A family of anisotropic integral operators
and behavior of its maximal eigenvalue

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Abstract. We study the family of compact integral operators $K_\beta$ in $L^2(\mathbb{R})$ with the kernel

$$K_\beta(x, y) = \frac{1}{\pi} \frac{1}{1 + (x - y)^2 + \beta^2 \Theta(x, y)},$$

depending on the parameter $\beta > 0$, where $\Theta(x, y)$ is a symmetric non-negative homogeneous function of degree $\gamma \geq 1$. The main result is the following asymptotic formula for the maximal eigenvalue $M_\beta$ of $K_\beta$:

$$M_\beta = 1 - \lambda_1 \beta^{\frac{2}{\gamma + 1}} + o(\beta^{\frac{2}{\gamma + 1}}), \quad \beta \to 0,$$

where $\lambda_1$ is the lowest eigenvalue of the operator $A = |d/dx| + \Theta(x, x)/2$. A central role in the proof is played by the fact that $K_\beta, \beta > 0$, is positivity improving. The case $\Theta(x, y) = (x^2 + y^2)^2$ has been studied earlier in the literature as a simplified model of high-temperature superconductivity.

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1. Introduction and the main result

1.1. Introduction. The object of the study is the following family of integral operators on $L^2(\mathbb{R})$:

$$K_\beta u(x) = \int K_\beta(x, y) u(y) dy,$$  \hspace{1cm} (1)

(here and below we omit the domain of integration if it is the entire real line $\mathbb{R}$) with the kernel

$$K_\beta(x, y) = \frac{1}{\pi} \frac{1}{1 + (x - y)^2 + \beta^2 \Theta(x, y)},$$  \hspace{1cm} (2)
where $\beta > 0$ is a small parameter, and the function $\Theta = \Theta(x, y)$ is a homogeneous non-negative function of $x$ and $y$ such that

$$\Theta(tx, ty) = t^\gamma \Theta(x, y), \quad \gamma > 0,$$

for all $x, y \in \mathbb{R}$ and $t > 0$, and the following conditions are satisfied:

$$\begin{cases} c \leq \Theta(x, y) \leq C, \\ |x|^2 + |y|^2 = 1, \\ \Theta(x, y) = \Theta(y, x), \quad x, y \in \mathbb{R}. \end{cases}$$

By $C$ or $c$ (with or without indices) we denote various positive constants whose value is of no importance. The conditions (3) and (4) guarantee that the operator $K_\beta$ is self-adjoint and compact.

Such an operator, with $\Theta(x, y) = (x^2 + y^2)^2$ was suggested by P. Krotkov and A. Chubukov in [6] and [7] as a simplified model of high-temperature superconductivity. The analysis in [6] and [7] reduces to the asymptotics of the top eigenvalue $M_\beta$ of the operator $K_\beta$ as $\beta \to 0$. Heuristics in [6] and [7] suggest that $M_\beta$ should behave as $1 - w\beta^{\frac{2}{5}} + o(\beta^{\frac{2}{5}})$ with some positive constant $w$. A mathematically rigorous argument given by B. S. Mityagin in [9] produced a two-sided bound supporting this formula. The aim of the present paper is to find and justify an appropriate two-term asymptotic formula for $M_\beta$ as $\beta \to 0$ for a homogeneous function $\Theta$ satisfying (3), (4), and some additional smoothness conditions (see (8)).

As $\beta \to 0$, the operator $K_\beta$ converges strongly to the positive-definite operator $K_0$, which is no longer compact. The norm of $K_0$ is easily found using the Fourier transform

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int e^{-i\xi x} f(x) dx,$$

which is unitary on $L^2(\mathbb{R})$. Then one checks directly that

$$\begin{align*}
\text{the Fourier transform of } m_t(x) = \frac{1}{\pi(\frac{t}{t^2 + x^2}), \quad t > 0,} \\
\text{equals } \hat{m}_t(\xi) = \frac{1}{\sqrt{2\pi}} e^{-t|\xi|},
\end{align*}$$

and hence the operator $K_0$ is unitarily equivalent to the multiplication by the function $e^{-|\xi|}$, which means that $\|K_0\| = 1$.

### 1.2. The main result

For the maximal eigenvalue $M_\beta$ of the operator $K_\beta$ denote by $\Psi_\beta$ a corresponding normalized eigenfunction. Note that the operator $K_\beta$ is positivity improving, i.e. for any non-negative non-zero function $u$ the function $K_\beta u$ is positive a.a. $x \in \mathbb{R}$ (see [12], Chapter XIII.12). Thus, by [12], Theorem XIII.43 (or by [3], Theorem 13.3.6), the eigenvalue $M_\beta$ is non-degenerate and the eigenfunction $\Psi_\beta$ can be assumed to be positive a.a. $x \in \mathbb{R}$. From now on we always choose $\Psi_\beta$ in this
way. Note in passing that due to the continuity of the kernel $K_\beta(x, y)$ in the variable $x$ the function $\Psi_\beta$ is in fact continuous and strictly positive for all $x \in \mathbb{R}$.

The behavior of $M_\beta$ as $\beta \to 0$, is governed by the model operator

$$(Au)(x) = |D_x|u(x) + 2^{-1}\theta(x)u(x),$$

where

$$\theta(x) = \Theta(x, x) = \begin{cases} |x|^{\gamma}\Theta(1, 1), & x \geq 0; \\ |x|^{\gamma}\Theta(-1, -1), & x < 0. \end{cases}$$

