Nonlinear Singular First Order Partial Differential Equations Whose Characteristic Exponent Takes a Positive Integral Value

By

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Abstract

We consider nonlinear singular partial differential equations of the form \((tD_t - \rho(x))u = ta(x) + G_2 (x) (t, D_t u, u, D_{t u}, ..., D_{t u})\).

It has been proved by Gérard and Tahara that there exists a unique holomorphic solution with \(u(0, x) = 0\) if the characteristic exponent \(\rho(x)\) avoids positive integral values. In the present paper we consider what happens if \(\rho(x)\) takes a positive integral value at \(x = 0\). Generically, the solution \(u(t, x)\) is singular along the analytic set \((t = 0, \rho(x) \in \mathbb{N}^*)\), \(\mathbb{N}^* = \{1, 2, ...\}\), and we investigate how far it can be analytically continued.

§1. Introduction

In this paper we consider the following type of nonlinear singular partial differential equations:

\[(tD_t - \rho(x))u = ta(x) + G_2 (x) (t, D_t u, u, D_{t u}, ..., D_{t u}),\]

where \((t, x) \in \mathbb{C}_t \times \mathbb{C}_x, x = (x_1, ..., x_n), D_t = \partial/\partial t, D_i = \partial/\partial x_i\). We assume that \(\rho(x)\) and \(a(x)\) are holomorphic functions defined in a polydisk \(D\) centered at the origin of \(\mathbb{C}^n\) and

\[G_2 (x) (t, z, X_0, X_1, ..., X_n) = \sum_{p+q+|\alpha| \geq 2} a_{p\alpha} (x) t^p z^q X_0^{\alpha_0} ... X_n^{\alpha_n}, |\alpha| = \alpha_0 + ... + \alpha_n.\]

Here \(a_{p\alpha} (x)\) is holomorphic in \(D\) and \(\sum_{p+q+|\alpha| \geq 2} \sup_{x \in D} |a_{p\alpha} (x)| t^p z^q X_0^{\alpha_0} ... X_n^{\alpha_n}\) is a convergent power series in \((t, z, X_0, ..., X_n)\).

Now we look for a (necessarily unique) local holomorphic solution \(u(t, x)\) with \(u(0, x) = 0\). The right hand side of (1) is well-defined because of this...
initial condition.

The following theorem is proved in [1].

**Theorem 1 (Gérard-Tahara).** Let \( \xi \neq 0 \) be a point in \( D \). If \( \rho (\xi) \in \mathbb{N}^* = \{1, 2, 3, \ldots\} \), then the equation (1) has a unique holomorphic solution \( u(t, x) \) with \( u(0, x) \equiv 0 \) in a neighborhood of \( (0, \xi) \in C_1 \times C_2^* \).

In this paper we consider what happens if \( \rho (\xi) \) takes a positive integral value.

First we explain the calculation in [1]. They express \( u(t, x) \) as a power series:

\[
u(t, x) = \sum_{m \geq 1} u_m(x) t^m.
\]

Then \( \{u_m(x)\} \) satisfies the following recurrence formula:

\[
u_1(x) = \frac{a(x)}{1 - \rho(x)},
\]

and for \( m \geq 2 \)

\[
(m - \rho(x)) u_m(x) = f_m(u_1(x), 2u_2(x), \ldots, (m - 1)u_{m-1}(x), u_1(x), \ldots, u_{m-1}(x), D_1u_1, \ldots, D_{m-1}u_1, \ldots, D_{n+m-1}, (a_{p+q+|a|+m}) \).
\]

Here \( f_m \) is a polynomial whose coefficients are 1.

The assumption \( \rho (\xi) \in \mathbb{N}^* \) guarantees that \( u_m(x) \) is holomorphic in a common neighborhood of the origin of \( C_2^* \) for all \( m \). On the other hand, if \( \rho (\xi) \in \mathbb{N}^* \) then \( u_m(x) \) may be singular at \( x = \xi \) for some \( m \), and there exists no holomorphic solution with \( u(0, x) \equiv 0 \) in any neighborhood of \( (0, \xi) \in C_1 \times C_2^* \). This situation is what we would like to consider in this paper.

**Example.** The equation

\[
(tD_1 - (1-x^g)) u = tx^g + G_2(x) (t, tD_1u, u, D_1u, \ldots, D_nu), \quad g, h \in \mathbb{N}^*
\]

has a (unique) holomorphic solution \( u(t, x) = \sum_{m \geq 1} u_m(x) t^m \) if and only if \( g \leq h \).

