-1; j = 1, 2)$ and $\delta_{2h,1}$
on the compact leaves $L$ of $F'$ in $\pi^{-1}(A_i)$, $\pi^{-1}(A_{ij})$ and $\pi^{-1}(A_{2h,1})$ respectively such that $[\delta_j]=0$, $[\delta_{ij}]=0$ and $[\delta_{2h,1}]=0$ in $H_1(L; \mathbb{Z})$ and $\langle \delta_j, \alpha_i \rangle = \langle \delta_{ij}, \varphi_i(\alpha_l) \rangle = 0$, $\langle \delta_{ij}, \alpha_l \rangle = \langle \varphi_{ij}(\alpha_l) \rangle = 0$ $(i=2, 3, \ldots, 2h-1; j=1, 2)$ and $\langle \delta_{2h,1}, \alpha_l \rangle = \langle \varphi_{2h,1}(\alpha_l) \rangle = 0$ (see Plante [10]). $[\delta_j]$, $[\delta_{ij}]$ and $[\delta_{2h,1}]$ can be considered to be expressed as linear combinations of $[\alpha_l]$ and $[\beta_i]$. Take diffeomorphisms $G_1$, $G_{ij}$ $(i=2, 3, \ldots, 2h-1; j=1, 2)$ and $G_{2h,1}: B \to B$ satisfying the properties $P(A_i, r)$, $P(A_{ij}, r)$ and $P(A_{2h,1}, r)$ respectively. Then, by the similar argument to Step 2 in the proof of (2), we can perturb $F'$ on $\pi^{-1}(A_i) \cong L \times A_i$, $\pi^{-1}(A_{ij}) \cong L \times A_{ij}$ and $\pi^{-1}(A_{2h,1}) \cong L \times A_{2h,1}$ using $\delta_1$, $\delta_{ij}$ and $\delta_{2h,1}$ to obtain a required foliation $F_1$.

**Lemma 19.** There exist simple closed curves $\eta_i$ on $L_{p_l}$ $(i=1, 2, \ldots, 2h-1)$ such that $[\eta_i]=0$ in $H_1(L_{p_l}; \mathbb{Z})$ and
\[
\begin{align*}
(1-1) & \quad \langle \eta_i, \alpha_l \rangle = \langle \eta_i, \delta_1 \rangle = \langle \eta_i, \delta_{ij} \rangle = 0 \quad (j=1, 2) \\
(1-2) & \quad \langle \eta_i, \varphi_{ij}(\delta_{ij}) \rangle = \langle \eta_i, \varphi_{2j}(\delta_{2j}) \rangle = 0 \quad (j=1, 2) \\
(1-3) & \quad \langle \eta_i, \varphi_{ij}(\alpha_{ij}) \rangle = \langle \eta_i, \varphi_{2j} \circ \varphi_{ij}(\alpha_{ij}) \rangle = \langle \eta_i, \varphi_{ij} \circ \varphi_{2j} \circ \varphi_{ij}(\alpha_{ij}) \rangle = 0 \\
(i-1) & \quad \langle \eta_i, \alpha_l \rangle = \langle \eta_i, \delta_{i+1,j} \rangle = \langle \eta_i, \delta_{i+1,j} \rangle = 0 \quad (j=1, 2) \\
(i-2) & \quad \langle \eta_i, \varphi_{ij}(\delta_{ij}) \rangle = \langle \eta_i, \varphi_{i+1,j}(\delta_{i+1,j}) \rangle = 0 \quad (j=1, 2) \\
(i-3) & \quad \langle \eta_i, \varphi_{ij}(\alpha_{ij}) \rangle = \langle \eta_i, \varphi_{i+1,j} \circ \varphi_{ij}(\alpha_{ij}) \rangle = \langle \eta_i, \varphi_{i+1,j} \circ \varphi_{ij}(\alpha_{ij}) \rangle = 0 \\
(2h-1-1) & \quad \langle \eta_{2h-1}, \alpha_l \rangle = \langle \eta_{2h-1}, \delta_{2h-1} \rangle = \langle \eta_{2h-1}, \delta_{2h,1} \rangle = 0 \quad (j=1, 2) \\
(2h-1-2) & \quad \langle \eta_{2h-1}, \varphi_{2j-1}(\delta_{2j-1}) \rangle = \langle \eta_{2h-1}, \varphi_{2h,1}(\delta_{2h,1}) \rangle = 0 \quad (j=1, 2) \\
(2h-1-3) & \quad \langle \eta_{2h-1}, \varphi_{2j-1}(\alpha_{ij}) \rangle = \langle \eta_{2h-1}, \varphi_{2h,1} \circ \varphi_{2j-1}(\alpha_{ij}) \rangle \\
& \quad = \langle \eta_{2h-1}, \varphi_{2j-1} \circ \varphi_{2h,1} \circ \varphi_{2j-1}(\alpha_{ij}) \rangle = 0
\end{align*}
\]