This operator is understood as the pseudo-differential operator $Op(a)$ with the symbol

$$a(x, \xi) = |\xi| + 2^{-1}\theta(x).$$

For the sake of completeness recall that $P = Op(p)$ is a pseudo-differential operator with the symbol $p = p(x, \xi)$ if

$$(Pu)(x) = \frac{1}{2\pi} \int \int e^{i(x-y)\xi} p(x, \xi)u(y)dyd\xi$$

for any Schwartz class function $u$. The operator $A$ is essentially self-adjoint on $C_0^\infty(\mathbb{R})$, and has a purely discrete spectrum (see e.g. [14], Theorems 26.2, 26.3). Using the von Neumann Theorem (see e.g. [11], Theorem X.25), one can see that $A$ is self-adjoint on $D(A) = D(|D_x|) \cap D(|x|^{\gamma})$, i.e $D(A) = H^1(\mathbb{R}) \cap L^2(\mathbb{R}, |x|^{2\gamma})$. Denote by $\lambda_l > 0, l = 1, 2, \ldots$ the eigenvalues of $A$ arranged in ascending order, and by $\varphi_l$ the corresponding normalized eigenfunctions. As shown in Lemma 2, the lowest eigenvalue $\lambda_1$ is non-degenerate and its eigenfunction $\varphi_1$ can be chosen to be non-negative a.a. $x \in \mathbb{R}$. From now on we always choose $\varphi_1$ in this way.

The main result of this paper is contained in the next theorem.

**Theorem 1.** Let $K_\beta$ be an integral operator defined by (1) with $\gamma \geq 1$. Suppose that the function $\Theta$ satisfies conditions (3), (4), and the following Lipshitz conditions:

$$\begin{cases} |\Theta(t, 1) - \Theta(1, 1)| \leq C|t - 1|, & t \in (1 - \varepsilon, 1 + \varepsilon), \\ |\Theta(t, -1) - \Theta(-1, -1)| \leq C|t + 1|, & t \in (-1 - \varepsilon, -1 + \varepsilon), \end{cases}$$

with some $\varepsilon > 0$. Let $M_\beta$ be the largest eigenvalue of the operator $K_\beta$ and let $\Psi_\beta$ be the corresponding eigenfunction. Then

$$\lim_{\beta \to 0} \beta^{-\frac{2}{\nu+1}} (1 - M_\beta) = \lambda_1.$$

Moreover, the rescaled eigenfunctions $\alpha^{-\frac{1}{2}} \Psi_\beta(\alpha^{-1} \cdot), \alpha = \beta^{-\frac{2}{\nu+1}},$ converge in norm to $\varphi_1$ as $\beta \to 0$. 
The eigenvalue $M_\beta$ was studied by B. Mityagin in [9] for $\Theta(x, y) = (x^2 + y^2)^\sigma$, $\sigma > 0$. It was conjectured that $\lim_{\beta \to 0} \beta^{-2/2\sigma + 1} (1 - M_\beta) = L$ with some $L > 0$, but only the two-sided bound

$$c\beta^{-2/2\sigma + 1} \leq 1 - M_\beta \leq C\beta^{-2/2\sigma + 1},$$

with some constants $0 < c \leq C$ was proved. It was also conjectured that in the case $\sigma = 2$ the constant $L$ should coincide with the lowest eigenvalue of the operator $|D_x| + 4x^4$. Note that for this case the corresponding operator (6) is in fact $|D_x| + 2x^4$. J. Adduci found an approximate numerical value $\lambda_1 = 0.978 \ldots$ in this case, see [1].

Similar eigenvalue asymptotics were investigated by H. Widom in [15] for integral operators with difference kernels. Some ideas of [15] are used in the proof of Theorem 1.

Let us now establish the non-degeneracy of the eigenvalue $\lambda_1$.

**Lemma 2.** Let $A$ be as defined in (6). Then

1. the semigroup $e^{-tA}$ is positivity improving for all $t > 0$,
2. the lowest eigenvalue $\lambda_1$ is non-degenerate, and the corresponding eigenfunction $\varphi_1$ can be chosen to be positive a.a. $x \in \mathbb{R}$.

**Proof.** The non-degeneracy of $\lambda_1$ and positivity of the eigenfunction $\varphi_1$ would follow from the fact that $e^{-tA}$ is positivity improving for all $t > 0$, see [12], Theorem XIII.44. The proof of this fact is done by comparing the semigroups for the operators $A$ and $A_0 = |D_x|$. Using (5) it is straightforward to find the integral kernel of $e^{-tA_0}$:

$$m_t(x - y) = \frac{1}{\pi} \frac{t}{t^2 + (x - y)^2}, \quad t > 0,$$

which shows that $e^{-tA_0}$ is positivity improving. To extend the same conclusion to $e^{-tA}$ let

$$V_n(x) = \begin{cases} 2^{-1} \theta(x), & |x| \leq n, \\ 2^{-1} \theta(\pm n), & \pm x > n, \end{cases} \quad n = 1, 2, \ldots .$$

Since $(A_0 + V_n)f \to Af$ and $(A - V_n)f \to A_0 f$ as $n \to \infty$ for any $f \in C_0^\infty(\mathbb{R})$, by [10], Theorem VIII.25a, the operators $A_0 + V_n$ and $A - V_n$ converge to $A$ and $A_0$ resp. in the strong resolvent sense as $n \to \infty$. Thus by [12], Theorem XIII.45, the semigroup $e^{-tA}$ is also positivity improving for all $t > 0$, as required.

**1.3. Rescaling.** As a rule, instead of $K_\beta$ it is more convenient to work with the operator obtained by rescaling $x \to \alpha^{-1}x$ with $\alpha > 0$. Precisely, let $U_\alpha$ be the unitary operator on $L^2(\mathbb{R})$ defined as $(U_\alpha f)(x) = \alpha^{-\frac{1}{2}} f(\alpha^{-1}x)$. Then $U_\alpha K_\beta U_\alpha^*$ is the integral operator with the kernel

$$\frac{\alpha}{\pi} \frac{1}{\alpha^2 + (x - y)^2 + \beta^2 \alpha^{-2} \Theta(x, y)}.$$
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Under the assumption $\beta^2 = \alpha^{\gamma + 1}$, this kernel becomes
\[
B_\alpha(x, y) = \frac{\alpha}{\pi} \frac{1}{\alpha^2 + (x - y)^2 + \alpha^3 \Theta(x, y)}.
\] (9)

Thus, denoting the corresponding integral operator by $B_\alpha$, we get
\[
K_\beta = U_\alpha^* B_\alpha U_\alpha, \quad \alpha = \beta^{\frac{2}{\gamma + 1}}.
\] (10)

Henceforth the value of $\alpha$ is always chosen as in this formula.