**Remark.** It sometimes happens, as is shown in the example above, that \( u_m(x) \) determined by (3) and (4) is holomorphic for all \( m \) even when \( \rho (0) \in \mathbb{N}^* \). In [1] it is proved that \( u(t, x) = \sum_{m \geq 1} u_m(x) t^m \) is convergent in a neighborhood of the origin in such a case.

Now we assume the following:

\[
\rho (0) \in \mathbb{N}^* = \{1, 2, 3, \ldots\}, \quad \rho (x) \neq \rho (0).
\]

Under this assumption, the set \( V = \{\rho (x) = \rho (0)\} \subset C_2^* \) is an analytic set of codimension 1. The equation (1) has a unique holomorphic solution \( u(t, x) \)
with \( u(0, x) \equiv 0 \) outside \( V \), and there may be no such solution near any point in \( V \). Now we put

\[
d(x) = \text{dist}(x, V) = \text{dist}(x, \partial V),
\]

where \( \text{dist}(x, Z) \) is the distance from \( x \) to a subset \( Z \subset \mathbb{C}^n \). The second equality holds if \( x \) belongs to a sufficiently small neighborhood of the origin.

We claim that the solution \( u(t, x) \) is holomorphic in an open set of the following form:

\[
|t| < Cd(x)^p. \quad x \text{ is sufficiently close to the origin},
\]

where \( p \) and \( C \) are positive constants. A striking feature of \( p \) is that it is completely determined by \( \rho(x) \) and nothing else. A more precise statement will be given later.

Assume that \( \rho(x) - \rho(0) \) has a zero exactly of order \( g \) at \( x = 0 \). Then we have the following estimate:

\[
\left| \frac{1}{\rho(x) - \rho(0)} \right| \leq C'd(x)^{-g}.
\]

where \( C' \) and \( g \) are positive constants. The proof of this estimate will be given in Appendix.

Now we announce

**Theorem 2 (Main Theorem).**

(i) If \( \rho(0) \geq g + 2 \), then the solution \( u(t, x) \) of (1) with \( u(0, x) \equiv 0 \) is holomorphic in a domain of the form

\[
|t| < Cd(x).
\]

(ii) If \( \rho(0) < g + 2 \), then the solution \( u(t, x) \) of (1) with \( u(0, x) \equiv 0 \) is holomorphic in a domain of the form

\[
|t| < Cd(x)^{\frac{g+2}{\rho(0)}}.
\]

In both cases \( C \) is a constant \( > 0 \) determined by \( \rho(x), a(x) \) and \( G_2(t, z, X_0, X_1, ..., X_m) \).

**§2. Proof of the Main Theorem**

We express the solution \( u(t, x) \) in the form of a power series in \( t \):

\[
u(t, x) = \sum_{m=1}^{\infty} u_m(x) t^m.
\]

Then \( \{u_m(x)\} \) satisfies the following recursive formula:

\[
u_1(x) = \frac{a(x)}{1 - \rho(x)}.
\]
and for \( m \geq 2 \)
\[
(m - \rho(x)) u_m(x) = f_m(u_1(x), u_2(x), \ldots, (m - 1) u_{m-1}(x), u_1(x), u_2(x), \ldots, u_{m-1}(x), D_t u_1, \ldots, D_t u_{m-1}, D_n u_m, a_{pq}(x))_{p+q+|\alpha| \leq m}.
\]
(8)
Here \( f_m \) is a polynomial whose coefficients are 1. Put \( \rho(0) = M \in \mathbb{N}^* = \{1, 2, \ldots\} \). Then \( u_m(x) \) (\( m \geq M \)) may be singular along \( V = \{ x \in \mathbb{C}^n : \rho(x) = M \} \). It is easy to see that we have the following type of estimate:
\[
|u_m(x)| \leq C_m d(x)^{-s_m} \quad (m \geq M)
\]
in a common neighborhood of the origin, where \( C_m \) is a positive constant and \( s_m \) (\( m \geq M \)) is a positive integer. Obviously we can take \( s_M = g \). (The first \( M-1 \) terms \( s_1, \ldots, s_{M-1} \) will be given later in a technical fashion).

