Criteria for Hypoellipticity of Differential Operators

By

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Introduction and Main Results

In the previous paper [12] (see also [10]), the author has given a sufficient condition for the hypoellipticity of differential operators of second order. It was also proved there that the sufficient condition is necessary for a special class of differential operators. The result about the necessity was extended to operators of higher order. The main purpose of the present paper is to extend the sufficient condition given in [12] to be applicable for differential operators of higher order.

Let \( P = p(x, D_x) \) be a differential operator of order \( m \geq 1 \) with coefficients in \( \mathcal{B}^m(\mathbb{R}^n) \). Here \( \mathcal{B}^m(\mathbb{R}^n) \) denotes the set of \( C^\infty(\mathbb{R}^n) \)-functions whose derivatives of any order are all bounded in \( \mathbb{R}^n \). Let \( A \) and \( \log A \) denote pseudodifferential operators with symbols \( <f> \) and \( \log <\xi> \), respectively, where \( <\xi> = (1 + |\xi|^2)^{1/2} \). Set \( ||u||_s = ||A^s u|| \) for real \( s \) and \( u \in C^\infty_0(\mathbb{R}^n) \), where \( ||\cdot|| \) denotes the \( L^2 \) norm. We write \( p[a,b](x, \xi) = \partial_x^\alpha \partial_\xi^\beta p(x, \xi) \) for multi-indices \( \alpha \) and \( \beta \). About the other notations used in the present paper we refer to Kumano-go [5].

Theorem 1. Assume that for any \( \varepsilon > 0 \) and any compact set \( K \) of \( \mathbb{R}^n \) there exists a constant \( C_{\varepsilon, K} \) such that

\[
\|(\log A)^m u\| + \sum_{0 < |\alpha + \beta| < m} ||(\log A)^{\alpha + \beta} P^{[s]}(\beta) u|| < \varepsilon ||Pu|| + C_{\varepsilon, K} ||u|| , \quad u \in C^\infty_0(K),
\]

where \( m \) is the order of \( P \) and \( P^{[s]}(\beta) = p^{[s]}(x, D_x) \). Then \( P \) is hypoelliptic in \( \mathbb{R}^n \).

Furthermore we have

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As a corollary to our main theorem, we have Theorem 1 of [12].

**Corollary 2.** Let $P$ be a differential operator of second order, that is,

$$P = \sum_{j,k} a_{jk}(x)D_jD_k + \sum_j b_j(x)D_j + c(x), \quad D_j = -i\partial_{x_j}.$$  

We assume that

\begin{equation}
\left\{ \begin{array}{l}
a_{jk} \text{ and } b_j \text{ are real valued,} \\
\sum_{j,k} a_{jk}(x)\xi_j\xi_k \geq 0 \quad \text{for any } (x, \xi) \in \mathbb{R}^{2n}.
\end{array} \right.
\end{equation}

If for any $\varepsilon > 0$ and any compact set $K$ of $\mathbb{R}^n$ the estimate

\begin{equation}
\| (\log A)^2 u \| \leq \varepsilon \| Pu \| + C_{\varepsilon,K} \| u \|, \quad u \in C^\infty_0(K)
\end{equation}

holds with a constant $C_{\varepsilon,K}$ then we have (2).

Theorem 1 seems to be applicable to a large class of hypoelliptic operators with finite degeneracy because for such operators we generally have the subelliptic estimate

$$\| u \|_x + \sum_{0 < |\alpha| + |\beta| < \infty} \| P^{(\beta)} u \|_{x-|\beta|} \leq C_{\varepsilon,K}(\| Pu \| + \| u \|), \quad u \in C^\infty_0(K)$$
for some $\varepsilon > 0$. (cf. [2], [13]). Theorem 1 also applies to a class of elliptic operators with infinite degeneracy. In fact, as discussed in [12] and [11], Theorem 1 (Corollary 2) is applicable to show the hypoellipticity of a second order differential operator

$$L_1 = D^2_{x_1} + D^2_{x_2} + \exp(-1/|x_1|^{\delta})D^2_{x_3}, \quad 0 < \delta < 1.$$  

Now we have:

**Proposition 3.** Let $L$ be a differential operator

$$D^{2l}_{x_1} + D^{2l}_{x_2} + \exp(-1/|x_1|^{\delta})D^{2l}_{x_3} \quad \text{in } \mathbb{R}^3,$$
where $\delta > 0$ and $l = 1, 2, \ldots$. Then $L$ is hypoelliptic in $\mathbb{R}^3$ if (and only if)

\begin{equation}
\delta < 1.
\end{equation}

**Remark 1.** The proposition for the case $l = 1$ was first proved by Kusuoka-Strook [6] (see Theorem 8.41 of [6]), by using the Malliavin calculus. About the Malliavin calculus we refer to Malliavin [7] and Ikeda-Watanabe [4].

**Remark 2.** The necessity of (4) follows from Theorem 3 of [9]. For
the proof of the necessity of (4) we can also employ Theorem 3 of [12] and Theorem 2 of Hoshiro [3].

**Remark 3.** It should be noted that a differential operator $\mathcal{A} = D_{x_1}^{2l} + \exp(-1/|x_1|^8)D_{x_1}^{2l}$ is hypoelliptic in $\mathbb{R}^2$ for any $\delta > 0$ (see Theorem 1.1 of [8], cf. Fedii [1]).

Unfortunately, we cannot apply Theorem 1 directly to the proof of Proposition 3, because it is quite hard to check the hypothesis (1) for $L$, more precisely, to show for the case $l \geq 2$

$$\|(\log A)D_{x_1}^{2l-1}u\| \leq \varepsilon\|Lu\| + C_{s,\varepsilon}||u||, \quad u \in C^n(K).$$

So, we need the following amelioration of Theorem 1 under an additional assumption.

**Theorem 4.** Assume that the principal symbol $p_m(x, \xi)$ of $P$ satisfies

(5) $p_m(x, \xi) \neq 0$ for $x' \neq 0$, where $x = (x', x'')$.

Then the conclusion (2) of Theorem 1 still holds when for any $\varepsilon > 0$ and any compact set $K$ of $\mathbb{R}^n$ the estimate

(6) $\|(\log A)^m u\| + \sum_{0 < |\alpha| + |\beta| < m} \|(\log A)^{|\alpha|+|\beta|} P_{(\beta)} u\| \leq \varepsilon\|Pu\| + C_{s,K}||u||, \quad u \in C^n(K)$

holds with a constant $C_{s,K}$.

The hypoelliptic operator $\mathcal{A}$ with $\delta \geq 1$ in Remark 3 is not covered by Theorem 4, because the estimate (6) does not hold for some small $\varepsilon > 0$ (see [9] and Remark 3.1 of [12]). To cover this exceptional example we give another criterion of hypoellipticity.

**Theorem 5.** Assume that the principal symbol $p_m(x, \xi)$ of $P$ satisfies (5). If for any compact set $K$ of $\mathbb{R}^n$ there exist a $\kappa_0 > 0$ and a constant $C_K$ such that

(7) $\|u\| + \sum_{0 < |\alpha| + |\beta| \leq m} \|P_{(\beta)} u\| \kappa_{0^{-1}\beta 1} \leq C_K(||Pu|| + ||u||_{-1}), \quad u \in C^n_0(K)$,

then we have (2).

This paper consists of three sections. In Section 1 we prove Theorem 1 and Corollary 2. Section 2 is devoted to proofs of Theorems 4 and 5.
Section 3 we give the proof of Proposition 3 and show that the condition (7) is satisfied for $A$ given in Remark 3.

§ 1. Proofs of Theorem 1 and Corollary 2

First we shall prove Corollary 2, admitting Theorem 1. By Theorem 1 it suffices to show that $\|((\log A)P)^{\alpha}u\| (|\alpha|=1)$ and $\|((\log A)P)^{\beta}u\|_{-1} (|\beta|=1)$ are estimated above by the right hand side of (1). As proved in [13], it follows from the assumption (*) that for any compact set $K$ we have

$$\sum_{|\alpha|=1} \|P^{(\alpha)}u\|^2 \leq C_K(\text{Re}(Pu, u)+\|u\|^2), \quad u \in C^\infty_0(K).$$

Indeed, this estimate follows from (2.6.6) and (2.6.9) of [13]. For $u \in C^\infty_0(K)$, take $\phi, \psi \in C^\infty_0(R^n)$ such that $\phi = 1$ in a neighborhood of $K$ and $\psi \in C^\infty_0(R^n)$ (that is, $\psi = 1$ in a neighborhood of $\text{supp }\phi$). Replace $u$ in the above estimate by $\psi(\log A)\phi u \in C^\infty_0(K_0)$, where $K_0 = \text{supp }\psi$. Then we have

$$\sum_{|\alpha|=1} \|P^{(\alpha)}(\log A)u\|^2 \leq C_K(\text{Re}(P(\log A)u, (\log A)u)
+ \|((\log A)u\|^2+\|u\|^2), \quad u \in C^\infty_0(K),$$

because $(1-\psi(\log A)\phi$ belongs to $S^\infty_0$ (see Chapter 2 of [5]). Since the principal symbol of $[P, \log A]$ is purely imaginary we have

$$\text{Re}([P, \log A]u, (\log A)u) \leq C(||(\log A)^2u||^2+||u||^2), \quad u \in C^\infty_0(K).$$

Above two estimates, together with Schwartz's inequality, show that the estimate

$$\sum_{|\alpha|=1} ||(\log A)P^{(\alpha)}u||^2 \leq C_K(\text{Re}(Pu, (\log A)^2u)+ ||(\log A)^2u||^2+||u||^2)$$

$$\leq \mu ||Pu||^2+C_{\mu,A}((||\log A\|^2||^2+||u||^2), \quad u \in C^\infty_0(K)$$

holds for any small $\mu>0$. Set $\epsilon = (\mu/2C_{\mu,A})^{1/2}$ in (3). Then we obtain

$$\sum_{|\alpha|=1} ||(\log A)P^{(\alpha)}u||^2 \leq 2\mu ||Pu||^2+C_{\mu,A}||u||^2, \quad u \in C^\infty_0(K).$$