Denote by $\mu_\alpha$ the maximal eigenvalue of the operator $B_\alpha$, and by $\psi_\alpha$ – the corresponding normalized eigenfunction. By the same token as for the operator $K_\beta$, the eigenvalue $\mu_\alpha$ is non-degenerate and the choice of the corresponding eigenfunction $\psi_\alpha$ is determined uniquely by the requirement that $\psi_\alpha > 0$. Moreover,
\[
\mu_\alpha = M_\beta, \quad \psi_\alpha(x) = (U_\alpha \Psi_\beta)(x) = \alpha^{-\frac{1}{\gamma}} \Psi_\beta(\alpha^{-1}x), \quad \alpha = \beta^{\frac{2}{\gamma + 1}}.
\] (11)

This rescaling allows one to rewrite Theorem 1 in a somewhat more compact form.

**Theorem 3.** Let $\gamma \geq 1$ and suppose that the function $\Theta$ satisfies conditions (3), (4), and (8). Then
\[
\lim_{\alpha \to 0} \alpha^{-1}(1 - \mu_\alpha) = \lambda_1.
\]
Moreover, the eigenfunctions $\psi_\alpha$, converge in norm to $\varphi_1$ as $\alpha \to 0$.

The rest of the paper is devoted to the proof of Theorem 3, which immediately implies Theorem 1.

2. “De-symmetrization” of $K_\beta$ and $B_\alpha$

First we de-symmetrize the operator $K_\beta$. Denote
\[
K^{(l)}_\beta u(x) = \int K^{(l)}_\beta(x, y)u(y)dy,
\]
with the kernel
\[
K^{(l)}_\beta(x, y) = \frac{1}{\pi} \frac{1}{1 + (x - y)^2 + \beta^2 \theta(x)}.
\]

**Lemma 4.** Let $\beta \leq 1$ and $\gamma \geq 1$. Suppose that the conditions (3), (4), and (8) are satisfied. Then
\[
\| K^{(l)}_\beta - K_\beta \| \leq C \gamma \beta^\frac{2}{\gamma}.
\] (12)
Proof. Due to (3) and (4),
\[ c(|t| + 1)^\gamma \leq \Theta(t, \pm 1) \leq C(|t| + 1)^\gamma, \quad t \in \mathbb{R}. \] (13)
Also,
\[ \begin{cases}
|\Theta(t, 1) - \Theta(1, 1)| \leq C(|t| + 1)^{\gamma-1}|t - 1|, \\
|\Theta(t, -1) - \Theta(-1, -1)| \leq C(|t| + 1)^{\gamma-1}|t + 1|,
\end{cases} \] (14)
for all \( t \in \mathbb{R} \). Indeed, (8) leads to the first inequality (14) for \(|t - 1| < \varepsilon\). For \(|t - 1| \geq \varepsilon\) it follows from (13) that
\[ |\Theta(t, 1) - \Theta(1, 1)| \leq C(|t| + 1)^\gamma \leq C'\varepsilon^{-1}(|t| + 1)^{\gamma-1}|t - 1|. \]
The second bound in (14) is checked similarly.

Now we can estimate the difference of the kernels
\[ K_\beta(x, y) - K_\beta^{(l)}(x, y) \]
\[ = \frac{1}{\pi} \frac{\beta^2(\Theta(x, x) - \Theta(x, y))}{(1 + (x - y)^2 + \beta^2(\Theta(x, x) - \Theta(x, y)))(1 + (x - y)^2 + \beta^2(\Theta(x, x)))}. \] (15)
It follows from (14) with \( t = y|x|^{-1} \) that
\[ |\Theta(x, x) - \Theta(y, x)| \leq C(|x| + |y|)^{\gamma-1}|x - y|. \]
Substituting into (15), we get
\[ |K_\beta(x, y) - K_\beta^{(l)}(x, y)| \leq C \frac{|x - y|}{(1 + (x - y)^2)^2-\delta} \frac{\beta^2(|x| + |y|)^{\gamma-1}}{(1 + \beta^2(|x| + |y|)^{\gamma})^\delta}, \]
for any \( \delta \in (0, 1) \). The second factor on the right-hand side does not exceed
\[ \beta^2 \max_{t \geq 0} \frac{t^{\gamma-1}}{(1 + t)^\delta} = C\beta^2 \frac{t^{\gamma-1}}{(1 + t)^\delta}, \]
under the assumption that \( \delta \geq 1 - \gamma^{-1} \). Therefore
\[ |K_\beta(x, y) - K_\beta^{(l)}(x, y)| \leq C\beta^2 \frac{|x - y|}{(1 + (x - y)^2)^2-\delta}. \]
For any \( \delta \in (0, 1) \) the right hand side is integrable in \( x \) (or \( y \)). Now, estimating the norm using the standard Schur Test, see Proposition 15, we conclude that
\[ \|K_\beta - K_\beta^{(l)}\| \leq C\beta^2 \int \frac{|t|}{(1 + t^2)^{2-\delta}}dt \leq C'\beta^2, \]
which is the required bound. \( \square \)
Similarly to the operator $K_\beta$, it is readily checked by scaling that the operator $K^{(l)}_\beta$ is unitarily equivalent to the operator $B^{(l)}_\alpha$ with the kernel

$$B^{(l)}_\alpha(x, y) = \frac{\alpha}{\pi \alpha^2 + (x - y)^2 + \alpha^3 \theta(x)}.$$ \hspace{1cm} (16)

Thus the bound (12) ensures that

$$\|B_\alpha - B^{(l)}_\alpha\| = \|K_\beta - K^{(l)}_\beta\| \leq C \alpha^{1 + \frac{1}{\gamma}}, \quad \alpha \leq 1,$$ \hspace{1cm} (17)

see (10) for the definition of $\alpha$.