We have

**Proposition 1.** We can choose \( s_m = m + g - M \) (\( m \geq M \)) if \( M \geq g + 2 \). On the other hand, we can choose \( s_{M+k} = g + 2 + k - 2 \) (\( 1 \leq k \leq M - 1 \)) if \( M < g + 2 \).

**Proof.**
Obviously we have
\[
|D^k u_m(x)| \leq C_m d(x)^{-(s_m+1)} \quad m \geq M, k = 1, \ldots, n
\]
for some positive constant \( C_m \). Hence we may set, for \( m \geq M+1 \),
\[
s_m = \max \left\{ s_{m+1}, 1 \leq m \leq m - 1 \right\}
\]
\[
\cup \left\{ \frac{j}{2} (s_{m+r} + 1) ; 2 \leq j \leq m, m_j \geq 1 (1 \leq j' \leq j), \sum_{j' = 1}^j m_j' \leq m \right\}
\]
(11)

Here we set \( s_m = -1 \) (\( 1 \leq m \leq M - 1 \)). This technical choice is made in order to deal with the exceptional cases \( m = 1, \ldots, M - 1 \) where \( u_m \) is holomorphic and so \( u_m \) and its derivatives are bounded. In these cases we don’t need the kind of estimate like (10), in which the term +1 was necessary because of the singularity of \( u_m (m \geq M) \). The quantity \( s_m + 1 \) comes from \( G_2 \)'s terms which are linear in \( u_1, \ldots, u_{m-1}, D_t u_1, \ldots, D_t u_{m-1}, D_n u_m \) and \( \sum_{j' = 1}^j (s_{m+j'} + 1) \) comes from \( G_2 \)'s \( j \)-th degree terms in \( u_1, \ldots, u_{m-1}, D_t u_1, \ldots, D_t u_{m-1}, D_n u_m : \ldots, D_n u_{m-1} \).

We can simplify (11) slightly. Since \( s_m \geq s_{m-1} + 1 \) (\( m \geq M + 1 \)) follows immediately from (11), we have \( s_m \geq s_{m-1} \) (\( m \geq 2 \)). Hence we obtain
\[
s_m = \max \left\{ s_{m-1} + 1 \right\} \cup \left\{ \frac{j}{2} (s_{m+j'} + 1) ; 2 \leq j \leq m, m_j \geq 1 (1 \leq j' \leq j), \sum_{j' = 1}^j m_j' = m \right\}
\]
Moreover, by using the fact
we obtain
\[
\sum_{j=2}^{m} \left( \sum_{r=1}^{j} (s_{m_r} + 1) \right) \geq \sum_{r \in A} m_r + (\text{card}A) (g - M + 1),
\]
where \( A = \{j'; m_{j'} \geq M\} \subseteq \{1, \ldots, j\} \).

If \( A = \emptyset \), then \( \sum_{j'=1}^{j} (s_{m_r} + 1) = 0 \leq m + g - M \), where the last inequality follows from the assumption \( m \geq M + 1 \).

Next if \( \text{card}A = 1 \), we have, since each \( m_{j'} \geq 1 \),
\[
\sum_{j' \in A} m_{j'} \leq m - \text{card}A \leq m - j + 1.
\]
Here we have used the notation \( A^c = \{1, 2, \ldots, j\} \backslash A \). So it follows from (13)
\[
\sum_{j'=1}^{j} (s_{m_r} + 1) \leq m - j + 1 + (g - M + 1),
\]
because \( j \geq 2 \).

Last, if \( \text{card}A \geq 2 \),
\[
\sum_{j' \in A} m_{j'} + (\text{card}A) (g - M + 1) \leq m + 2(g - M + 1)
\]
\[
\leq m + g - M,
\]
because \( g - M \leq -2 \).

The case \( M \geq g + 2 \) has now been proved.

Next, we prove the case \( M < g + 2 \). First we assume that \( l = 1 \). The case \( k = 0 \) obviously holds. It is easy to see that \( s_m = s_{m-1} + 1 \), \( M + 1 \leq m \leq 2M - 1 \). Hence the case \( l = 1 \) is proved.