**Proof.** We prove for each $i$. Put $\eta = \sum_{i=3}^{2h-1} m_i \alpha_i + \sum_{i=1}^g n_i \beta_i$ $(m_i, n_i \in \mathbb{Z})$. Then the equations $(i-1)$ are satisfied. So we consider the system equations $(i-2)$ and $(i-3)$. If $g \geq 6$, we can solve the equations $(i-2)$ and $(i-3)$ over integers. Let $\eta_0$ be a non-trivial general solution of $(i-2)$ and $(i-3)$. Then we can choose a simple closed curve $\eta$ such that $\eta_0$ is homologous to a multiple of $\eta$ because $\eta = \sum_{i=3}^{2h-1} m_i \alpha_i + \sum_{i=1}^g n_i \beta_i$ $(m_i, n_i \in \mathbb{Z})$ is realizable by a simple closed curve on $L_{p_l}$ if $m_3, \ldots, m_g, n_3, \ldots, n_g$ are relatively prime.

**Proof of Theorem 17 (3) continued.** By Lemma 18, it is sufficient to perturb the compact leaves $L_{p_i}$ $(i=1, 2, \ldots, 2h-1)$. Take small neighborhoods $C_i$ of $p_i$ such that they are disjoint. From Lemma 19, there exist simple closed curves $\eta_i$ on $L_{p_l}$ satisfying the conditions $(i-1)$, $(i-2)$ and $(i-3)$. Thus
by the similar argument to Step 3 in the proof of (2), we can perturb \(F_i\) on \(\pi^{-1}(C_i)\) using \(\eta_i\) to obtain a foliation which has no compact leaves. This completes the proof.

§4. Instability of Foliations with Singular Leaves

In this section we consider the case \(F\) has singular leaves.

**Proposition 20.** Let \(F\) be a foliation of \(M\) such that \(B_0\) has \(m+n\) boundaries. If \(g \geq 2\), then there exists a foliation \(F_1\) of \(M\) such that \(F_1\) is sufficiently \(C^{r}\)-close to \(F\) and \(F_1\) has no compact leaves on \(\pi^{-1}(\hat{B}_0 - \{p_1, \ldots, p_s\})\), where \(s = 2h + m + n - 1\) if \(B\) is an orientable surface of genus \(h\) and \(s = 2h_1 + m + n\) if \(B\) is homeomorphic to \(\sum_{i=1}^{h} P^n_{i} - \bigcup_{i=1}^{m+n} D^2_i\) or \(\sum_{i=1}^{h_1} P^{2} - \bigcup_{i=1}^{m+n} D^2_i\) for an orientable surface of genus \(h_1\), \(\sum_{i=1}^{h_1}\).