Since it follows from (2.16.14) of [13] that we have

$$\sum_{|\beta|=1} ||A^{-1}P^{(\beta)}u||^2 \leq C(\sum_{j=1} \text{Re}(A^{-1}D^jPu, A^{-1}D^ju)+||u||^2), \quad u \in C^\infty_0(K),$$

the similar discussion as above shows that $\|((\log A)P^{(\beta)}u\|_{-1} (|\beta|=1)$ is estimated above by the right hand side of (1). We have completed the proof of Corollary 2.
From now on we shall prove Theorem 1. Let $h(x)$ be a $C_0^\infty$-function such that $h=1$ on $|x| \leq 1/5$ and $\text{supp } h \subset \{|x| < 7/24\}$. Write

$$p(x, \xi) = \sum_{k=0}^m p_k(x, \xi),$$

where $p_k$ is positively homogeneous in $\xi$ of degree $k$. For $\tau \equiv (x_0, \xi_0) \in \mathbb{R}^n \times S^{n-1}$ we consider a microlocalized pseudodifferential operator of $P$ as follows:

$$P_\tau = p_\tau(y, D_y; \lambda) = \sum_{k=0}^m p_k(x_0 + \lambda y, \xi_0 + \lambda D_y) h(\lambda D_y/3) \lambda^{-2k}$$

where $\lambda$ is a positive small parameter (see Hörmander [2] and Section 1 of [12]). It is clear that the symbol of $P_\tau$, $p_\tau(y, \eta; \lambda) \equiv \sigma(P_\tau)$ satisfies for any $\alpha$ and $\beta$

$$|\partial^\alpha_x D^\beta_{\eta} p_\tau(y, \eta; \lambda)| \leq C_{\alpha\beta} \lambda^{-2m + |\alpha + \beta|},$$

where $C_{\alpha\beta}$ is a constant independent of $\lambda$. From the estimate (1) we have:

Lemma 1.1. For any $\varepsilon > 0$ and any $\tau \equiv (x_0, \xi_0) \in \mathbb{R}^n \times S^{n-1}$ there exists a constant $C(\varepsilon, \tau)$ such that

$$||\log(\lambda^{-1})^m ||Hv|| + \sum_{0<|\alpha + \beta| < \xi} \lambda^{1/2}(\log \lambda^{-1})^{1/2} ||H \sigma(P_\tau)v||$$

$$\leq \varepsilon ||H_0 P_\tau v|| + C(\varepsilon, \tau) \sum_{0<|\alpha + \beta| < \xi} \lambda^{1/2} ||(H_0 P_\tau v)|| ||v||,\quad v \in \mathcal{S}_x, \quad 0<\lambda \leq 1,$$

where $H = h(\lambda D_y)h(\lambda y)$ and $H_0 = h(\lambda D_y/2)h(\lambda y/2)$. Here $P_{\tau}^{(\xi)}$ are pseudodifferential operators with symbols $\sigma(P_{\tau}^{(\xi)}) \equiv \sigma^\alpha_x D^\beta_{\eta} p_\tau(y, \eta; \lambda)$.

Proof. Note that $\{h(\lambda^{2}\xi - \xi_0); 0<\lambda \leq 1\}$ is a bounded set in $S^1_0$, as a pseudodifferential operator in $R^*_x$, because $\lambda^2 < (4/3) |\xi|^{-1}$ on supp $h(\lambda^{2}\xi - \xi_0)$. Replace $u$ in (1) by $h(x-x_0)h(\lambda^{2}D_x - \xi_0)w = h(x-x_0)h(\lambda^{2}D_x - \xi_0)h(\lambda^{2}D_x - \xi_0/3)w)$, $w \in \mathcal{S}_x$. Then, by means of the symbolic calculus of pseudodifferential operators we have

$$||(\log A)^m h(\lambda^{2}D_x - \xi_0)h(x-x_0)w||$$

$$+ \sum_{0<|\alpha + \beta| < \xi} ||A^{-\beta} (\log A)^{\alpha + \beta} h(\lambda^{2}D_x - \xi_0)h(x-x_0)P_{\tau}^{(\xi)} h(\lambda^{2}D_x - \xi_0/3)w||$$

$$\leq \varepsilon ||h((\lambda^{2}D_x - \xi_0)/2)h((x-x_0)/2)P_{\tau}^{(\xi)} h(\lambda^{2}D_x - \xi_0/3)w||$$

$$+ C_{\alpha\beta} \sum_{0<|\alpha + \beta| < \xi} ||A^{-|\beta|} h((\lambda^{2}D_x - \xi_0)/2)h((x-x_0)/2)P_{\tau}^{(\xi)} h(\lambda^{2}D_x - \xi_0/3)w||,$$

because $(\log A)^j A^{-1} \in S^0_{1,0}$ for $j = 1, \ldots, m$ and $h(x/2) = 1$ on supp $h(x)$. Set
\[ w(x) = (\exp(i \lambda^{-2} x \cdot \xi_0) v(\lambda^{-1}(x-x_0))) \text{ for } v(y) \in \mathcal{S}_\gamma. \] Then we have

\[ \exp(-i \lambda^{-2} x \cdot \xi_0) \rho(x, D_x) h((\lambda^2 D_x - \xi_0)/3) w(x) = (P_\eta v)(\lambda^{-1}(x-x_0)) \]

by noting the change of variables.

\[ x-x_0 = \lambda y, \quad \xi - \lambda^{-2} \xi_0 = \lambda^{-1} \eta. \]

Similarly we have for any real \( s \)

\[ \exp(-i \lambda^{-2} x \cdot \xi_0) A^s h(\lambda^2 D_x - \xi_0) w(x) \]

\[ = \lambda^{-2s} (q(D_x; \lambda) h(\lambda D_x) v)(\lambda^{-1}(x-x_0)). \]

where \( q(\eta; \lambda) = (\lambda^4 + |\lambda \eta + \xi_0|)^{1/2} \). It is clear that \( q(\eta; \lambda) h(\lambda \eta) \geq (2/3)^j h(\lambda \eta) \) and \( \{q(\eta; \lambda) h(\lambda \eta/2); 0 < \lambda \leq 1\} \) is a bounded set in \( S^1_{\delta_0} \), as a pseudodifferential operator in \( \mathbb{R}^n \).

Furthermore we have

\[ \exp(-i \lambda^{-2} x \cdot \xi_0) (\log A^s h(\lambda^2 D_x - \xi_0)) w(x) \]

\[ = ((\log \lambda^{-4} + r(D_x; \lambda)) h(\lambda D_x) v)(\lambda^{-1}(x-x_0)), \]

where \( r(\eta; \lambda) = \log q(\eta; \lambda) \). Since \( \{r(\eta; \lambda) h(\lambda \eta/2); 0 < \lambda \leq 1\} \) \( (j = 1, \ldots, m) \) are bounded sets in \( S^1_{\delta_0} \) we have

\[ ||r(D_x; \lambda) h(\lambda D_x) v|| \leq m^{-2m}(\log \lambda^{-1})^j ||h(\lambda D_x) v||, \quad v \in \mathcal{S}_\gamma, \]

if \( 0 < \lambda \leq \lambda_1 \) for a sufficiently small \( \lambda_1 \). In view of (1.5)-(1.8), it follows from (1.4) that

\[ \log \lambda^{-1} ||Hv|| + \sum_{0 < |\omega + \beta| \leq n} 2|\beta||H(P^{(\eta)}_{\gamma}) v|| \]

\[ \leq m \epsilon ||P_{\gamma} v|| + C \delta \sum_{0 < |\omega + \beta| \leq n} \lambda^{|\beta||H(P^{(\eta)}_{\gamma}) v|| ||v||}, \quad \epsilon \in \mathcal{S}_\gamma, \text{ if } 0 < \lambda \leq \lambda_1, \]

where \( (P^{(\eta)}_{\gamma}) v \) is defined by the same formula as (1.1) with \( p(x, \xi) \) replaced by \( p^{(\eta)}(x, \xi) \). Note that

\[ \sigma(P^{(\eta)}_{\gamma}) = \sigma^c(D_x^a p_{\gamma}(y, \eta; \lambda) = \lambda^{-|\omega|+|\beta|} \sigma((P^{(\eta)}_{\gamma}) y) \text{ if } |\lambda \eta| \leq 3/5. \]

Since \( \text{supp } h(\lambda \eta) \) and \( \text{supp } h(\lambda \eta/2) \) are contained in \( \{\eta; |\lambda \eta| \leq 3/5\} \), the symbolic calculus of pseudodifferential operators shows that (1.3) follows from (1.9) if \( 0 < \lambda \leq \lambda_1 \). The estimate (1.3) for \( \lambda_1 < \lambda \leq 1 \) is trivial. Q.E.D.

For a real \( \kappa > 0 \) and an integer \( k > 0 \) we denote by \( A_{\kappa,k} \) a pseudodifferential operator with a symbol \( (1+\kappa \xi)^{-k} \). It is easy to check that for any \( \alpha \) the estimate
\[
\text{(1.10)} \quad |\partial_x^\alpha((1 + \kappa \langle \xi \rangle)^{-\lambda})| \leq C_\alpha \langle \xi \rangle^{-\alpha} (1 + \kappa \langle \xi \rangle)^{-\lambda}
\]
holds with a constant \(C_\alpha\) independent of \(\kappa\). Set
\[

\mathcal{A}_\alpha(\eta; \lambda) = (1 + \kappa \langle \xi \rangle)^{-\lambda} h(\lambda \eta).
\]
Then it follows from (1.10) that for any \(\alpha\) the estimate
\[
\text{(1.12)} \quad |\partial_x^\alpha \mathcal{A}_\alpha(\eta; \lambda)| \leq C_\alpha \lambda^{\alpha+1} k_\alpha(\eta; \lambda), \quad \lambda |\eta| \leq 1/5
\]
holds with another constant \(C_\alpha\) independent of \(\kappa\) and \(\lambda\).

Set \(h_b(x) = h(x/\delta)\) for a small \(0 < \delta \leq 1/10\). Fix an integer \(N > m\). Take a sequence \(\{h_{b_j}\}_{j=1}^{N-m+1} \subset C_0^\infty(\mathbb{R}^n)\) such that
\[
\text{(1.13)} \quad h_{b_8} = h_{b_1} \subset h_{b_2} \subset \cdots \subset h_{b_8}^{N-m+1} = h_{b_8}
\]
and for any \(\alpha\) the estimate
\[
\text{(1.14)} \quad |D_x^\alpha h_{b_8}(x)| \leq C_\alpha \lambda^{\alpha+1},
\]
holds with a constant \(C_\alpha\) independent of \(N\) and \(j\) (\(C_\alpha = 1\)).

**Lemma 1.2.** Write
\[
\text{(1.15)} \quad h_{b_j}(\lambda D_x) \mathcal{A}_\alpha(D_x; \lambda) h_{b_j}(\lambda y) h_{b_j}^{\alpha+1}(\lambda D_x) h_{b_j}^{\alpha+1}(\lambda y)
\]
\[
= h_{b_j}(\lambda D_x) \mathcal{A}_\alpha(D_x; \lambda) h_{b_j}(\lambda y) + r(y, D_x; \lambda).
\]
Then for any integer \(l > 0\) there exists a constant \(C_l\) independent of \(\lambda\), \(\kappa\) and \(N\) such that
\[
\text{(1.16)} \quad ||r(y, D_x; \lambda)v|| \leq C_l \lambda^{2l} N^{2l+2m+2} ||v||, \quad v \in S_y.
\]

**Proof.** Note (1.12) and (1.14). Then (1.16) follows from the symbolic calculus of pseudodifferential operators and the Carderon-Vaillancourt theorem (See Chapter 7 of [5]). Q.E.D.