Now suppose that the claim has been shown for \( s_1, s_2, \ldots, s_{lM+k-1} \), \( l \geq 2 \), \( 0 \leq k \leq M - 1 \). Then we have
\[
(s_{(l-1)M} + 1) + (s_{M+k} + 1) = ((l-1)(g+2) - 1) + \{(g+2) + k - 1\}
\]
\[
= l(g+2) + k - 2.
\]
We are going to prove that this attains the maximum of the right hand side of (12). When \( l_1 + \cdots + l_j = l - l' \), \( k_1 + \cdots + k_j = Ml' + k \),

\[
(s_{l_1M+k_1}+1) + \cdots + (s_{l_jM+k_j}+1) = \sum_{j' \in A} (l_{j'} (g+2) + k_{j'} - 1)
\]

\[
= (g+2) \sum_{j' \in A} l_{j'} + \sum_{j' \in A} k_{j'} - \text{card}A
\]

for \( A = \{ j' ; l_{j'} \geq 1 \} \subset \{ 1, 2, \ldots, l \} \). Remark that \( \sum_{j' \in A} l_{j'} = \sum_{j=1}^{l-1} l_{j'} = l - l' \) and that \( k_{j'} \geq 1 \) if \( l_{j'} = 0 \). If \( A = \emptyset \), then the right hand side is equal to \( 0 \leq (g+2) + k - 2 \). The claim obviously holds in this case.

Next if \( A \neq \emptyset \), then

\[
(s_{l_1M+k_1}+1) + \cdots + (s_{l_jM+k_j}+1) = (l-l') (g+2) + \sum_{j' \in A} k_{j'} - \text{card}A.
\]

Let us estimate the second term in the right hand side. If \( j' \notin A \) then \( l_{j'} = 0 \) and \( k_{j'} \geq 1 \). So it follows that

\[
\sum_{j' \in A} k_{j'} = (Ml' + k) - \sum_{j' \in A} k_{j'}
\]

\[
\leq (Ml' + k) - \text{card}A
\]

\[
= (Ml' + k) - (j - \text{card}A).
\]

By combining (14) and (15) we obtain

\[
(s_{l_1M+k_1}+1) + \cdots + (s_{l_jM+k_j}+1)
\]

\[
\leq (l-l') (g+2) + (Ml' + k) - j
\]

\[
= l(g+2) + l'(g+2) + M + k - j
\]

\[
\leq l(g+2) + k - j
\]

\[
\leq l(g+2) + k - 2.
\]

Thus we have proved that \( s_{lM+k} = l(g+2) + k - 2 \).

We will make use of the following

**Lemma 1.** Let \( \Omega \) be a domain in \( \mathbb{C}^n \), \( x = (x_1, \ldots, x_n) \), and assume that a

holomorphic function \( u(x) \) in \( \Omega \) satisfies

\[
|u(x)| \leq C(r), \quad a \in \mathbb{N} = \{ 0, 1, 2, \ldots \},
\]

where \( r = \text{dist}(x; \partial \Omega) \) is the distance from \( x \) to the boundary \( \partial \Omega \) of \( \Omega \) and \( C(r) \) is a

polynomial in \( r \) of degree \( \leq a \) with non-negative coefficients. Then we have

\[
|\partial_i u(x)| \leq e^{(a+1)C(r)}, \quad i = 1, \ldots, n.
\]
Proof.
We may assume that \( i = 1 \) without loss of generality. Goursat's formula implies that
\[
D_t u(x) = \frac{1}{2\pi \sqrt{-1}} \oint_\Gamma \frac{u(y, x_2, \ldots, x_n)}{(x_1 - y)^2} \, dy,
\]
where \( \Gamma = \{ y - x_1 = \frac{1}{a+1} r \} \subset C_x \). Since
\[
\text{dist}((y, x_2, \ldots, x_n); \partial \Omega) \geq r - \frac{r}{a+1} = \frac{a}{a+1} r \quad \text{for} \quad y \in \Gamma,
\]
we have by writing \( C(r) = \sum_{j=0}^{a} C_j r^j \).
\[
\sup_{y \in \Gamma} |u(y, x_2, \ldots, x_n)| \leq \sum_{j=0}^{a} C_j \left( \frac{a}{a+1} r \right)^{r-a} \leq \left( \frac{a}{a+1} \right)^{r-a} \sum_{j=0}^{a} C_j r^{r-a} \leq e \frac{C(r)}{r^a}.
\]
Hence we get the following estimate:
\[
|D_t u(x)| \leq \frac{1}{2\pi} \cdot 2\pi \frac{r}{a+1} \cdot \frac{1}{\left( \frac{a}{a+1} r \right)^2} \cdot e \frac{C(r)}{r^a} = e \frac{(a+1) C(r)}{r^a+1}.
\]