**Proof.** It is similarly proved as in the proof of Lemma 18. Moreover we can perturb \(F\) on \(\pi^{-1}(\hat{B}_0)\) using \(\alpha_i, \pi^{-1}(\hat{e}_j)\) using \(\delta_i, \pi^{-1}(\hat{e}_{ij})\) using \(\delta_{ij}\) \((i = 2, \ldots, 2h - 2; j = 1, 2)\), \(\pi^{-1}(\hat{e}_{2h-1,j})\) using \(\delta_{2h-1,j}\) \((j = 1, \ldots, m + n + 1)\) and \(\pi^{-1}(\hat{d}_j)\) using \(\delta'_j\) \((j = 1, \ldots, m + n)\), where \([\delta]\)'s are expressed as linear combinations of \([\alpha_2]\) and \([\beta_3]\).

A) Case of foliations with only rotation leaves as singular leaf.

Note that \(F\) is a foliation which satisfies \(n = 0\) in §1 and \(B\) is a \(V\)-manifold without boundary.

**Theorem 21.** Let \(F\) be a foliation of \(M\) such that \(F\) has \(m\) rotation leaves \(L_1, \ldots, L_m\) with holonomy order \(k_1, \ldots, k_m\) respectively. If \(g \geq 3\max (k_i; 1 \leq i \leq m) + 1\), then \(F\) is \(C^{r}\)-unstable \((r \geq 0)\).

**Proof.** First we perturb \(F_i\) in Proposition 20 on each \(U(L_j) \simeq \pi^{-1}(V_j)\). Let \(\alpha_i = p_j(\alpha_i)\) and \(\delta'_j = p_j(\delta'_j)\) \((j = 1, \ldots, m)\), where \(p_j: L \to L_j\) is the covering map. Let \(\tau_j\) \((j = 1, 2, \ldots, m)\) be simple closed curves on \(L_j\) representing generators of the holonomy group \(\pi_k\) of \(L_j\). If \(g(L_j) \geq 4\), then there exist simple closed curves \(\eta_j\) on \(L_j\) such that \([\eta_j] \neq 0\) in \(H_1(L_j; \mathbb{Z})\) and \(\langle \eta_j, \alpha_i \rangle = \langle \eta_j, \delta'_j \rangle = \langle \eta_j, \tau_j \rangle = 0\) respectively. Note that \(p_j \circ p_j(\alpha_i) = p_j(\alpha_i)\). Take tubular neighborhoods \(S_j\) of \(\eta_j\) in \(L_j\) which are homeomorphic to \(\eta_j \times (0, 1)\) respectively. By Proposition 3 (a) of Vogt [17], the normal disk bundles \(U(L_j)\) are trivial. Hence the bundles over \(S_j\) \((j = 1, 2, \ldots, m)\) are the products \(S_j \times D^3\) respectively. Thus by the similar argument to Step 2 in the proof of Theorem 17 (2), we
perturb $F_1$ on $U(L_j)$ to obtain a foliation $F_2$ which has no compact leaves on $\pi^{-1}(B-\{p_1, \ldots, p_s\})$. We easily see that $g(L_j) \geq 4$ if and only if $g \geq 3k_j+1$ ($j=1, \ldots, m$). Finally we perturb $L_{p_j}=\pi^{-1}(p_j)$ ($i=1, \ldots, s$) similarly as in Step 3 of the proof of Theorem 17 (2). For this purpose, it is sufficient to take simple closed curves $r_i$ on $L_{p_j}$ ($i=1, \ldots, s$) such that the holonomy along each $r_i$ is trivial. This is possible because $3\max(k_i; 1 \leq i \leq m)+1 \geq 6$ (see Lemma 19). This completes the proof.

**B) Case of foliations with only reflection leaves as singular leaf.**

Note that $F$ is a foliation which satisfies $m=0$, $n=n''$ in §1 and $B$ is a smooth manifold with $n$ boundary components.