As in [12] we state the following simple proposition:

**Proposition 1.3.** Let \(N\) be a fixed positive integer and let \(\lambda\) satisfy \(0 < \lambda \leq 1\). For any finite sequence of positive numbers \(\{\lambda_1\}_{i=1}^l\) there exists a constant \(C_l\) such that
\[
\text{(1.17)} \quad \sum_{i=1}^l C_l (N\lambda)^{2i} \leq 1 + C_l (N\lambda)^{2i}.
\]

**Proof.** is omitted. (See [12]).

**Lemma 1.4.** Set \(\bar{H}_b = h_b(\lambda D_x) \mathcal{A}(D_x; \lambda) h_b(\lambda y)\) for \(j = 1, \cdots, N-m+1\). Then
for any \(s > 0\) and any \(\tau = (x_0, \xi_0) \in \mathbb{R}^d \times S^{d-1}\) there exists a constant \(C(s, \tau)\) independent of \(\lambda, \kappa, j\) and \(N\) such that

\[
(1.18) \quad \log \lambda^{-s}n \|\hat{H}_b \delta v\| + \sum_{0 < |\alpha| < s \leq \lambda} \log \lambda^{-s}n \|P_{\gamma(x)} \hat{H}_b \delta v\| 
\]

\[
\leq \|P_\tau \hat{H}_b \delta v\| + C(s, \tau) \lambda^s N^{s+2n+3m} \|v\|, \\
v \in S_y, \text{ if } 0 < \lambda \leq \lambda_0(s, \tau),
\]

where \(\lambda_0(s, \tau)\) is a sufficiently small positive number.

**Proof.** Note that \(\|v\| \leq \|Hv\| + \|(1-H)v\|\) and that there exists a small \(\lambda_0(\epsilon, \tau) (\leq 1/3)\) such that

\[
(1.19) \quad \log \lambda^{-s}n \|\hat{H}_b \delta v\| \leq C(\epsilon, \tau)/2 \quad \text{if } 0 < \lambda \leq \lambda_0(\epsilon, \tau),
\]

where \(C(\epsilon, \tau)\) is the same constant as in (1.3). If \(0 < \lambda \leq \lambda_0(\epsilon, \tau)\) we have the estimate (1.3) with \(\epsilon, C(\epsilon, \tau)\) and \(\|v\|\) in the right hand side replaced by \(2\epsilon, 2C(\epsilon, \tau)\) and \(\|(1-H)v\|\), respectively. Substitute \(\hat{H}_b \delta v\) into this modified estimate. Then, in view of (1.19) we obtain (1.18) by setting \(\epsilon = \lambda^{s-m}/2\). Indeed, as in the proof of Lemma 1.2, it is easy to see that for any real \(s > 0\) there exists a \(\lambda_0\) such that

\[
\|(1-H)\hat{H}_b \delta v\| + \|(1-H)P_{\gamma(x)} \hat{H}_b \delta v\| + \|(1-H_0)P_{\gamma(x)} \hat{H}_b \delta v\| 
\]

\[
+ \|(1-H_0)P_\tau \hat{H}_b \delta v\| \leq C_\lambda \lambda^s N^{s+2n+3m} \|v\|, \\
v \in S_y.
\]

Q.E.D.

**Lemma 1.5.** For any \(s > 0\) there exists a constant \(M\) independent of \(s, \kappa, \lambda, and N\) such that for \(j = 1, \ldots, N - m\)

\[
(1.20) \quad \|P_\gamma \hat{H}_b \delta v\| \leq M \|\hat{H}_b \delta v\| + MN(\log \lambda^{-s})^{-1} \|P_\tau \hat{H}_b \delta v\| 
\]

\[
+ C_\lambda \lambda^s N^{s+2n+3m} \|v\|, \\
v \in S_y, \quad \text{if } \log \lambda^{-s} \geq MN \text{ and } 0 < \lambda \leq \lambda_0(\epsilon, \tau),
\]

where \(C_\lambda\) is a constant independent of \(\lambda, \kappa, \lambda,\) and \(N\), and \(\lambda_0(\epsilon, \tau)\) is the same as in Lemma 1.4. Here \(\hat{H}_b = h_{28}(\lambda D_x) h_{28}(\lambda D_x) \beta h_{28}(\lambda D_y).

**Proof.** It follows that

\[
(1.21) \quad \|P_\gamma \hat{H}_b \delta v\| \leq \|\hat{H}_b \delta v\| + \|P_\tau \hat{H}_b \delta v\|.
\]

Noting \(h_{28}(x) = h_{28}(x) h_{28}(x)\) and considering the expansion formula of the simplified symbol of \(\hat{H}_b\), we have
for some constants $C_\epsilon$ and $C_s$. Using Proposition 1.3 we have

$$||\tilde{H} P_\gamma v|| \leq 2||\tilde{H} h_{28}(\lambda^2) P_\gamma v|| + C_s \lambda^{s+2n+3m} ||v||$$

Here and in what follows we denote by the same notation $C_s$ different constant independent of $\lambda$, $\kappa$ and $N$, depending on $s$. We shall estimate the second term of the right hand side of (1.21). In view of Lemma 1.2 it suffices to estimate $||[P_\gamma, \tilde{H}^j h_{j+1}^{i+1}]v||$, where $H^j_{i+1} = h^j_{i+1}(\lambda D_j)h^i_{j+1}(\lambda y)$. Write

$$[P_\gamma, \tilde{H}^j] = [P_\gamma, h^j_{i+1}(\lambda D_j)\partial_\gamma]h^i_{j+1}(\lambda y) + \partial_\gamma h^j_{i+1}(\lambda D_j)[P_\gamma, h^i_{j+1}(\lambda y)].$$

Note that the expansion formula

$$[P_\gamma, h^j_{i}(\lambda D_j)\partial_\gamma] = \sum_{0<|\beta|<|\gamma|/2} (-1)^{|\beta|} (h^j_{i})(\lambda D_j)\partial_\gamma)^{|\beta|} P_\gamma(\partial_\gamma)^{|\beta|} R(y, D_j; \lambda),$$

where $R$ is an operator negligible, in the sense of

$$||Rv|| \leq C_s \lambda^{N^{s+2n+3m}} ||v||.$$

In view of (1.2), (1.12) and (1.14), we see that there exists a constant $M_1$ independent of $s$, $\kappa$, $\lambda$ and $N$ such that

$$||[P_\gamma, h^j_{i+1}(\lambda D_j)\partial_\gamma]h^i_{j+1}(\lambda y)H^j_{i+1} v|| \leq (M_1 N)^{|\alpha|} \lambda^{N^{s+2n+3m}} ||v||$$

holds with some constants $C'_\epsilon$. Consider the expansion formula of the simplified symbol of $\partial_\gamma h^i_{j+1}(\lambda y)$ and use Proposition 1.3. Then the second term of the right hand side of (1.22) is estimated above by

$$2(M_1 N)^m||\tilde{H}^{i+1} v|| + C_s \lambda^{s+2n+3m} ||v||.$$

Together with Proposition 1.3, the symbolic calculus shows that for any $\beta$ ($0<|\beta|<n$) the estimate

$$\lambda^{|\beta|} ||\partial_\gamma (\partial_\gamma)^{|\beta|} (M_2 N)^{|\beta|} ||\tilde{H}^{i+1} v|| \leq \sum_{0<|\beta|<n} \lambda^{N^{s+2n+3m}} ||v||$$

holds with some constants $C'_\epsilon$.
holds with a constant \( M_2 \) independent of \( s, \kappa, \lambda \) and \( N \). Note (1.22)-(1.24) and use (1.18) with \( j = j + 1 \). Then, taking a suitable \( M \) larger than \( M_1 \) and \( M_2 \), we see that \( \| [P_\gamma, h^{(2)}_i(\lambda D_y)h^{(2)}_i(\lambda y)H^{(j+1)}_6] \| \) is estimated above by the right hand side of (1.20) because \( (MN)^j \leq (\log \lambda^{-i})^jMN(\log \lambda^{-i})^{-1}, j = 1, \cdots, m \) if \( \log \lambda^{-i} \geq MN \). Noting \( \| h^{(2)}_i(\lambda D_y)v \| \leq \| h^{(2)}_i(\lambda D_y)v \| \) for \( v \in S_0 \), we can also estimate \( \| [P_\gamma, h^{(2)}_i(\lambda D_y)H^{(j+1)}_6] \| \) by means of (1.18). We have estimated the second term of the right hand side of (1.21). So we obtain the desired estimate (1.20).

Q.E.D.

By Lemmas 1.4 and 1.5 we have

**Lemma 1.6.** For any real \( s > 0 \), \( \kappa > 0 \) and any integer \( N > m \) there exists a constant \( M \) independent of \( s, \kappa, \lambda \) and \( N \) such that

\[
(1.25) \quad (\log \lambda^{-i})^N\| H_6 v \| \leq (\log \lambda^{-i})^N\| H_{28} v \| + 2(MN)^N\lambda^{-2m}\| v \| \\
+ C_4 N!N^{r+2m+3m}\| v \|, \quad v \in S_0, \text{ if } 0 < \lambda \leq \lambda_0(s, \tau),
\]

where \( C_4 \) is a constant independent of \( \lambda, \kappa \) and \( N \). Here \( H_6 = h_6(\lambda D_y)h_6(D_y; \lambda) \times h_6(\lambda y) \) and \( \lambda_0(s, \tau) \) is the same as in Lemma 1.4.

**Proof.** In view of \( H_6 = H_{1\frac{1}{2}} \) it follows from (1.18) that

\[
(\log \lambda^{-i})^N\| H_{1\frac{1}{2}} v \| \leq (\log \lambda^{-i})^N\| H_{28} v \| + C_4 \lambda^N S^{r+2m+3m}\| v \|.
\]

Applying (1.20) to the first term of the right hand side. Then we have

\[
(\log \lambda^{-i})^N\| H_6 v \| \leq M(\log \lambda^{-i})^{N-m}\| H_{28} v \| + MN(\log \lambda^{-i})^{N-m-1}\| P_\gamma H_6 v \| \\
+ 2C_4(\log \lambda^{-i})^{N-m}\lambda^N S^{r+2m+3m}\| v \| \quad \text{if } \log \lambda^{-i} \geq MN.
\]

Use (1.20) for the second term of the right hand side and moreover use (1.20) repeatedly \((N-m-2)\) times. Then we obtain

\[
(1.26) \quad (\log \lambda^{-i})^N\| H_6 v \| \\
\leq M\sum_{j=0}^{N-m-1}(\log \lambda^{-i})^{N-j-m}(MN)^j\| H_{28} v \| \\
+ (MN)^{N-m}\| P_\gamma H_{N-m+1} \| \\
+ (\log \lambda^{-i})^{N-m}(1 + \sum_{j=0}^{N-m-1}(\log \lambda^{-i})^{-j}(MN)^j)C_4 \lambda^N S^{r+2m+3m}\| v \|, \quad \text{if } \log \lambda^{-i} \geq MN.
\]

Since \( h_6^{N-m+1} = h_6 \) and \( \{ \lambda^m a(P_\gamma H_6^{N-m+1}); 0 < \lambda \leq 1 \} \) is a bounded set in \( S_0^0 \), we have

\[
\| P_\gamma H_{N-m+1} v \| \leq M\lambda^{-2m}\| v \|,
\]
by taking another large $M$ if necessary. If $\log \lambda^{-s} \geq MN$ it follows from (1.26) that

$$ (\log \lambda^{-s})^N ||\vec{H}_g v|| \leq (\log \lambda^{-s})^{N-1} ||\vec{H}_{28} P_{\gamma} v|| + (MN)^N (\lambda^{-2m+1}) ||v|| + C_s (\log \lambda^{-s})^{N-1} \lambda^s N^{r+2m+3m} ||v||. $$

When $\log \lambda^{-s} \leq MN$ this estimate still holds because of the second term of the right hand side. Noting

$$ (\log \lambda^{-s})^N \lambda^s = (\log \lambda^{-s})^N \exp (-\log \lambda^{-s}) \leq N!, $$
we obtain (1.25).