### 3. Approximation for $B^{(l)}_\alpha$

#### 3.1. Symbol of $B^{(l)}_\alpha$

Now our aim is to show that the operator $I - \alpha A$ is an approximation of the operator $B^{(l)}_\alpha$, defined above. To this end we need to represent $B^{(l)}_\alpha$ as a pseudo-differential operator. Rewriting the kernel (16) as

$$B^{(l)}_\alpha(x, y) = \frac{1}{m_\alpha (x - y)}, \quad t = g_\alpha(x),$$

with

$$g_\alpha(x) = \sqrt{1 + \alpha \theta(x)},$$ \hspace{1cm} (18)

and using (5), we can write for any Schwartz class function $u$:

$$(B^{(l)}_\alpha u)(x) = \frac{1}{2\pi} \int \int e^{i(x-y)\xi} b^{(l)}_\alpha(x, \xi) u(y) dy d\xi,$$

where

$$b^{(l)}_\alpha(x, \xi) = \frac{1}{g_\alpha(x)} e^{-\alpha |\xi| g_\alpha(x)}.$$

Thus $B^{(l)}_\alpha = \text{Op}(b^{(l)}_\alpha)$.

#### 3.2. Approximation for $B^{(l)}_\alpha$

Let the operator $A$ and the symbol $a(x, \xi)$ be as defined in (6) and (7). Our first objective is to check that the error

$$r_\alpha(x, \xi) = b^{(l)}_\alpha(x, \xi) - (1 - \alpha a(x, \xi))$$

is small in a certain sense. The condition $\gamma \geq 1$ will allow us to use standard norm estimates for pseudo-differential operators. Using the formula

$$e^{-\alpha y} = 1 - \alpha y + \alpha \int_0^y (1 - e^{-\alpha t}) dt, \quad y > 0,$$
we can split the error as follows:

\[ r_\alpha(x, \xi) = r_\alpha^{(1)}(x) + r_\alpha^{(2)}(x, \xi), \]

\[ r_\alpha^{(1)}(x) = \frac{1}{g(x)} + \alpha 2^{-1} \theta(x) - 1, \]

\[ r_\alpha^{(2)}(x, \xi) = \frac{\alpha}{g(x)} \int_0^{\|\xi\|g(x)} (1 - e^{-\alpha t}) dt, \]

where we have used the notation \( g(x) = g_\alpha(x) \) with \( g_\alpha \) defined in (18). Since \( \gamma \geq 1 \), we have

\[ |g'(x)| \leq C g(x), \quad C = C(\gamma), \quad x \neq 0, \]

for all \( \alpha \leq 1 \). Introduce also the function \( \zeta \in C^\infty(\mathbb{R}_+) \) such that

\[ \zeta'(x) \geq 0, \quad \zeta(x) = \begin{cases} x, & 0 \leq x \leq 1; \\ 2, & x \geq 2. \end{cases} \]

Note that

\[ \zeta(x_1 x_2) \leq 2 \zeta(x_1) x_2, \quad x_1 \geq 0, \quad x_2 \geq 1. \]

We study the above components \( r^{(1)}, r^{(2)} \) separately and introduce the function

\[ e_\alpha^{(1)}(x) = \frac{1}{\langle x \rangle^\gamma \zeta(\langle x \rangle^\gamma)} r_\alpha^{(1)}(x), \]

and the symbol

\[ e_\alpha^{(2)}(x, \xi) = g_\alpha(x)^{-\gamma} (\zeta(\langle \xi \rangle^\gamma))^{-1} r_\alpha^{(2)}(x, \xi), \]

where \( \gamma \in (0, 1] \) is a fixed number. To avoid cumbersome notation the dependence of \( e_\alpha^{(2)} \) on \( \gamma \) is not reflected in the notation. We denote the operators \( \text{Op}(r_\alpha) \) and \( \text{Op}(e_\alpha) \) by \( R_\alpha \) and \( E_\alpha \) respectively (with or without superscripts).

**Lemma 5.** Let \( \gamma \geq 1 \). Then for all \( \alpha > 0 \),

\[ \|e_\alpha^{(1)}\|_{L^\infty} \leq C \alpha. \]

**Proof.** Estimate the function \( r_\alpha^{(1)} \):

\[ |r_\alpha^{(1)}(x)| \leq \begin{cases} \frac{C \alpha^2}{\alpha \theta(x)}, & \alpha \theta(x) \leq 1/2, \\ \frac{C \alpha}{\alpha \theta(x)}, & \alpha \theta(x) > 1/2, \end{cases} \]

with a constant \( C \) independent of \( x \). The second estimate is immediate, and the first one follows from the Taylor’s formula

\[ \frac{1}{\sqrt{1 + t}} = 1 - \frac{t}{2} + O(t^2), \quad 0 \leq t \leq \frac{1}{2}. \]
Thus

$$|r^{(1)}_\alpha(x)| \leq C\alpha|x|^\gamma \zeta(\alpha|x|\gamma).$$

This leads to the proclaimed estimate for $e^{(1)}_\alpha$. \qed

**Lemma 6.** Let $\gamma \geq 1$. Then for all $\alpha > 0$ and any $\lambda \in (0, 1]$,\n
$$\|E^{(2)}_\alpha\| \leq C_\lambda \alpha.$$\n
**Proof.** To estimate the norm of $\text{Op}(e^{(2)}_\alpha)$ we use Proposition 16. It is clear that the distributional derivatives $\partial_x, \partial_\xi, \partial_x \partial_\xi$ of the symbol $e^{(2)}_\alpha(x, \xi)$ exist and are given by

$$\partial_x r^{(2)}_\alpha(x, \xi) = -\frac{\alpha}{g^2} g' \int_0^{||\xi||g} (1 - e^{-\alpha t}) dt + \frac{\alpha}{g} ||\xi||g'(1 - e^{-\alpha||\xi||g}),$$