Now we come back to the proof of the Main Theorem.
In a sufficiently small neighborhood of the origin, we may assume that the following estimates hold:
\[
|j u_j(x)| \leq A_j, \quad |D_i u_i(x)| \leq A_i, \quad (j = 1, \ldots, M - 1, \ i = 1, \ldots, n),
\]
\[
|u_M(x)| \leq A d(x)^{-\gamma}, \quad |D_i u_M(x)| \leq A d(x)^{-\gamma - (p+1)}, \quad (i = 1, \ldots, n),
\]
\[
s_m + 1 \leq N (m \geq M), \quad |m - \rho (x)| \geq N \sigma m (m \geq M + 1),
\]
\[
|a_{\rho m} (x)| \leq A_{\rho m}.
\]
where \( A, N, \sigma \) and \( A_{\rho m} \) are positive constants and \( \sum_{\rho+q+|\alpha| \geq 2} A_{\rho q m} t^\rho x_0^\sigma x_1^{\alpha_1} \cdots x_n^{\alpha_n} \) is a convergent power series.
Consider now the following analytic equation (with parameter \( d > 0 \)):
\[
\sigma Y = \sigma (A_1 + A t^2 + \cdots + A t^{M-1} + \frac{A}{d^{p+1}} t^p)
\]
\[
+ \frac{1}{d} \sum_{\rho+q+|\alpha| \geq 2} A_{\rho q m} t^\rho Y^\sigma (e^Y)^{\alpha_1} \cdots (e^Y)^{\alpha_n}
\]
\[
- \frac{1}{d} \sum_{m=2}^{M} B_m t^m,
\]
where $B_m$ is the coefficient defined by the following identity:

$$
\sum_{m=2}^{\infty} B_m t^m = \sum_{p+q+|\alpha| \geq 2} A_{p,q,\alpha} \alpha_1^{1+\cdots+\alpha_q^{p}} (At + A t^2 + \cdots + A t^{M-1})^{q+|\alpha|},
$$

$$|\alpha| = \alpha_0 + (\alpha_1 + \cdots + \alpha_n).$$

By the implicit function theorem, this analytic equation has a unique holomorphic solution $Y$ of the form

$$Y = \sum_{m \geq 1} Y_m(d) t^m.$$

Here $Y_m(d)$ is determined by the following type of recursive formula:

$$Y_1 = \cdots = Y_{M-1} = A, \quad Y_M = \frac{A}{d^{g+1}},$$

and for $m \geq M+1$,

$$\sigma Y_m = \frac{1}{d} F_m(Y_1, \ldots, Y_{M-1}; e Y_1, \ldots, e Y_{M-1}; \{A_{p,q,\alpha}\}_{p+q+|\alpha| \leq m}),$$

where $F_m$ is a polynomial with positive coefficients.

It is easy to see that $Y_m(d)$ is of the form

$$Y_m(d) = \frac{C_m(d)}{d^{t_m}},$$

where $C_m$ is a polynomial of order $\leq t_m$ with non-negative coefficients. Here $t_m = 0 (1 \leq m \leq M-1)$, $t_M = g + 1$ and for $m \geq M+1$

$$t_m = 1 + \max \left\{ t_{m_i}; 1 \leq m_i \leq m-1 \right\}$$

$$\cup \left\{ \sum_{j=1}^{t_{m_i}} 2 \leq j \leq m, m_{j'} \geq 1 (1 \leq j' \leq j), \sum_{j=1}^{t_{m_i}} m_{j'} \leq m \right\}. \quad (16)$$

It is obvious that $t_m = s_m + 1 (m \geq 1)$. So we have

$$t_m = m + g - M + 1 (m \geq M) \text{ if } M \geq g + 2,$$

$$t_{M+k} = (g + 2) + k - 1 (l \geq 1, 0 \leq k \leq M-1) \text{ if } M < g + 2.$$