Q.E.D.

By means of (1.18) with $j=1$ we see that there exists a constant $M'$ independent of $s$, $\epsilon$ and $\lambda$ such that for any $N=0, 1, \ldots, m$

$$ (\log \lambda^{-s})^N ||\vec{H}_g v|| \leq M' \lambda^{-2m} ||v|| + C_s N^{r+2m+3m} ||v||. $$

Therefore, by taking another large $M$ if necessary, we may assume that (1.25) holds for any $N=0, 1, \ldots$. Let $\tau$ be a small parameter chosen later on. Multiply $\tau^N!$ by both sides of (1.25) and sum up with respect to $N=0, 1, \ldots$. Then we obtain

$$ \lambda^{-s\tau} ||\vec{H}_g v|| \leq \lambda^{-s\tau} ||\vec{H}_{28} P_{\gamma} v|| + (2\lambda^{-2m} \sum_{N=0}^{\infty} (MN^r)^N/N! + C_s \sum_{N=0}^{\infty} \tau^N N^{r+2m+3m}) ||v|| $$

because $\sum (\log \lambda^{-s})^N/N! = \lambda^{-s\tau}$. Choose $\tau$ such that $Me^\epsilon < 1$ and $0 < \tau < 1$. Then, by means of the stirling formula $N^N/N! \leq e^N$ we have

$$ \lambda^{-s\tau} ||\vec{H}_g v|| \leq \lambda^{-s\tau} ||\vec{H}_{28} P_{\gamma} v|| + C' \lambda^{-2m} ||v||, \quad v \in S_\gamma, $$

for another constant $C'$. Note that $\tau$ is independent of $s$ because $M$ is so. Hence we can replace $s\tau$ in (1.27) by $2s' + 2s'' + 2m$ for any real $s'$, $s'' > 0$. Multiply $\lambda^{2s'+2m}$ by (1.27) with $s\tau$ replaced by $2s' + 2s'' + 2m$. Then we see that there exists a constant $C_0 = C_0(s', s'', \tau)$ independent of $\epsilon$ and $\lambda$ such that

$$ \lambda^{-2s'} ||\vec{H}_g v|| \leq \lambda^{-2s'} ||\vec{H}_{28} P_{\gamma} v|| + C_0 \lambda^{2s''} ||v||, \quad v \in S_\gamma \text{ if } \lambda \text{ is sufficiently small.} $$

Taking another large $C_0$ if necessary, we may assume that (1.28) holds for $0 < \lambda \leq 1$.

Note that for any $\tilde{\xi}_0 \in S^{s-1}$, any $0 < \delta' \leq 1$ and any real $\tilde{s}$ the estimate

$$ C^{-1} ||h_{g'}(\lambda D_\gamma) v|| \leq ||q(D_\gamma; \lambda) \tilde{h}_{g'}(\lambda D_\gamma) v|| \leq C ||h_{g'}(\lambda D_\gamma) v||, \quad v \in S_\gamma $$

holds for some constant $C = C(\tilde{s}, \delta')$ because
\[ C^{-1} \leq q(\eta; \lambda) \leq C \quad \text{on supp } h_{\gamma}(\lambda \eta). \]

Substitute \( v(y) = h(\lambda D_x) v(y) \) into (1.28) for \( \tilde{v}(y) = \exp(-i \lambda^{-2} x \cdot \tilde{\xi}_0) w(x) \big|_{x = \lambda y + x_0}, w \in S_x \). Then in view of (1.5), (1.6) and (1.11) we see that there exists a constant \( C'_0 \) such that

\[
||(h(\lambda D_x - \tilde{\xi}_0) A^\prime x_s h_2(x - x_0) w)||^2 \\
\leq C'_0 ||h_2(\lambda D_x - \tilde{\xi}_0) A^\prime x_s h_2(x - x_0) P(x, D_x) w||^2 \\
+ ||h(\lambda D_x - \tilde{\xi}_0) A^\prime s' w||^{2 + \lambda ||w||^2_{s', r}},
\]

\[ w \in S_x \quad \text{if } 0 < \lambda \leq 1. \]

Here we used the fact that
\[{\lambda^{-1} h_2(\lambda^2 D_x - \tilde{\xi}_0) h_2(x - x_0) (1 - h(\lambda^2 D_x - \tilde{\xi}_0)); 0 < \lambda \leq 1}, \]
\[{\lambda^{-1} h_2(\lambda^2 D_x - \tilde{\xi}_0) h_2(x - x_0) P(x, D_x) (1 - h((\lambda^2 D_x - \tilde{\xi}_0)/3)); 0 < \lambda \leq 1}\]
are contained in a bounded set of \( S_{1, 0}'' \).

To complete the proof of Theorem 1 we define the following:

\textbf{Definition 1.7.} For \( \delta > 0 \) and \( \tilde{\xi}_0 \in S^{n-1} \) we say that a function \( \gamma(\tilde{\xi}) \in C^\infty(R^n) \) belongs to \( \Psi_{\delta, \tilde{\xi}_0} \) if \( \gamma \) satisfies

\[
0 \leq \gamma \leq 1, \quad \gamma(\xi) = 1 \quad \text{for } |\xi|/|\xi| - \tilde{\xi}_0| \leq \delta/12 \text{ and } |\xi| \leq 1/2, \]
\[
\gamma(\xi) = 0 \quad \text{for } |\xi|/|\xi| - \tilde{\xi}_0| \geq \delta/10 \text{ or } |\xi| \leq 1/3, \]
\[
\gamma(t \xi) = \gamma(\xi) \quad \text{for } t \geq 1 \text{ and } \xi \in S^{n-1}. \]

Divide both sides of (1.30) by \( \lambda \) and integrate with respect to \( \lambda \) from 0 to 1. Then, we see that for any \( r = (x_0, \tilde{\xi}_0) \in R^n \times S^{n-1} \), any real \( s', s'' > 0 \), any integer \( k > 0 \) and any \( \kappa > 0 \) there exists a constant \( C'' = C''(r, s', s'', k) \) independent of \( \kappa \) such that

\[
||\gamma_2(D_x) A_{s, k} h_2(x - x_0) w||^2 \]
\[
\leq C''(||\tilde{\gamma}_2(D_x) A_{s, k} h_2(x - x_0) P(x, D_x) w||^{2 + ||w||^2_{s', r}},
\]

\[ w \in S_x, \]

if \( \gamma_2(\xi) \in \Psi_{\delta, \tilde{\xi}_0} \) and \( \tilde{\gamma}_2(\xi) \in \Psi_{148, \tilde{\xi}_0} \). (See Proposition 1.7 of [12]).

We shall prove (2). Let \( (x_0, \tilde{\xi}_0) \in T^*(R^n) \setminus 0 \) and let \( u \in D'(R^n) \). Set \( \tilde{\xi}_0 = \xi_0/|\xi_0| \). Suppose that \( (x_0, \tilde{\xi}_0) \in WF Pu \). Then there exists a \( \delta > 0 \) such that \( \tilde{\gamma}_2(D_x) h_2(x - x_0) Pu \in H_{r'} \) for any real \( s' > 0 \) if \( \tilde{\gamma}_2(\xi) \in \Psi_{148, \tilde{\xi}_0} \). Since \( h_4(x - x_0) u \in \mathcal{E}' \) we have \( h_4(x - x_0) u \in H_{r''} \) for some \( s'' > 0 \). Choose \( k > 0 \) in (1.31) such that \( k \geq s' + s'' + m \). Then, by taking a sequence \( \{w_k\}_{k=1} \subset S_x \) such
that
\[ w_j \to h_4(x-x_0)u \quad \text{in} \quad H_{-s''}, \]
from (1.31) we see that
\[ (1.32) \quad \| A_{\alpha,k} \psi_\theta(D_x) h_0(x-x_0)u \|_{s'}^2 \leq C'' \left( \| \psi_\theta(D_x) h_0(x-x_0)P u \|_{s''}^2 + \| h_4(x-x_0)u \|_{s',s''}^2 \right), \]
if \[ \psi_\theta(e) \in \mathcal{W}_{s',\tilde{e}_0} \quad \text{and} \quad \tilde{\psi}_\theta(e) \in \mathcal{W}_{148,\tilde{e}'_0}. \]
Here we used the fact that \[ \| A_{\alpha,k} w \| \leq \| w \| \]
for \[ w \in L^2. \]
Letting \( \kappa \) tend to 0 in (1.32), we have
\[ \psi_\theta(D_x) h_0(x-x_0)u \in H_s. \]
Since \( s' \) is arbitrary, we have \( (x_0, \tilde{\xi}_0) \in \mathcal{WF} \ u. \) Now the proof of Theorem 1 has been completed.