$$\partial_\xi r^{(2)}_\alpha(x, \xi) = \alpha \text{sign } \xi (1 - e^{-\alpha||\xi||g}),$$

$$\partial_x \partial_\xi r^{(2)}_\alpha(x, \xi) = \alpha^2 \xi g' e^{-\alpha||\xi||g},$$

for all $x \neq 0, \xi \neq 0$. For any $\lambda \in (0, 1]$ the elementary bounds hold:

$$\int_0^{||\xi||g} (1 - e^{-\alpha t}) dt \leq ||\xi||g \zeta((\alpha||\xi||g)^\lambda) \leq 2||\xi||g^{1+\lambda} \zeta((\alpha||\xi||)^\lambda),$$

$$|1 - e^{-\alpha||\xi||g}| \leq \zeta((\alpha||\xi||g)^\lambda) \leq 2g^\lambda \zeta((\alpha||\xi||)^\lambda),$$

$$\alpha||\xi||g e^{-\alpha||\xi||g} \leq \zeta((\alpha||\xi||g)^\lambda) \leq 2g^\lambda \zeta((\alpha||\xi||)^\lambda).$$

Here we have used (20). Thus, in view of (19),\n
$$|r^{(2)}_\alpha(x, \xi)| + |\partial_\xi r^{(2)}_\alpha(x, \xi)| + |\partial_x r^{(2)}_\alpha(x, \xi)| \leq C\alpha(\xi) g^{\lambda} \zeta((\alpha||\xi||)^\lambda).$$

Also,

$$|\partial_x \partial_\xi r^{(2)}_\alpha(x, \xi)| \leq \frac{\alpha |g'|}{g} (\alpha||\xi||g e^{-\alpha||\xi||g}) \leq C\alpha g^{\lambda} \zeta((\alpha||\xi||)^\lambda).$$

Now estimate the derivatives of the weights:

$$|\partial_x g^{-\lambda}| = \lambda g^{-\lambda-1} g' \leq C g^{-\lambda}, \quad x \neq 0,$$

$$|\partial_\xi (\zeta(\xi((\alpha(\xi)^\lambda))^{-1})| \leq C \frac{1}{(\xi)^2 \zeta((\alpha(\xi)^\lambda))}, \quad \xi \in \mathbb{R}.$$\n
Thus the symbol $e^{(2)}_\alpha(x, \xi)$ as well as its derivatives $\partial_x, \partial_\xi, \partial_x \partial_\xi$ are bounded by $C\alpha$ for all $\alpha > 0$ uniformly in $x, \xi$. Now the required estimate follows from Proposition 16. \qed

We make a useful observation.
Corollary 7. Let γ ≥ 1 and x ∈ (0, 1]. Then for any function f ∈ D(A),

\[ \alpha^{-1} \| R^{(1)}_\alpha f \| \to 0, \quad \alpha \to 0, \]

\[ \alpha^{-1} \| E^{(2)}_\alpha (D_x) \xi ((\alpha (D_x))^\xi) f \| \to 0, \quad \alpha \to 0. \] (23)

Proof. Rewrite:

\[ \| R^{(1)}_\alpha f \| = \| E^{(1)}_\alpha (x)^\gamma \xi (\alpha (x)^\gamma) f \| \leq \| E^{(1)}_\alpha \| \| (x)^\gamma \xi (\alpha (x)^\gamma) f \|. \] (25)

By Lemma 5 the norm of \( E^{(1)}_\alpha \) on the right-hand side is bounded by \( C\alpha \). The function \( (x)^\gamma \xi (\alpha (x)^\gamma) f \) tends to zero as \( \alpha \to 0 \) a.a. \( x \in \mathbb{R} \), and it is uniformly bounded by the function \( (x)^\gamma |f| \), which belongs to \( L^2 \), since \( f \in D(A) \). Thus the second factor in (25) tends to zero as \( \alpha \to 0 \) by the Dominated Convergence Theorem. This proves (23).

Proof of (24). Estimate:

\[ \| E^{(2)}_\alpha (D_x) \xi ((\alpha (D_x))^\xi) f \| \leq \| E^{(2)}_\alpha \| \| (\xi)^\gamma \xi (\alpha (\xi)^\gamma) f \|. \]

By Lemma 6 the norm of the first factor on the right-hand side is bounded by \( C\alpha \). The second factor tends to zero as \( \alpha \to 0 \) for the same reason as in the proof of (23).

4. Norm-convergence of the extremal eigenfunction

Recall that the maximal positive eigenvalue \( \mu_\alpha \) of the operator \( B_\alpha \) is non-degenerate, and the corresponding (normalized) eigenfunction \( \psi_\alpha \) is positive a.a. \( x \in \mathbb{R} \).

The principal goal of this section is to prove that any infinite subset of the family \( \psi_\alpha, \alpha \leq 1 \) contains a norm-convergent sequence. We begin with an upper bound for \( 1 - \mu_\alpha \) which will be crucial for our argument.

Lemma 8. If \( \gamma \geq 1 \), then

\[ \limsup_{\alpha \to 0} \alpha^{-1} (1 - \mu_\alpha) \leq \lambda_1. \] (26)

Proof. Denote \( \varphi = \varphi_1 \). By a straightforward variational argument it follows that

\[ \mu_\alpha \geq (B_\alpha \varphi, \varphi) \geq |(B^{(l)}_\alpha \varphi, \varphi)| - \| B_\alpha - B^{(l)}_\alpha \| \]

\[ \geq ((I - \alpha A) \varphi, \varphi) - |(R_\alpha \varphi, \varphi)| + o(\alpha) \]

\[ = 1 - \alpha \lambda_1 - |(R_\alpha \varphi, \varphi)| + o(\alpha), \]

where we have also used (17). By definitions (21) and (22),

\[ |(R_\alpha \varphi, \varphi)| \leq \| R^{(1)}_\alpha \| + \| E^{(2)}_\alpha (D_x) \xi ((\alpha (D_x))^\xi) \varphi \| \| g^{\xi}_\alpha \varphi \|, \]

where \( \kappa \in (0, 1] \). It is clear that \( g^{\xi}_\alpha \varphi \in L^2 \) and its norm is bounded uniformly in \( \alpha \leq 1 \). The remaining terms on the right-hand side are \( o(\alpha) \) due to Corollary 7. This leads to (26).
The established upper bound leads to the following property.