We are going to prove that $Y$ is a majorant power series of $u$ if $d = d(x)$. More precisely, we want to show that for $m \geq 1$

$$|u_m(x)| \leq |mu_m(x)| \leq Y_m(d) \quad (17)$$

$$|D^i u_m(x)| \leq i Y_m(d), \quad i = 1, 2, \ldots, n. \quad (18)$$

The cases $m = 1, 2, \ldots, M$ are obviously true. We will prove the remaining cases by induction on $m$. Suppose that the above inequalities have been shown to be true for $u_1, u_2, \ldots, u_{m-1}$. Then we have
\[ |u_m(x)| \]
\[ \leq \frac{1}{m - \rho(x)} f_m \left( |u_1|, 2|u_2|, \ldots, (m - 1)|u_{m-1}|, |u_1|, |u_2|, \ldots, |u_{m-1}| \right) \]
\[ |D_1 u_1|, \ldots, |D_1 u_4|, \ldots, |D_1 u_{m-1}|, \ldots, |D_1 u_{m-1}|; \{ |a_{p+q}| \} \]
\[ \leq \frac{1}{N m} f_m \left( |u_1|, 2|u_2|, \ldots, (m - 1)|u_{m-1}|, |u_1|, |u_2|, \ldots, |u_{m-1}|, \right) \]
\[ |D_1 u_1|, \ldots, |D_1 u_4|, \ldots, |D_1 u_{m-1}|, \ldots, |D_1 u_{m-1}|; \{ |a_{p+q}| \} \]
\[ \leq \frac{1}{N m} f_m \left( Y_1, Y_2, \ldots, Y_{m-1}, Y_1, Y_2, \ldots, Y_{m-1}, \right) \]
\[ e Y_1, \ldots, e Y_1, \ldots, e Y_{m-1}, \ldots, e Y_{m-1}; \{ A_{p+q} \} \]
\[ \text{It's an equality!} \]
\[ = \frac{1}{N m} \right. \]
\[ \text{Therefore we obtain} \]
\[ |m u_m(x)| \leq \frac{d}{N m} Y_m(d) \leq Y_m(d). \]

Here we assume that \( x \) is in a ball of radius \( <N \) centered at the origin. Hence \( 0 < d < N \). Moreover, since
\[ |u_m(x)| \leq \frac{1}{N m} d Y_m(d) = \frac{C_m(d)}{N m \ d^{m-1}}, \]
we deduce by using the lemma that
\[ |D_1 u_m(x)| \leq \frac{1}{N m} e C_m(d) \leq e C_m(d) = e Y_m(d). \]

So induction proceeds. We have proved that \( u \ll Y \).

Our next investigation is about the convergence of \( Y = \sum_{m \geq 1} Y_m(d) t^m \) and \( u(t, x) = \sum_{m \geq 1} u_m(x) t^m \).

Fix some sufficiently small \( d_0 > 0 \). Then for some \( T > 0 \), the series \( \sum_{m \geq 1} Y_m(d_0) T^m \) is convergent by the implicit function theorem.

Let us consider the case \( M \geq g + 2 \), where \( t_m = m + g - M + 1 \ (m \geq M) \). We have
\[ \sum_{m \geq M} Y_m(d_0) T^m = \sum_{m \geq M} \frac{C_m(d_0)}{d_0^{m+g-M+1}} T^m = \frac{1}{d_0^{g-M+1}} \sum_{m \geq M} C_m(d_0) \left( \frac{T}{d_0} \right)^m. \]

and if \( |t/d| < |T/d_0| \) and \( 0 < d \leq d_0 \), then
\[ u \ll Y = \sum_{m \geq 1} Y_m(d) t^m \]
\[ = \sum_{1 \leq m \leq M-1} Y_m(d) t^m + \sum_{m \geq M} \frac{C_m(d)}{d_0^{m+g-M+1} t^m} \]
\[ = \sum_{1 \leq m \leq M-1} Y_m(d) t^m + \frac{1}{d_0^{g-M+1}} \sum_{m \geq M} C_m(d) \left( \frac{t}{d} \right)^m. \]
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So for \( x \) sufficiently close to the origin, there exists a positive constant \( C \) such that \( u(t, x) \) is holomorphic in \( |t| < Cd(x)^{q+2} \). (In fact \( C = |T/d_0| \).