\section{2. Proofs of Theorems 4 and 5}

Throughout this section, \( P \) satisfies the assumption (5). It is clear that
for \((x_0, \xi_0) \in T^*R^n \setminus 0 \) and \( u \in \mathcal{D}'(R^n) \)
\[ (x_0, \xi_0) \in \mathcal{WF} \ P u \Rightarrow (x_0, \xi_0) \in \mathcal{WF} \ u \]
if \( x_0 \neq (0, x'_0). \) So, we shall consider the case \( x_0 = (0, x'_0). \) Since our consideration is local near \( x_0 \) we may assume that for any \( \mu > 0 \) there exists a constant \( c_\mu > 0 \) such that
\[ (2.2) \quad | p_m(x, \xi) | \geq c_\mu \quad \text{for all} \quad (x, \xi) \in R^n \times S^{n-1} \]
if \( | x' | \geq \mu. \)
Let \( h(x) \in C_c^\infty(R^n) \) be the same as in Section 1. Noting \( \sqrt{2}/5 < 7/24, \) we may assume that \( h(x) \) is written as
\[ (2.3) \quad h(x) = h_1(x') h_2(x'') \]
for \( C_c^\infty \)-functions \( h_1 \) and \( h_2 \) satisfying
\[
\begin{cases}
  h_1(x') = 1 & \text{on} \quad | x' | \leq 1/5 \\
  h_2(x'') = 1 & \text{on} \quad | x'' | \leq 1/5.
\end{cases}
\]
First we shall prove Theorem 4. We prepare four lemmas. If \( P_\gamma = p_\gamma(y, D_j; \lambda) \) is the same as in Section 1, from (6) we have:

**Lemma 2.1.** For any \( \varepsilon > 0 \) and any \( \tau = (x_0, \tilde{\xi}_0) \in R^n \times S^{n-1} \) with \( x_0 = (0, x'_0) \) there exists a constant \( C(\varepsilon, \tau) \) such that
\[
H = h(\lambda D_y^2)h(y) \quad \text{where } \sum' \text{ means the summation which is taken over all } \alpha = (\alpha', \alpha''), \beta \text{ satisfying } 0 < |\alpha + \beta| < m \text{ and } \alpha' = 0.
\]

Proof. Set \( q^q(x, \xi) = (1 - h_1(2x'))/p(x, \xi) \). Then it follows from (2.2) and the symbolic calculus that for \( Q^0 = q^q(x, D_y) \) we have

\[
Q^0 P = 1 - h_1(2x') - R, \quad R \in S_{1,0}^{-1}.
\]

Setting \( Q = \sum_{j=0}^{m-1} R^j Q^0 \) we have

\[
QP = 1 - \sum_{j=0}^{m-1} R^j h_1(2x') - R^m
\]

\[
\equiv 1 + R_1 + R_2, \quad R_2 \in S_{1,0}^{-m}.
\]

For \( 0 \leq |\alpha + \beta| < m \), note

\[
[P_{q^q}^{(0)}, h(x-x_0)] = [P_{q^q}^{(0)}, h_1(x')]h_2(x''-x_0') + h_1(x')(P_{q^q}^{(0)}, h_2(x''-x_0')]
\]

and

\[
\begin{cases}
[P_{q^q}^{(0)}, h(x')]Q, & [P_{q^q}^{(0)}, h_1(x')]R_2 \in S_{1,0}^{-1} \\
[P_{q^q}^{(0)}, h_1(x')]R_3 \in S_{-\infty}^{m}
\end{cases}
\]

because \( \text{supp } h_1(2x') \cap \text{supp } D^\alpha h_1(x') = \emptyset \) for \( \alpha' \neq 0 \). It follows from (2.6) that the estimate

\[
||[P_{q^q}^{(0)}, h_1(x')]w|| \leq C(||Pw||_{-1} + ||w||_{-1})
\]

\[
\leq \epsilon||Pw|| + C_\epsilon||w||, \quad w \in S_x
\]

holds for any \( \epsilon > 0 \) and some \( C_\epsilon \). As in the proof of Lemma 1.1, replace \( u \) in (6) by \( h(x-x_0)h(\lambda^2 D_x - \tilde{\xi}_0)w = h(x-x_0)h(\lambda^2 D_x - \tilde{\xi}_0) h((\lambda^2 D_x - \tilde{\xi}_0)/3w), \quad w \in S_x \). Then, using (2.5) and (2.7) together with symbolic calculus, we obtain

\[
||A^m h(\lambda^2 D_x - \tilde{\xi}_0)w(x-x_0)|| + \sum' ||A^{-1}|| (\log A)^{\alpha + \beta} h(\lambda^2 D_x - \tilde{\xi}_0)h(x-x_0) P_{q^q}^{(0)}h((\lambda^2 D_x - \tilde{\xi}_0)/3w)||
\]

\[
\leq \epsilon||h(\lambda^2 D_x - \tilde{\xi}_0)/2)h((\lambda^2 D_x - \tilde{\xi}_0)/3w)|| + C_\epsilon \sum' ||A^{-1}|| h((\lambda^2 D_x - \tilde{\xi}_0)/2)P_{q^q}^{(0)}h((\lambda^2 D_x - \tilde{\xi}_0)/3w)|| + ||w||, \quad w \in S_x.
\]

Set \( w(x) = (\exp i\lambda^{-2}x \cdot \tilde{\xi}_0)\nu(\lambda^{-1}(x-x_0)) \) for \( \nu(y) \in S_y \). From (2.8) together with (1.5)--(1.8) we obtain (2.4), similarly as in the proof of Lemma 1.1 Q.E.D.
For $h(x) = h_1(x') h_2(x'')$ and a small $0 < \delta \leq 1/10$ set $h_3(x) = h(x/\delta)$, $h_4(x) = h_1(x'/\delta)$ and $h_5(x') = h_3(x' / \delta)$). Fix an integer $N > m$. Take a sequence 
\[ \{h_6(x) = h_1(x') h_2(x'') \} \in C_0^\infty(\mathbb{R}^n) \] such that
\[
\begin{cases}
0 \leq h_6(x) \leq 1, \\
h_6 = h_1 \subset h_2 \subset \cdots \subset h_{N-m+1} = h_{28},
\end{cases}
\]
and for any $\alpha'$ and $\alpha''$ estimates
\[
|D_\alpha'' h_6(x')| \leq C_\alpha'' N^{1/4}, \quad |D_\alpha'' h_6(x'')| \leq C_\alpha'' N^{1/4}
\]
hold with constants $C_\alpha''$ and $C_\alpha'''$ independent of $N$ and $j$.

**Lemma 2.2.** Set $\tilde{H}_6 = \tilde{h}_6(\lambda D_y) h_6(D_x; \lambda) h_6(\lambda y)$ for $j = 1, \ldots, \lambda - m + 1$. Then for any $\tau > 0$ and any $\tau = (x_0, x') \in \mathbb{R}^2 \times S^{n-1}$ with $x_0 = (0, x''_0)$ there exists a constant $C(s, \tau)$ independent of $\lambda$, $\kappa$, $j$, and $N$ such that
\[
(\log \lambda)^{-m} ||H_6|v|| + \sum' (\log \lambda^{-\tau})^{1/2} \sum \lambda^{1/2} \|P_{\tau} h_6 \| ||v|| \leq ||P_{\tau} H_6 |v|| + C(s, \tau) \lambda^2 N^{1/2} ||v||, \quad v \in S_y,
\]
if $0 < \lambda \leq \lambda_0(s, \tau)$,

where $\lambda_0(s, \tau)$ is a sufficiently small positive number. Here $\sum'$ means the same summation as in Lemma 2.1.

**Proof.** The estimate (2.11) follows from (2.4), by the quite same way as in the proof of Lemma 1.4. Q.E.D.

**Lemma 2.3.** Set $\tilde{h}_6(x') = 1 - h_6(2x')$ for $0 < \delta \leq 1/10$. Then for any $s > 0$ and any $d = (x_0, x') \in \mathbb{R}^2 \times S^{n-1}$ with $x_0 = (0, x''_0)$ there exist a constant $C(s, \tau)$ such that
\[
\lambda^{-2m} ||h_6(\lambda y') h(\lambda D_x) |v|| \leq C'(s, \tau) \lambda^2 ||P_{\tau} |v|| + \lambda^2 ||v||, \quad v \in S_y, \quad 0 < \lambda \leq 1.
\]

**Proof.** Set $q_0^0(\eta, \eta; \lambda) = (1 - h_6(3y') h(\lambda \eta/2)/p_\lambda(\eta, \eta; \lambda)$. Then \( \{\lambda^{-2m} q_0^0(\eta, \eta; \lambda) ; 0 < \lambda \leq 1 \} \) is a bounded set in \( S_{0,0}' \). The symbolic calculus shows that, if \( Q_0^0 = Q_0^0(y, D_x; \lambda) \), we have
\[
Q_0^0 P = (1 - h_6(\lambda y') h(\lambda D_x)) - R
\]
for $R = r(\lambda, D_x; \lambda)$ such that \( \{\lambda^{-2m} (\eta, \eta; \lambda) ; 0 < \lambda \leq 1 \} \) is a bounded set in \( S_{0,0}' \). Take an integer $l$ satisfying $2l \geq s + 2m$ for a fixed $s > 0$. Set $Q_0 = \sum_{i=0}^{l-1} R_i Q_0^0$. Then we obtain
\[ Q_c \Gamma = 1 + \sum_{j=0}^{l-1} R^j ((1 - h_1(3\lambda y' / \delta))h(\lambda D_y))^2 - 1 - R^l \]

Note that \( \{ \lambda^{-2l} \sigma(R); 0 < \lambda \leq 1 \} \) is a bounded set in \( S_{0,0}^0 \) and that, for any \( s > 0 \), \( \{ \lambda^{-s} \sigma(h_0(\lambda y'))h(\lambda D_y)R; 0 < \lambda \leq 1 \} \) is a bounded set in \( S_{0,0}^0 \) because \( 1 - h_1(3x' / \delta)h(\xi(2) - 1 = 0 \) on \( \supp h_0(x')h(\xi) \). From (2.13) we obtain (2.12) because \( \{ \lambda^{-2m} \sigma(h_0(\lambda y')h(\lambda D_y)Q); 0 < \lambda \leq 1 \} \) is a bounded set in \( S_{0,0}^0 \).

Q.E.D.

Lemma 2.4. For any \( s > 0 \) there exists a constant \( M \) independent of \( s, \kappa, \lambda \) and \( N \) such that for \( j = 1, \ldots, N - m \)

\[ ||P \tilde{\Gamma}^{j} v|| \leq M (||H^{j+1} v|| + \lambda \sum \lambda^{j+1} (\lambda D_y)j^{1+1}(\lambda y)) \]

where \( C_\kappa^\kappa \) is a constant independent of \( \lambda, \kappa \) and \( N \), and \( \lambda_0(s, r) \) is a positive number smaller than \( \lambda_0(s, r) \) in Lemma 2.2. Here \( H^{j+1} = \lambda D_y)^j(x'_y, x'_y)h_0(\lambda y) \) and \( r = (x_0, \xi_0) \in R^d \times S^{n-1} \) with \( x_0 = (0, x'_0) \).