**Lemma 9.** For any \( x \in (0, 1) \),

\[
\| g^x \psi_\alpha \| \leq C
\]

uniformly in \( \alpha \leq 1 \).

**Proof.** By definition of \( \psi_\alpha \),

\[
g^x \psi_\alpha = \mu^{-1}_\alpha g^x \mathbf{B}_\alpha \psi_\alpha.
\]

In view of (4), by definition (18) we have \( \Theta(x, y) \geq C|x|^{\gamma} \geq c\theta(x) \), so that the kernel \( B_\alpha(x, y) \) is bounded from above by

\[
B_\alpha(x, y) \leq \frac{\alpha}{\pi} \frac{C}{(x - y)^2 + \alpha^2 g_\alpha(x)^2},
\]

and thus the kernel \( \tilde{B}_\alpha(x, y) = g_\alpha(x)^{\alpha} B_\alpha(x, y) \) satisfies the estimate

\[
\tilde{B}_\alpha(x, y) \leq \frac{C}{\pi \alpha} \frac{1}{(1 + \alpha^{-2}(x - y)^2)^{1 - \frac{x}{2}}}.
\]

Since \( x < 1 \), by Proposition 15 this kernel defines a bounded operator with the norm uniformly bounded in \( \alpha > 0 \). Thus

\[
\| g^x \psi_\alpha \| \leq C \mu^{-1}_\alpha \| \psi_\alpha \| \leq C \mu^{-1}_\alpha.
\]

It remains to observe that by Lemma 8 the eigenvalue \( \mu_\alpha \) is separated from zero uniformly in \( \alpha \leq 1 \). \( \square \)

Now we obtain more delicate estimates for \( \psi_\alpha \). For a number \( h \geq 0 \) introduce the function

\[
S_\alpha(t; h) = \frac{\alpha}{\pi} \frac{1}{\alpha^2 + t^2 + h}, \quad t \in \mathbb{R},
\]

and denote by \( S_\alpha(h) \) the integral operator with the kernel \( S_\alpha(x - y; h) \). Along with \( S_\alpha(h) \) we also consider the operator

\[
T_\alpha(h) = S_\alpha(0) - S_\alpha(h).
\]

Due to (5) the Fourier transform of \( S_\alpha(t; h) \) is

\[
\hat{S}_\alpha(\xi; h) = \frac{\alpha}{\sqrt{2\pi} \sqrt{\alpha^2 + h}} e^{-|\xi|\sqrt{\alpha^2 + h}}, \quad \xi \in \mathbb{R},
\]

so that

\[
\| S_\alpha(h) \| = \frac{\alpha}{\sqrt{\alpha^2 + h}}, \quad \| T_\alpha(h) \| = 1 - \frac{\alpha}{\sqrt{\alpha^2 + h}}.
\]

Denote by \( \chi_R \) the characteristic function of the interval \( (-R, R) \).
Lemma 10. For sufficiently small $\alpha > 0$ and $\alpha R \leq 1$,

$$\|\hat{\psi}_\alpha \chi_R\|^2 \geq 1 - \frac{4\lambda_1}{R}. \quad (30)$$

Proof. Since $B_\alpha(x, y) < S_\alpha(x - y; 0)$ (see (9) and (27)) and $\psi_\alpha \geq 0$, we can write, using (28):

$$\mu_\alpha = (B_\alpha \psi_\alpha, \psi_\alpha) \leq \int_{\mathbb{R}} \int_{\mathbb{R}} S_\alpha(x - y; 0) \psi_\alpha(x) \psi_\alpha(y) dx dy = \int_{\mathbb{R}} e^{-\alpha|\xi|} |\hat{\psi}_\alpha(\xi)|^2 d\xi$$

$$\leq \int_{|\xi| \leq R} |\hat{\psi}_\alpha(\xi)|^2 d\xi + e^{-\alpha R} \int_{|\xi| > R} |\hat{\psi}_\alpha(\xi)|^2 d\xi$$

$$= (1 - e^{-\alpha R}) \int_{|\xi| \leq R} |\hat{\psi}_\alpha(\xi)|^2 d\xi + e^{-\alpha R}.$$

Due to (26), $\mu_\alpha \geq 1 - 2\alpha \lambda_1$ for sufficiently small $\alpha$, so

$$1 - e^{-\alpha R} - 2\alpha \lambda_1 \leq (1 - e^{-\alpha R}) \|\hat{\psi}_\alpha \chi_R\|^2,$$

which implies that

$$\|\hat{\psi}_\alpha \chi_R\|^2 \geq 1 - \frac{2\alpha \lambda_1}{1 - e^{-\alpha R}}.$$

Since $e^{-s} \leq (1 + s)^{-1}$ for all $s \geq 0$, we get $(1 - e^{-s})^{-1} \leq 2s^{-1}$ for $0 < s \leq 1$, which entails (30) for $\alpha R \leq 1$. \[\square\]

Lemma 11. For sufficiently small $\alpha > 0$ and any $R > 0$,

$$\|\psi_\alpha \chi_R\| \geq 1 - 4\alpha \lambda_1 - \frac{C}{R^\gamma}, \quad (31)$$

with some constant $C > 0$ independent of $\alpha$ and $R$.