**Theorem 22.** If $g \geq 7$, then $F$ is $C'$-unstable ($r \geq 0$).

**Proof.** We take conic tubular neighborhoods $A_j$ of $\partial_j B-q_j$ ($j=1, \ldots, n$) in $B$ and want to perturb $F_1$ in Proposition 20 on each $\pi^{-1}(A_j)$ which is homeomorphic to $L \times \hat{D}^2$, where for $g \in D, g \neq 1$, $g: \hat{D}^2 \to \hat{D}^2$ is defined by $g(x, y)=(x, -y)$ for $(x, y) \in \hat{D}^2$. Let $p: L \to L_0=\pi^{-1}(\partial_j B-q_j)$ be the covering map and $\alpha_i=p(\alpha_i)$. We put $\alpha_i=\alpha$ if $p(\alpha_i)$ is homologous to a twice of a simple closed curve $\alpha$. We take a simple closed curve $\delta_j'$ on $L_0$ such that $[\delta_j']=0$ in $H_1(L_0; \mathbb{Z})$ and $\langle \delta_j', \alpha_i \rangle = 0$. We can assume that $[\delta_j']$ is expressed as a linear combination of $[\bar{\alpha}_2]$ and $[\bar{\beta}_2]$, where $[\bar{\alpha}_i]$ and $[\bar{\beta}_i]$ form a symplectic basis for $H_1(L_0; \mathbb{Z})$. Then taking a diffeomorphism $G_j: \hat{D}^2 \to \hat{D}^2$ satisfying the property $P(D^2, r)$ and $G_j(x, y)=(G'_j(x, y), y)$, $G_j(x, -y)=G'_j(x, y)$, we can perturb $F_1$ on $\pi^{-1}(A_j)$ using $\delta_j'$ to obtain a foliation $F_2$ which has no compact leaves on $\pi^{-1}(B-\{p_1, \ldots, p_s, q_1, \ldots, q_n\})$. We can perturb the compact leaves $L_{p_i}=\pi^{-1}(p_i)$ ($i=1, \ldots, s$) in the usual way. Finally we want to perturb the compact leaves $L_{q_j}=\pi^{-1}(q_j)$ ($j=1, \ldots, n$). It is sufficient to take simple closed curves $\eta_j$ on $L_{q_j}$ such that $[\eta_j]=0$ in $H_1(L_{q_j}; \mathbb{Z})$ and $\langle \eta_j, \alpha_i \rangle = \langle \eta_j, p(\delta_j') \rangle = \langle \eta_j, \psi_j(\alpha_i) \rangle = \langle \eta_j, \psi_j(\delta_j') \rangle = 0$. This is possible because of $g(L_0) \geq 4$. $g(L_0) \geq 4$ if and only if $g \geq 7$. This completes the proof.

**C) Case of foliations with dihedral leaves.**

We consider the case $F$ has no rotation leaves and some points of each boundary $\partial_j B$ of $B$ correspond to dihedral leaves $L_{j,k}$ with holonomy groups $D_{k,j,k}$. We take $\eta_j$ on $L_{q_j}$ such that $[\eta_j]=0$ in $H_1(L_{q_j}; \mathbb{Z})$ and $\langle \eta_j, \alpha_i \rangle = \langle \eta_j, p(\delta_j') \rangle = \langle \eta_j, \psi_j(\alpha_i) \rangle = \langle \eta_j, \psi_j(\delta_j') \rangle = 0$. This is possible because of $g(L_0) \geq 4$. $g(L_0) \geq 4$ if and only if $g \geq 7$. This completes the proof.

**Theorem 23.** If $g \geq 8\max(l_j, k; 1 \leq j \leq n; 1 \leq k \leq m_j)+1$, then $F$ is $C'$-unstable ($r \geq 0$).
Proof. We can perturb $F_1$ in Proposition 20 on each $\pi^{-1}(\partial_j B)-\bigcup_{k=1}^{m_j} L_{j,k}$ using a simple closed curve $\delta'_{j,k}$ on a reflection leaf in $\pi^{-1}(\partial_j B)-\bigcup_{k=1}^{m_j} L_{j,k}$ as in the proof of Theorem 22, where $[\delta'_{j,k}]$ can be considered to be expressed as linear combinations of $[\alpha_2]$ and $[\beta_2]$. We let $F_2$ be the resulting foliation. Next we want to perturb $F_2$ on saturated tubular neighborhoods $U(L_{j,k})$ of $L_{j,k}$. Let $p_{j,k}: L \to L_{j,k}$ be the covering map and $\lambda$ and $\mu$ simple closed curves on $L_{j,k}$ which represent generators of the holonomy group of $L_{j,k}$. We need the following lemma.