Proof. It follows that

\[ ||P \tilde{\Gamma}^{j} v|| \leq ||H^{j+1} v|| + ||(P \tilde{\Gamma}^{j} \tilde{\Gamma}) v|| \]

The estimation of the first term of the right hand side of (2.15) is the quite same as in the proof of Lemma 1.5. We shall estimate the second term. In view of Lemma 1.2 it suffices to estimate \( ||(P \tilde{\Gamma}^{j} \tilde{\Gamma}) v|| \), where \( H^{j+1} = h_0^{j+1}(\lambda D_y)j^{1+1}(\lambda y) \). Note

\[ ||(P \tilde{\Gamma}^{j} \tilde{\Gamma}) H^{j+1}|| \]

\[ \leq ||(P \tilde{\Gamma}^{j} \tilde{\Gamma}) h_0(\lambda D_y)j^{1+1}(\lambda y)|| \]

We shall consider the first term of the right hand side of (2.16). As in the proof of Lemma 1.5 we have (1.22). The estimation of the second term of the right hand side of (1.22) is the quite same as there. About the first term, instead of (1.24), we see that for any \( \beta (0 < |\beta| < m) \) the estimate

\[ \lambda^{j||\beta||} ||(P \tilde{\Gamma}^{j} \tilde{\Gamma}) h_0(\lambda y')h_0^{j+1}(\lambda y)|| \leq \sum_{|\beta| + |\beta| = m} \lambda^{j||\beta||} \]

holds with a constant \( M \) independent of \( s, \kappa, \lambda \) and \( N \), where \( h_0(x') \) is the
same as in Lemma 2.3. In fact, the estimate (2.17) follows from the symbolic calculus if we note

\[
(2.18) \quad [P_{y(x)}, h^j(y)] = [P_{y(x)}, h^j(x')] \circ h^j(x') + [P_{y(x)}, h^j(y')] [P_{y(x)}, h^j(y')]
\]

and \( h^j(x') = 1 \) on \( \text{supp} \ D_{x'}^\alpha h^j(x') \) for \( \alpha' \neq 0 \). It follows from (2.11) with \( j = j + 1 \) that the first and the second terms of the right hand side of (2.17) multiplied by \((M_1N)'^1\) are estimated above by the right hand side of (2.14). We shall estimate the third term of the right hand side of (2.17) multiplied by \((M_1N)'^1\), that is,

\[
J \equiv \sum_{0 < \rho < |x| - 1} \lambda^{2|\beta| - 2m + 2\rho} (M_3N)'\beta + \| \tilde{h}^j(x') H_0^j v \|
\]

where \( M_3 = \max(M_1, M_2) \). In view of \( H_0^j v = h(y) H_0^j v \), it follows from (2.12) that

\[
J \leq C'(s, \tau) \sum_{0 < \rho < |x| - 1} \lambda^{2|\beta| + \phi} (M_3N)'\beta + \|P_{y} H_0^j v \| + \lambda^\phi \|v\|.
\]

Take \( \tilde{\rho}_0(s, \tau) \) small enough so that

\[
\lambda^2 \max(C'(s, \tau), 1) \leq (\log \lambda^{-1})^{-1} \quad \text{if} \quad 0 < \lambda \leq \tilde{\rho}_0(s, \tau).
\]

Then \( J \) is estimated above by the right hand side of (2.14) when \( 0 < \lambda \leq \tilde{\rho}_0(s, \tau) \) and \( \log \lambda^{-1} \geq MN \) with a suitable \( M > M_3 \). We have estimated the first term of the right hand side of (2.16). Noting (2.18) with \( \beta = 0 \), by means of (2.11) and (2.12) we can also estimate the second term of the right hand side of (2.16). The estimation of the right hand side of (2.15) is completed. So we obtain (2.14). Q.E.D.

From (2.11) and (2.14), we obtain (1.25) in Lemma 1.6. In fact, in the proof of Lemma 1.6 we did not use the fact that the second term of the left hand side of (1.18) is estimated above by its right hand side. By mean of (1.25) we can also prove (2.1) for \( (x_0, \xi_0) \in T^*R^n \setminus 0 \) with \( x_0 = (0, x'_0) \), by the same way as the one after Lemma 1.6. The proof of Theorem 4 is completed.

In the rest of this section we shall prove Theorem 5. We state three lemmas. By the same way as in the proof of Lemma 2.1, from (7) we have:

**Lemma 2.5.** For any \( \tau = (x_0, \xi_0) \in R^n \times S^{n-1} \) with \( x_0 = (0, x'_0) \) there exist a \( 0 < \varepsilon_0 < 1 \) and a constant \( C_\tau \) such that
(2.19) $||Hy|| + \sum' \lambda^{-2\kappa_0 + 1}\beta || Hf^{(\alpha')} || \\
\leq C||h(\lambda D_s/2)P_{\alpha'}|| + \sum' \lambda^{1+\beta} || Hf^{(\alpha')} || + \lambda^l||v||$,
$v \in S$ , if 0 $<\lambda \leq 1$ ,

where $H=h(\lambda D_y)h(\lambda y)$. Here $\sum'$ means the summation which is taken over all $\alpha=(\alpha', \alpha'')$, satisfying $0<|\alpha+\beta| \leq m$ and $\alpha'=0$.

Proof. We also have (2.5)-(2.7) for $0<|\alpha+\beta| \leq m$. By means of (2.5) and the first inequality of (2.7), it follows from (7) that

$$||h(\lambda D_x-\xi_0)h(x-x_0)w|| + \sum' ||A^{-1}\beta || h(\lambda D_x-\xi_0)h((\lambda D_x-\xi_0)/3)w||$$
$$\leq C|\lambda||h((\lambda D_x-\xi_0)/2)h((\lambda D_x-\xi_0)/3)w|| + ||A^{-1}w||,$

In view of (1.5)-(1.8), the estimate (2.19) immediately follows from the above estimate. Q.E.D.

Let $N$ be a sufficiently large integer chosen later on. If $\{h^{(\nu)}_j\}_{j=1}^{N-3}$ is the same as the one in the proof of Theorem 4, from (2.19) we have

Lemma 2.6. Set $\bar{H}^{(\nu)}_j = h^{(\nu)}_j(\lambda D_y)\varphi_\kappa(D_y; \lambda h^{(\nu)}_j(\lambda y))$ for $j=1, \cdots, N-m+1$. Then for any $s>0$ and any $\tau = (x_0, \xi_0) \in R^n \times S^{n-1}$ with $x_0=(0, x_0')$, there exists a constant $C'(s, \tau)$ independent of $K$ and $\kappa$ such that

(2.20) $||\bar{H}^{(\nu)}_j|| + \sum' \lambda^{2s_0 + 1}\beta || P^{(\alpha')}_{\gamma} \bar{H}^{(\nu)}_j || \leq C'(s, \tau)(||P_{\alpha'} \bar{H}^{(\nu)}_j || + \lambda^l||v||),$  
$v \in S$, if 0 $<\lambda \leq \lambda_0(\tau), j = 1, \cdots, N-m+1,$

where $\lambda_0(\tau)$ is a sufficiently small positive number. Here $s_0$ and $\sum'$ are the same as in Lemma 2.5.

Proof. Take a small $\lambda_0(\tau)$ such that

$$1/2 \geq \lambda^{2s_0} C_\gamma$$

where $C_\gamma$ is the same constant as in (2.19). Then we have (2.20) from (2.19), by the similar way as in the proof of Lemma 1.4. Q.E.D.

The above lemma together with Lemma 2.3 leads us to the following:

Lemma 2.7. For any $s>0$ there exists a constant $C_s$ independent of $\kappa$ and $\lambda$ such that for $j=1, \cdots, N-m$
(2.21) $||P_\gamma \hat{H}_\delta v|| \leq C_s(||\hat{H}_{2\delta} P_\gamma v|| + \lambda^{2s} ||P_\gamma \hat{H}_\delta^{1 + \nu} v|| + \lambda^s ||v||),$  
$v \in \mathcal{S}_\gamma,$ if $0 < \lambda \leq \lambda_1(r),$

where $r = (x_0, \xi_0) \in \mathbb{R}^n \times S^{n-1}$ with $x_0 = (0, x_0').$

**Proof:** As in the proof of Lemma 2.4, it suffices to estimate $||[P_\gamma, \hat{H}_\delta^{1 + \nu}] v||.$ We shall consider the first term of the right hand side of (2.16). Noting the expansion formula of $[P_\gamma, h_\delta](\lambda D_\omega)\phi_\omega,$ we can also have

(2.22) $||[P_\gamma, h_\delta](\lambda D_\omega)\phi_\omega||$ 
$\leq C(\sum_{0 < |\beta| \leq m} \lambda^{|\beta|} ||\phi_\omega P_\gamma h_\delta(\lambda \omega)\hat{H}_\delta^{1 + \nu} v||$ 
$+ \lambda^2 \sum_{0 < |\beta| \leq m+1} \lambda^{2|\beta|} ||h_\delta(\lambda \omega)\hat{H}_\delta^{1 + \nu} v|| + \lambda^s ||v||).$

In the proof of the lemma we denote by the same notation $C$ different constants independent of $\lambda$ and $\epsilon$ (and $j$). The second term of the right hand side is estimated above by a constant times of $\lambda^2 ||\hat{H}_\delta^{1 + \nu} v|| + \lambda^s ||v||.$ So we can estimate it by the right hand side of (2.21) by means of (2.20) with $j = j + 1.$

About the first term of the right hand side of (2.22), we have

(2.23) $\lambda^{|\beta|} ||\phi_\omega P_\gamma h_\delta(\lambda \omega)\hat{H}_\delta^{1 + \nu} v|| \leq C(\sum_{0 < |\beta| \leq m} \lambda^{|\beta| + 2s} ||P_\gamma \hat{H}_\delta^{1 + \nu} v||$ 
$+ \lambda^2 ||\hat{H}_\delta^{1 + \nu} v|| + \lambda^{2|\beta| - 2s} ||h_\delta(\lambda \omega')\hat{H}_\delta^{1 + \nu} v|| + \lambda^s ||v||),$  

where $h_\delta(\omega')$ is the same as in Lemma 2.3. It follows again from (2.20) with $j = j + 1$ that the first and the second terms of the right hand side of (2.23) are estimated above by a constant times of the right hand side of (2.21). Note that Lemma 2.3 still holds because it was obtained only under the hypothesis of (5) ((2.2)). By means of (2.12) we can also estimate the third term of the right hand side of (2.23). Now, the first term of the right hand side of (2.16) has been estimated. The estimation of its second term is similarly performed by using (2.20) and (2.12). So we obtain (2.21). Q.E.D.

For any $s > 0$ take an integer $N$ such that

(2.24) $N - m + 1 > s \kappa_0.$

Then, by means of (2.20) and the repeated use of (2.21) we easily see that for any $s > 0$ there exists a constant $C_s'$ independent of $\lambda$ and $\epsilon$ such that

(2.25) $||\hat{H}_\delta v|| \leq C'_s( ||\hat{H}_{2s} P_\gamma v|| + \lambda^s ||v||),$  
$v \in \mathcal{S}_\gamma,$ if $0 < \lambda \leq 1,$

where $r = (x_0, \tilde{\xi}_0) \in \mathbb{R}^n \times S^{n-1}$ with $x_0 = (0, x_0').$ Setting $s = 2s' + 2s''$ for real
s', s' > 0 we have the similar estimate as (1.28). So we can prove (2.1) for 
\((x_0, \xi_0) \in T^*\mathbb{R}^n \setminus 0\) with \(x_0 = (0, x_0')\). Proof of Theorem 5 is completed.