Proof. It follows from (4) that $\Theta(x, y) \geq c|x|^\gamma$, so that the kernel $B_\alpha(x, y)$ satisfies the bound

$$B_\alpha(x, y) \leq S_\alpha(x - y; c\alpha^3 R^\gamma), \quad \text{for } |x| \geq R > 0.$$

Since $\psi_\alpha \geq 0$,

$$\mu_\alpha = (B_\alpha \psi_\alpha, \psi_\alpha) \leq (S_\alpha(0) \psi_\alpha, \psi_\alpha \chi_R) + (S_\alpha(c\alpha^3 R^\gamma) \psi_\alpha, \psi_\alpha (1 - \chi_R))$$

$$= (T_\alpha(c\alpha^3 R^\gamma) \psi_\alpha, \psi_\alpha \chi_R) + (S_\alpha(c\alpha^3 R^\gamma) \psi_\alpha, \psi_\alpha).$$

In view of (29),

$$\mu_\alpha \leq \|T_\alpha(c\alpha^3 R^\gamma)\| \|\psi_\alpha \chi_R\| + \|S_\alpha(c\alpha^3 R^\gamma)\|$$

$$= (1 - \frac{1}{\sqrt{1 + c\alpha R^\gamma}}) \|\psi_\alpha \chi_R\| + \frac{1}{\sqrt{1 + c\alpha R^\gamma}}.$$
Using, as in the proof of the previous lemma, the bound (26), we obtain that

\[
1 - \frac{1}{\sqrt{1 + c\alpha R^\gamma}} - 2\alpha \lambda_1 \leq \left(1 - \frac{1}{\sqrt{1 + c\alpha R^\gamma}}\right) \|\psi \chi_R\|,
\]

so

\[
1 - \frac{4\lambda_1(1 + c\alpha R^\gamma)}{c R^\gamma} \leq \|\psi \chi_R\|.
\]

This entails (31).

Now we show that any sequence from the family $\psi_\alpha$ contains a norm-convergent subsequence. The proof is inspired by [15], Lemma 7. We precede it with the following elementary result.

**Lemma 12.** Let $f_j \in L^2(\mathbb{R})$ be a sequence such that $\|f_j\| \leq C$ uniformly in $j = 1, 2, \ldots$, and $f_j(x) = 0$ for all $|x| \geq \rho > 0$ and all $j = 1, 2, \ldots$. Suppose that $f_j$ converges weakly to $f \in L^2(\mathbb{R})$ as $j \to \infty$, and that for some constant $A > 0$, and all $R \geq R_0 > 0$,

\[
\|\hat{f}_j\| \geq A - CR^{-\delta}, \quad \delta > 0,
\]

uniformly in $j$. Then $\|f\| \geq A$.

**Proof.** Since $f_j$ are uniformly compactly supported, the Fourier transforms $\hat{f}_j(\xi)$ converge to $\hat{f}(\xi)$ a.a. $\xi \in \mathbb{R}^d$ as $j \to \infty$. Moreover, the sequence $\hat{f}_j(\xi)$ is uniformly bounded, so $\hat{f}_j \chi_R \to \hat{f} \chi_R$, $j \to \infty$ in $L^2(\mathbb{R})$ for any $R > 0$. Therefore (32) implies that

\[
\|\hat{f} \chi_R\| \geq A - CR^{-\delta}.
\]

Since $R$ is arbitrary, we have $\|f\| = \|\hat{f}\| \geq A$, as claimed.

**Lemma 13.** For any sequence $\alpha_n \to 0$, $n \to \infty$, there exists a subsequence $\alpha_{n_k} \to 0$, $k \to \infty$, such that the eigenfunctions $\psi_{\alpha_{n_k}}$ converge in norm as $k \to \infty$.

**Proof.** Since the functions $\psi_\alpha, \alpha \geq 0$ are normalized, there is a subsequence $\psi_{\alpha_{n_k}}$ which converges weakly. Denote the limit by $\psi$. From now on we write $\psi_k$ instead of $\psi_{\alpha_{n_k}}$ to avoid cumbersome notation. In view of the relations

\[
\|\psi_k - \psi\|^2 = 1 + \|\psi\|^2 - 2 \text{Re}(\psi_k, \psi) \to 1 - \|\psi\|^2, \quad k \to \infty,
\]

it suffices to show that $\|\psi\| = 1$.

Fix a number $\rho > 0$, and split $\psi_k$ in the following way:

\[
\psi_k(x) = \psi_{k,\rho}^{(1)}(x) + \psi_{k,\rho}^{(2)}(x), \quad \psi_{k,\rho}^{(1)}(x) = \psi_k(x) \chi_\rho(x).
\]
Clearly, $\psi_{k,\rho}^{(1)}$ converges weakly to $w_\rho = \psi \chi_\rho$ as $k \to \infty$. Assume that $\alpha_{n_k} \leq \rho^{-\gamma}$, so that by (31),

$$\|\psi_{k,\rho}^{(1)}\|^2 \geq 1 - \frac{C}{\rho^{\gamma}}, \quad \|\psi_{k,\rho}^{(2)}\|^2 \leq \frac{C}{\rho^{\gamma}}.$$ 

Therefore, for any $R > 0$,

$$\|\hat{\psi}_{k,\rho}^{(1)} \chi_R\| \geq \|\hat{\psi}_{k,\rho} \chi_R\| - \|\psi_{k,\rho}^{(2)}\| \geq 1 - 4\lambda_1 R^{-1} - C\rho^{-\gamma},$$

where we have used (30). By Lemma 12,

$$\|w_\rho\| \geq 1 - C\rho^{-\gamma}.$$ 

Since $\rho$ is arbitrary, $\|\psi\| \geq 1$, and hence $\|\psi\| = 1$. As a result, the sequence $\psi_k$ converges in norm, as claimed.}

5. Asymptotics of $\mu_\alpha, \alpha \to 0$: proof of Theorem 1

As before, by $\lambda_1, l = 1, 2, \ldots$ we denote the eigenvalues of $A$ arranged in ascending order, and by $\varphi_l$ the corresponding normalized eigenfunctions. Recall that the lowest eigenvalue $\lambda_1$ of the model operator $A$ is non-degenerate and its (normalized) eigenfunction $\varphi_1$ is chosen to be positive a.a. $x \in \mathbb{R}$. We begin with proving Theorem 3.