Lemma 24. There exist simple closed curves $\eta_{j,k}$ on $L_{j,k}$ such that 1) $[\eta_{j,k}] \neq 0$ in $H_1(L_{j,k}; \mathbb{Z})$, 2) $\langle \eta_{j,k}, p_{j,k}(\alpha_i) \rangle = \langle \eta_{j,k}, p_{j,k}(\alpha_2) \rangle = \langle \eta_{j,k}, p_{j,k}(\beta_2) \rangle = \langle \eta_{j,k}, \lambda \rangle = \langle \eta_{j,k}, \mu \rangle = 0$ and 3) $U(L_{j,k})$ restricted to $\eta_{j,k}$ are trivial disk bundles.

Proof. By the result of Vogt [17], List 1, Propositions 5 and 6, $(U(L_{j,k}), F)$ is represented by the vector $(v, 1, u, 1, \cdots, 1)$ or $(v, u^{j+1}, u, 1, \cdots, 1)$ (see [17] for details). We may assume that $p_{j,k}(\alpha_1), p_{j,k}(\alpha_2), p_{j,k}(\beta_2), \lambda$ and $\mu$ are represented by $[\alpha_i], [\beta_i]$ ($i=1, 2, 3, 4$) where $[\alpha_i], [\beta_i]$ ($i=1, 2, \cdots, g(L_{j,k}))$ form a canonical symplectic basis for $H_2(L_{j,k}; \mathbb{Z})$. We can take the simple closed curves $\alpha_5$ and $\beta_5$ which satisfy 1) and 2) since $g \geq 8\max(1, 1 \leq j \leq n, 1 \leq k \leq m_j)+1$ implies $g(L_{j,k}) \geq 5$. If $U(L_{j,k})$ restricted to $\alpha_5$ and $\beta_5$ are non-trivial respectively, there is a simple closed curve $\eta$ on $L_{j,k}$ such that $U(L_{j,k})$ restricted to $\eta$ is trivial and $[\eta] = [\alpha_5]+[\beta_5]$. We put $\eta_{j,k} = \eta$.

Now we continue the proof of Theorem 23. We perturb $F_2$ on $U(L_{j,k})$ using $\eta_{j,k}$ in the usual way to obtain a foliation $F_3$ which has no compact leaves on the $\pi^{-1}(B-\{p_1, \cdots, p_s\})$. It is easy to perturb the compact leaves $L_{p_i} = \pi^{-1}(p_i)$ $(i=1, \cdots, s)$. This completes the proof.

D) General case.

Combining Theorems 21, 22 and 23, we have the following.

Theorem 25. Let $F$ be a foliation of a closed 4-manifold $M$ by closed orientable surfaces and $B=M/F$ the leaf space. Suppose $F$ has $m$ rotation leaves with holonomy groups $\mathbb{Z}_{h_i}$ ($i=1, 2, \cdots, m$) and $m_j$ dihedral leaves with holonomy groups $D_{l_j,k}$ ($k=1, 2, \cdots, m_j$) which correspond to points of $\partial_j B$ for each $j$ ($1 \leq j \leq n'$).

If $g \geq 3\max(\{k_i; 1 \leq i \leq m\})+1, 8\max(l_j,k; 1 \leq j \leq n', 1 \leq k \leq m_j)+1, 7\varepsilon)$, then $F$ is $C'$-unstable ($r \geq 0$), where $\varepsilon=0$ or 1 and $F$ has no reflection leaves if
and only if \( s = 0 \).

**Remark 26.** If \( g \geq 2 \) and \( g \) is even, then \( F \) has only rotation leaves as singular leaf. Hence Theorem 25 reduces to Theorem 21.

**References**