Remark. The assumption (5) in Theorems 4 and 5 can be weakened to 
Hörmander's condition in the region \(\{(x, x'); x' \neq 0\}\) as follows: For 
y any \(\mu > 0\) estimates

\[
(2.26) \quad \left\{ \begin{array}{l}
| p(x, \xi) | \geq c_\mu > 0 , \\
| p^{(g)}_{\beta}(x, \xi) | \leq C_{\alpha, \beta, \mu} (|\xi|^{-\alpha - \beta} + |\xi|) \\
& (0 \leq \delta < 1/2 < \rho \leq 1) ,
\end{array} \right.
\]

if \(|x'| \geq \mu\) and \(|\xi|\) large enough,

hold with constants \(c_\mu\) and \(C_{\alpha, \beta, \mu}\). In fact, under this assumption we have

\[
| [P^{(g)}_{\beta}, h_\lambda(x')] | \leq C |Pw|_{\delta - \rho} + |w|_{\delta - \rho} , \quad w \in \mathcal{S}_x
\]

and

\[
\lambda^{\alpha + \beta} | [h_\lambda, P^{(g)}_{\beta}] v | \leq C (|Pv| + \lambda^\beta |v|) , \quad v \in \mathcal{S}_y ,
\]

\((\sigma = \min(1 - 2\delta, 2\rho - 1))\)

instead of (2.7) and (2.12), respectively. Using these we can prove Theorems
4 and 5. The detail is omitted.

\section{3. Proof of Proposition 3}

As stated in the introduction, the necessity of (4) was already proved in
[9] in a little more general form, see Theorem 3 of [9]). We can also prove
the sufficiency of (4) in a slightly more general form.

Let \(P_0\) be a differential operator of the form

\[
(3.1) \quad P_0 = D_{x_1}^{2l} + D_{x_2}^{2l} + g(x_1)D_{x_2}^{2l} \quad \text{in } \mathbb{R}^3 ,
\]

where \(l = 1, 2, \cdots\) and \(g(x_1)\) is a \(B^\omega\)-function such that \(g(x_1) > 0\) \((x_1 \neq 0)\) and \(g(0) = 0\). When \(l \geq 2\) we assume that for any integer \(j > 0\)

\[
(3.2) \quad | g^{(j)}(x_1) | \leq C (\frac{1}{1 + |x_1|^{\delta j}})
\]

in a neighborhood of \(x_1 = 0\),

where \(g^{(j)} = D_{x_1}^j g\) and \(\sigma\) is a number satisfying

\[
(3.3) \quad 0 < \sigma < 1/2^2 .
\]

It is clear that \(\exp(-1/|x_1|^\delta), \delta > 0\), satisfies (3.2) for any small \(\sigma > 0\).

\textbf{Proposition 3.1.} Let \(P_0\) be the above operator. When \(l \geq 2\) we assume
(3.2) with \(\sigma\) satisfying (3.3). If \(g(x_1)\) satisfies
(3.4) \[ \lim_{|x_1| \to 0} |x_1| |\log g(x_1)| = 0. \]

then \( P_0 \) is hypoelliptic in \( \mathbb{R}^3 \).

When \( 0 < \delta < 1 \), \( \exp(-1/|x_1|^\delta) \) satisfies (3.4). So the sufficiency of (4) follows from this proposition. Proposition 3.1 for the case \( l=1 \) is included in Theorem 1.1 of [11], where (3.2) is not assumed. We shall prove Proposition 3.1 when \( l \geq 2 \). Without loss of generality, we may assume that for any \( \mu > 0 \) there exists a constant \( c_\mu > 0 \) such that \( g(x_1) \geq c_\mu \) for \( |x_1| \geq \mu \). We prepare the following:

**Lemma 3.2.** (i) It follows that

(3.5) \[ ||D_{x_2}^2 u|| \leq ||P_0 u||, \quad u \in S. \]

(ii) For any \( \varepsilon > 0 \) there exists a constant \( C_\varepsilon \) such that

(3.6) \[ ||(\log A)\varepsilon u|| \leq \varepsilon ||P_0 u|| + C_\varepsilon ||u||, \quad u \in S. \]

(iii) Let \( \kappa \) be \( (1-\delta_0)/(l+1) \), where \( \delta_0 = -2l^2 \sigma \). There exists a constant \( C_0 \) such that

(3.7) \[ ||g(x_1)^{-1}u||_{l_1-1+\varepsilon} \leq C_0(||P_0 u|| + ||u||), \quad u \in S, \]

and moreover for any \( j=1, \cdots, 2l \)

(3.8) \[ ||g^{(j)}(x_1)u||_{l_1-1+j+\varepsilon} \leq C_0(||P_0 u|| + ||u||), \quad u \in S. \]

**Remark.** It follows from (3.3) that \( \delta_0 < 1 \) and hence \( 0 < \varepsilon < 1/(l+1) \).

Admitting this lemma for a while, we shall apply Theorem 4 to the proof of Proposition 3.1. We shall check the assumption (6). Set \( \varepsilon = -\varepsilon' \) in (3.6) for a small \( \varepsilon' > 0 \). Then we have

(3.9) \[ ||(\log A)\varepsilon' u|| \leq ||P_0 u|| + C_{\varepsilon'} ||u||, \quad u \in S. \]

From (3.5) and (3.9) we have

(3.10) \[ ||(\log A)\varepsilon' D_{x_2}^{2l-1} u|| \leq 2 ||P_0 u|| + (C_{\varepsilon'} + 1) ||u||, \quad u \in S \]

for \( j=1, \cdots, 2l-1 \). It follows from (3.7) and (3.8) that for any \( \varepsilon > 0 \)

(3.11) \[ ||(\log A)^{j+k} g^{(j)}(x_1)D_{x_2}^{2l-k} u||_{l_1-1+j+k} \leq 2 ||P_0 u|| + C_{\varepsilon'} ||u|| \]

\begin{align*}
(0 < j+k < 2l) \end{align*}

holds with a constant \( C_{\varepsilon'} \), because the left hand side of (3.11) is estimated above by
Summing up (3.6), (3.10) and (3.11) we have (6) for $P_0$. By means of Theorem 4 we have the hypoellipticity of $P_0$.

For the proof of Proposition 3.1 it remains to prove Lemma 3.2. Note that

$$\|D_{s_2}^l u\|^2 + \|D_{s_2}^l u\|^2 + \|g(x_j)^{1/2}D_f^l u\|^2 = (P_0 u, u), \quad u \in \mathcal{S}.$$  

Replacing $u$ by $D_{s_2}^l u$, we have

$$\|D_{s_2}^l u\|^2 \leq (P_0 D_{s_2}^l u, D_{s_2}^l u) = (P_0 u, D_{s_2}^l u) \leq \|P_0 u\|^2 + \|D_{s_2}^l u\|^2)/2, \quad u \in \mathcal{S},$$

so that we have (3.5). Let $f(\xi, \eta)$ be a symbol in $S^0_{1,0}$ such that

$$\begin{cases} f = 1 & \text{on } \{ |\xi| \leq |\eta| \} \cap \{ |\xi| + |\eta| \geq 1 \}, \\ \operatorname{supp} f \subset \{ |\xi| \leq 2 |\eta| \} \cap \{ |\xi| + |\eta| \geq 1/2 \} . \end{cases}$$

Since $P_0$ is microlocally elliptic on $\{ |\xi| \geq |\eta| \}$ it is easy to see

$$\| (1-f) u \|_{L_2} + \| [P_0, f] u \|_1 \leq C(\|P_0 u\| + \|u\|), \quad u \in \mathcal{S}.$$  

Hence, it suffices to show (3.6)–(3.8) with $u$ replaced by $f u$. Since $\{ |\xi| \leq 2 |\eta| \}$ on $\operatorname{supp} f$, instead of (3.6) it suffices to show

$$\| (\log \langle D_2 \rangle^{1/2} u \| \leq \varepsilon \|P_0 u\| + C_\varepsilon \|u\|, \quad u \in \mathcal{S}.$$  

If $f_1(\xi, \eta)$ is a symbol in $S^0_{1,0}$ such that $f_1 = 1$ on $\operatorname{supp} f$ and $\operatorname{supp} f_1 \subset \{ |\xi| \leq 3 |\eta| \}$, we have

$$f_1(D_2, D_2) g^{(j)}(x_j) f(D_2, D_2) \equiv g^{(j)}(x_j) f(D_2, D_2) \bmod S^{-\infty}.$$

Since $|\xi| \leq 3 |\eta|$ on $\operatorname{supp} f_1$, instead of (3.7) and (3.8), it also suffices to show

$$\| g(x_j) \langle D_2 \rangle^{2l-1\pm k} u \| \leq C(\|P_0 u\| + \|u\|), \quad u \in \mathcal{S}$$

and

$$\| g^{(j)}(x_j) \langle D_2 \rangle^{2l-1\pm k} u \| \leq C(\|P_0 u\| + \|u\|), \quad u \in \mathcal{S}$$

for $0 < j \leq 2l$, respectively. We shall prove (3.14)–(3.16), by using the similar method as in Section 5 of [8].

Let $\phi_k (\tau)$ ($k=0, 1, 2, 3$) be $C_0^\infty$-functions in $\mathcal{R}^1$ such that
\[ \begin{align*}
\text{supp } \phi_0 & \subset \{ |t| < 1 \}, \quad \phi_0 = 1 \quad \text{on } |t| \leq 1/2, \\
\text{supp } \phi_1 & \subset \{ |t| < 2 \}, \quad \phi_1 = 1 \quad \text{on } |t| \leq 1, \\
\text{supp } \phi_2 & \subset \{ |t| < 1 \}, \quad \phi_2 = 1 \quad \text{on } |t| \geq 2, \\
\text{supp } \phi_3 & \subset \{ |t| < 3 \}, \quad \phi_3 = 1 \quad \text{on } |t| \leq 2
\end{align*} \]

and \( \phi_1 + \phi_2 = 1 \) in \( \mathbb{R}^1 \). Let \((\xi, \eta)\) denote the dual variables of \((x, y)\) and set \( \lambda(\xi, \eta) = (|\xi|^{2(l+1)} + \eta^2 + 1)^{1/(l+1)} \). Then \( \lambda(\xi, \eta) \) satisfies (1.1) of Chapter 7 of [5], so it is a basic weight function.

**Lemma 3.3.** Set \( h(\xi, \eta) = \lambda(\xi, \eta)^{2l^2} \) and set \( x_k(x_1, \xi, \eta) = \phi_k(g(x_1)h(\xi, \eta)) \) \((k=0, 1, 2)\). Then

\[ x_1(x_1, D_x, D_y) + x_2(x_1, D_x, D_y) = I \]

and \( x_k(x_1, D_x, D_y) \) \((k=0, 1, 2)\) belongs to \( S^0_{\lambda, 1, \delta_0} \), that is, for any \( \alpha \) and \( j \)

\[ (3.17) \quad |(\Theta_{\xi} \partial_{\eta})^s D_{\xi}^i x_k(x_1, \xi, \eta)| \leq C_{\alpha, j} \lambda(\xi, \eta)^{-|\alpha|+s_0 j}, \]

where \( \delta_0 = 2l^2 \alpha < 1 \)

**Proof.** By means of (3.2) we have (3.17) by the quite same way as in the proof of Proposition 5.1 of [8], if we set \( \tau = 1/(l+1) \) there. The detail is omitted. Q.E.D.