Next, we consider the case \( M < g + 2 \), where \( t_{IM+k} = t\ (g+2) + k-1 \) \((i \geq 1, 0 \leq k \leq M-1)\). We have

\[
\sum_{m \geq M} Y_m(d_0) d^m + \sum_{k=0}^{M-1} \frac{1}{d_0^{g+1}} \sum_{m \geq M} C_m(d_0) \left( \frac{T}{d_0} \right)^m < \infty.
\]

Hence if \( |t^M/d^{g+2}| < |T^M/d^{g+2}| \) and \( 0 < d \leq d_0 \), then

\[
\sum_{m \geq 1} Y_m(d) t^m = \sum_{m \geq 1} Y_m(d) t^m + \sum_{k=0}^{M-1} \frac{1}{d_0^{g+1}} \sum_{m \geq M} C_m(d_0) \left( \frac{T}{d_0} \right)^m \leq \infty.
\]

So for \( x \) sufficiently close to the origin, there exists a positive constant \( C \) such that \( u(t, x) \) is holomorphic in \( |t^M| < Cd(x)^{q+2} \).

The proof of the Main Theorem has now been completed.

§3. Appendix

We give the proof due to T. Oaku of the estimate (6) in the Introduction. The author is very much grateful for him.

**Proposition 2.** Let \( f(x) \) be a holomorphic function defined in a neighborhood \( \Omega \) of the origin of \( \mathbb{C}^n_x \), \( x = (x_1, \ldots, x_n) \). Assume that \( f(x) \) has a zero exactly of order \( g \in \mathbb{N} \) at the origin. Denote by \( V \) the set \( \{ x \in \Omega : f(x) = 0 \} \), and by \( d(x) \) the distance from \( x \in \Omega \) to \( V \). Then there exist a neighborhood \( \Omega' \subset \Omega \) of the origin and a positive constant \( C > 0 \) such that \( |f(x)| \geq Cd(x)^g \), \( x \in \Omega' \).

**Proof.**

Let the Maclaurin expansion of \( f(x) \) be \( f(x) = \sum_{|\alpha| \geq g} f_\alpha x^\alpha \). Set \( f_\theta(x) = \sum_{|\alpha| = \theta} f_\alpha x^\alpha \), which is a nonzero homogeneous polynomial. By applying a suitable linear change of coordinates if necessary, we may assume that \( f_{\theta,0,\ldots,0} \neq 0 \). The preparation theorem of Weierstrass implies that \( f(x) \) can be written in the following form:
where \( c(x) \) is a holomorphic function defined near the origin 0 of \( \mathbb{C}^n \) and each \( a_i(x') (i = 1, 2, \ldots, g) \) is holomorphic near the origin 0' of \( \mathbb{C}^{r-1} \), \( x' = (x_2, x_3, \ldots, x_n) \). So there exist functions \( \phi_1(x'), \ldots, \phi_g(x') \) (which are not necessarily holomorphic) such that

\[
f(x) = c(x) \prod_{j=1}^g (x_1 - \phi_j(x')) , \quad \phi_1(0') = \cdots = \phi_g(0') = 0.
\]

If \( x \) is sufficiently close to the origin, we have

\[
\frac{1}{2}|c(0)| \prod_{j=1}^g |x_1 - \phi_j(x')| \geq \frac{1}{2} |c(0)| d(x)^g.
\]

This kind of estimate does not hold in the real analytic case. We give a counterexample.

Set \( f(x_1, x_2) = -x_1^2 + x_2^2, (x_1, x_2) \in \mathbb{R}^2 \). The function \( f \) has a zero of order 2 at \((0, 0)\). Set \( V^R = \{(x_1, x_2) \in \mathbb{R}^2; f(x_1, x_2) = 0\} \). If \( d > 0 \) then the distance to \( V^R \) from the point \((-d, 0)\) is \( d \). It is attained by \((0, 0)\). Note that \( f(-d, 0) = d^2 \).

This is not a paradox. There are some points in \( V^C = \{(z_1, z_2) \in \mathbb{C}^2; f(z_1, z_2) = 0\} \) that are closer to \((-d, 0)\) than the origin is. Such points are found, for example, on the intersection of \( V^C \) and \( \mathbb{R}_{x_1} \times i\mathbb{R}_{y_2} \subset \mathbb{C}^2_{(x_1, y_1, x_2, y_2)} = \mathbb{C}_{(x_1, x_2)} \). In fact, the equation \( f(x_1, iy_2) = -x_1^2 - y_2^2 = 0 \) defines a curve in \( \mathbb{R}^2_{(x_1, y_2)} \) \( \ni (-d, 0) \) which is very close to the point \((-d, 0)\) if \( d > 0 \) is sufficiently small.

The reader is referred to [5] for estimates of real analytic functions from below.

**References**