Set \( v_k = x_k(x_1, D_x, D_y)u \) \((k=1, 2)\) for \( u \in \mathcal{S} \). Let \( \tilde{v}_3(x, \eta) \) be the Fourier transform of \( v_3(x, y) \) with respect to \( y \). Setting \( x_3(x_1, \eta) = \phi_3(g(x_1)h(0, \eta)) \), we have \( \tilde{v}_3(x, \eta) = x_3(x, \eta) \tilde{v}_3(x, \eta) \) because \( x_3 = 1 \) on \( \text{supp } x_3(x_1, \xi, \eta) \). It follows from (3.12) that

\[ (3.18) \quad \|g(x_1)D_y^{2l-1+\kappa}v_3\|^2 \]

\[ = \|g(x_1)D_y^{12l}\langle \eta \rangle^{l-1+\kappa}x_3(x_1, \eta)g(x_1)D_y^{12l}\langle \eta \rangle^{\kappa}\tilde{v}_3\|^2 \]

\[ \leq 3\|P_0v_3\| + C\|v_3\|^2 \]

\[ \leq C'(\|P_0v_3\|^2 + \|v_3\|^2) \]

because \( \kappa = (1-\delta_0)/(l+1) \) and \( g(x_1)\langle \eta \rangle^{2l/(l+1)} < 3 \) on \( \text{supp } x_3 \). By means of (3.2), for \( 0 < j \leq 2l \) we have furthermore

\[ (3.19) \quad \|g^{(j)}(x_1)D_y^{2l-j+\kappa}v_3\| \]

\[ \leq C_j\|g(x_1)D_y^{12l-\kappa}\langle \eta \rangle^{l-j+\kappa}x_3(x_1, \eta)g(x_1)D_y^{12l}\langle \eta \rangle^\kappa\tilde{v}_3\| \]

\[ \leq C(\|P_0v_3\| + \|v_3\|). \]

It follows from the hypothesis (3.3) of Proposition 3.1 that for any \( \epsilon > 0 \) there exists a constant \( c_\epsilon > 0 \) such that
(3.20) \[ |x_1| \leq \varepsilon (\log \langle \eta \rangle)^{-1} \quad \text{on supp } x_3(x_1, \eta) \]

if \( \langle \eta \rangle \geq c_\varepsilon \),

because \((x_1, \eta) \in \text{supp } x_3\) implies \(g(x_1, \eta)^{2^{(t+1)}} < 3\). By means of Poincaré's inequality it follows from (3.20) that for a constant \(C_0\) independent of \(\varepsilon\) we have

(3.21) \[ \| (\log \langle \eta \rangle)^{t} \tilde{v}_1 \|_{L^2(R^2)} \leq C_0 e^{2t} \| D_t \tilde{v}_1 \|_{L^2(R^2)} \]

if \( \langle \eta \rangle \geq c_\varepsilon \),

because \(x_3 \tilde{v}_1 = \tilde{v}_1\) and so \(\text{supp } \tilde{v}_1(x, \eta) \cap \{ \langle \eta \rangle \geq c_\varepsilon \} \) is contained in \(\{ |x_1| \leq \varepsilon (\log \langle \eta \rangle)^{-1} \}\). Integrating (3.21) with respect to \(\eta\) we have

(3.22) \[ \| (\log \langle D_s \rangle)^{t} v_1 \|_{L^2(R^2)} \leq e \| D_t \tilde{v}_1 \|_{L^2(R^2)}^2 + C_0 \| v_1 \|_{L^2(R^2)}^2 , \]

by taking another small \(\varepsilon > 0\). Estimates (3.22) and (3.12) with \(u\) replaced by \(v_1\) give

\[ \| (\log \langle D_s \rangle)^{t} v_1 \|_{L^2(R^2)} \leq \varepsilon (P_0 v_1, v_1) + C_0' \| v_1 \|_{L^2(R^2)}^2 . \]

Replace \(v_1\) by \((\log \langle D_s \rangle)^{t} v_1\). Then, for any \(0 < \varepsilon \leq 1/2\) we have

(3.23) \[ \| (\log \langle D_s \rangle)^{2t} v_1 \|_{L^2(R^2)} \leq \varepsilon \| D_t \tilde{v}_1 \|_{L^2(R^2)}^2 + C_0' \| v_1 \|_{L^2(R^2)}^2 , \]

because for some \(C_0'\) we have

\[ C_0 \| (\log \langle D_s \rangle)^{t} v_1 \|_{L^2(R^2)} \leq (1/4) \| (\log \langle D_s \rangle)^{2t} v_1 \|_{L^2(R^2)}^2 + (C_0' / 2) \| v_1 \|_{L^2(R^2)}^2 . \]

To obtain estimates for \(v_2 = x_2(x_1, D_s, D_s)u\) we consider an operator \(\bar{P}_0 = \bar{P}_0(x_1, D_s, D_s)\) which is obtained by modifying \(P_0\) in "a neighborhood of \(x_1 = 0\)" as follows: Set

\[ \bar{P}_0(x_1, \xi, \eta) = \xi_1^{2t} + \xi_2^{2t} + (g(x_1)h(\xi, \eta) + \xi_0(x_1, \xi, \eta))h(\xi, \eta)^{-1} \eta^{2t} . \]

Then we have

**Lemma 3.4.** \(\bar{P}_0(x_1, \xi, \eta)\) belongs to \(S^{2t(t+1)}_{\lambda, \beta, \delta_0}\) and satisfies Hörmander's condition as follows:

i) There exists a constant \(c_\delta > 0\) such that

(3.24) \[ |\bar{P}_0(x_1, \xi, \eta)| \geq c_\delta \lambda(\xi, \eta)^{2t} \quad \text{for large } |\xi| + |\eta| . \]

ii) For any \(\alpha\) and \(\beta\) there exists a constant \(C_{\alpha \beta}\) such that

(3.25) \[ |\bar{P}_0^{(\alpha)}(x_1, \xi, \eta)| \leq C_{\alpha \beta} \lambda(\xi, \eta)^{\delta_0 |\beta| - |\alpha|} \quad \text{for large } |\xi| + |\eta| , \]

where \(\delta_0 = 2l_\sigma < 1\).
The proof is similar as the one of Proposition 5.3 of [8]. The detail is omitted.

In the consequence of this lemma, we have a parametrix \( Q \in S^{\frac{-2l}{\lambda+1},\delta_0} \) such that

\[
I = Q\tilde{P}_o + K, \quad K \in S^{-\infty},
\]

furthermore

\[
Q = Q_0Q_1, \quad Q_0 \in S^{\frac{-2l}{\lambda+1},\delta_0}, \quad Q_1 \in S^{\frac{2l+1}{\lambda+1},\delta_0}, \quad \sigma(Q_0) = \tilde{P}_0(x_1, \xi, \eta)^{-1} \text{ for large } |\xi| + |\eta|.
\]

Note that \( A^{2l/(l+1)}Q \in S^{\frac{2l+1}{\lambda+1},\delta_0} \) and \( \tilde{P}_0x_2 \equiv P_0x_2 \mod. S^{-\infty} \). Recall \( v_2 = x_2(x_1, D_x, D_y)u \) for \( u \in S \). It follows from (3.26) that

\[
\|v_2\|_{2l/(l+1)} \leq C(\|P_0v_2\| + \|u\|).
\]

In view of (3.27) and (3.25) it is easy to see \( g^{(2l)}(x_1)D_x^{2l-j}Q \in S^{\frac{2l+j}{\lambda+1},\delta_0} \). Noting \( \lambda(D_x, D_y)^{-1} \in S^{\frac{2l-(l+1)}{l+1},\frac{1}{\lambda+1}} \), we have

\[
\|g^{(2l)}(x_1)D_x^{2l-j}v_2\|_{\lambda} \leq C(\|P_0v_2\| + \|u\|)
\]

for \( 0 \leq j \leq 2l \). Note that, for \( k = 1, 2, [P_0, x_2]Q \in S^{\frac{2l+j}{\lambda+1},\delta_0} \) and

\[
[P_0, x_2] = [P_0, x_2]Q\tilde{P}_0 = [P_0, x_2]Q\tilde{P}_0 \equiv [P_0, x_2]Q\tilde{P}_0 \mod. S^{-\infty}.
\]

Then we have

\[
\|[P_0, x_2]u\|_{\lambda} \leq C(\|P_0u\| + \|u\|) \quad (k = 1, 2).
\]

Since \( x_1 + x_2 = I \), from (3.23), (3.28) and (3.30) we have (3.14). Furthermore, (3.15) follows from (3.18) and (3.29) together with (3.30), and we obtain (3.16) from (3.19), (3.29) and (3.30). Now, we have proved Lemma 3.2. So the proof of Proposition 3.1 for the case \( l \geq 2 \) is completed.

We remark that the proof of Proposition 3.1 for the case \( l = 1 \) also follows from the above discussion. It follows from the nonnegativeness of \( g(x_1) \) that (3.2) holds with \( \sigma = 1/2 \). Lemmas 3.3 and 3.4 still hold with \( \delta_0 = 1/2 \) if we set \( \lambda(\xi, \eta) = (|\xi|^6 + |\eta|^1 + 1)^{1/6} \) and \( h(\xi, \eta) = \lambda(\xi, \eta) \). We have (3.22) by the same way as in the case \( l \geq 2 \). We also have (3.28) with \( 2l/(l+1) \) replaced by \( 2/3 \) and (3.30) with \( \epsilon \) replaced by \( 1/3 \) because \( \lambda(D_x, D_y)^{-1} \) belongs to \( S^{\frac{1}{2l+1},\delta_0} \). Hence we have (3.6) with \( l = 1 \), which is nothing but (3) of Corollary 2. By Corollary 2 we see that Proposition 3.1 also holds when \( l = 1 \).

We finally remark that the condition (7) is satisfied for
Indeed, we have (3.7) and (3.8) with $P_0$ replaced by $\mathcal{A}$ because we did not use the assumption (3.4) in the derivation of (3.18), (3.19) and (3.28)-(3.30). So, we have (7) if we show

\[ |u| \leq C(|\mathcal{A}u| + |u|), \quad u \in S. \]

It follows from Poincaré’s inequality that

\[ |u|^2 \leq |D_{x_1} u|^2 \leq (\mathcal{A}u, u) \leq (|\mathcal{A}u|^2 + |u|^2)/2 \]

if $u \in S$ and if $\text{supp } u \subset \{ |x_1| \leq \varepsilon \}$ for a sufficiently small $\varepsilon > 0$. The estimate (3.31) easily follows from (3.32) and the elliptic estimate in the region $\{x_1 \neq 0\}$.

References

[3] Hoshiro, T., A property of operators characterized by iteration and a necessary condition for hypoellipticity, preprint in Kyoto Univ.