Proof of Theorem 3. The proof essentially follows the plan of [15]. It suffices to show that for any sequence $\alpha_n \to 0, n \to \infty$, one can find a subsequence $\alpha_{n_k} \to 0, k \to \infty$ such that

$$\lim_{k \to \infty} \alpha_{n_k}^{-1}(1 - \mu_{\alpha_{n_k}}) = \lambda_1,$$

and $\psi_{\alpha_{n_k}}$ converges in norm to $\varphi_1$ as $k \to \infty$. By Lemma 13 one can pick a subsequence $\alpha_{n_k}$ such that $\psi_{\alpha_{n_k}}$ converges in norm as $k \to \infty$. As in the proof of Lemma 13 denote by $\psi$ the limit, so $\|\psi\| = 1$ and $\psi \geq 0$ a.e. For simplicity we write $\psi_\alpha$ instead of $\psi_{\alpha_{n_k}}$. For an arbitrary function $f \in D(A)$ write

$$\mu_\alpha(\psi_\alpha, f) = (B_\alpha \psi_\alpha, f) = (\psi_\alpha, B_\alpha^{(l)} f) + (\psi_\alpha, (B_\alpha - B_\alpha^{(l)}) f)$$

$$= (\psi_\alpha, f) - \alpha(\psi_\alpha, A f) + (\psi_\alpha, R_\alpha f) + (\psi_\alpha, (B_\alpha - B_\alpha^{(l)}) f).$$

This implies that

$$\alpha^{-1}(1 - \mu_\alpha)(\psi_\alpha, f) = (\psi_\alpha, A f) - \alpha^{-1}(\psi_\alpha, R_\alpha f) - \alpha^{-1}(\psi_\alpha, (B_\alpha - B_\alpha^{(l)}) f). \quad (33)$$

In view of (17) the last term on the right-hand side tends to zero as $\alpha \to 0$. The first term trivially tends to $(\psi, A f)$. Consider the second term:

$$|((\psi_\alpha, R_\alpha f)| = |(\psi_\alpha, R_\alpha^{(1)} f) + (g_\alpha^{\infty} \psi_\alpha, E_\alpha^{(2)} \langle D_x \rangle \zeta((\alpha \langle D_x \rangle)^{\kappa}) f)|$$

$$\leq \|R_\alpha^{(1)} f\| + \|g_\alpha^{\infty} \psi_\alpha\| \|E_\alpha^{(2)} \langle D_x \rangle \zeta((\alpha \langle D_x \rangle)^{\kappa}) f\|.$$
Assume now that \( \kappa < 1 \). By Corollary 7 and Lemma 9, the right-hand side is \( o(\alpha) \), and hence, if \( (\psi, f) \neq 0 \), then passing to the limit in (33) we get

\[
\lim_{\alpha \to 0} \alpha^{-1}(1 - \mu_\alpha) = \frac{(\psi, A F)}{(\psi, f)}.
\]

Let \( f = \varphi_I \) with some \( I \), so that \( (\psi, A F) = \lambda_I (\psi, \varphi_I) \). Suppose that \( (\psi, \varphi_I) \neq 0 \), so that

\[
\lim_{\alpha \to 0} \alpha^{-1}(1 - \mu_\alpha) = \lambda_I.
\]

By the uniqueness of the above limit, \( (\psi, \varphi_j) = 0 \) for all \( j \)'s such that \( \lambda_j \neq \lambda_k \). Thus, by completeness of the system \( \{\varphi_k\} \), the function \( \psi \) is an eigenfunction of \( A \) with the eigenvalue \( \lambda_I \). In view of (26), \( \lambda_I \leq \lambda_1 \). Since the eigenvalues \( \lambda_j \) are labeled in ascending order we conclude that \( \lambda_I = \lambda_1 \). As this eigenvalue is non-degenerate and the corresponding eigenfunction \( \varphi_1 \) is positive a.e., we observe that \( \psi = \varphi_1 \). \( \square \)

**Proof of Theorem 1.** Theorem 1 follows from Theorem 3 due to the relations (11). \( \Box \)

### 6. Miscellaneous

In this short section we collect some open questions related to the spectrum of the operator (1).

#### 6.1. Theorems 1 and 3 give information on the largest eigenvalue \( M_\beta \) of the operator \( K_\beta \) defined in (1), (2). Let

\[
M_\beta \equiv M_{1, \beta} > M_{2, \beta} \geq \ldots
\]

be the sequence of all positive eigenvalues of \( K_\beta \) arranged in descending order. The following conjecture is a natural extension of Theorem 1.

**Conjecture 14.** For any \( j = 1, 2, \ldots \)

\[
\lim_{\beta \to 0} \beta^{-\frac{2}{\pi^2}}(1 - M_{j, \beta}) = \lambda_j,
\]

where \( \lambda_1 < \lambda_2 \leq \ldots \) are eigenvalues of the operator \( A \) defined in (6), arranged in ascending order.

For the case \( \Theta(x, y) = (x^2 + y^2)^2 \) the formula (35) was conjectured in [9], Section 7.1, but without specifying what the values \( \lambda_j \) are. As in [9], the formula (35) is prompted by the paper [15] where asymptotics of the form (35) were found for an integral operator with a difference kernel.
6.2. Although the operator $K_\beta$ converges strongly to the positive-definite operator $K_0$ as $\beta \to 0$, we can’t say whether or not $K_\beta \cdot \beta > 0$ has negative eigenvalues.

6.3. Suppose that the function $\Theta(x, y)$ in (2) is even, i.e. $\Theta(-x, -y) = \Theta(x, y)$, $x, y \in \mathbb{R}$. Then the subspaces $H^e$ and $H^o$ of $L^2(\mathbb{R})$ of even and odd functions are invariant for $K = K_\beta$. Consider restriction operators $K^e = K | H^e$ and $K^o = K | H^o$ and their positive eigenvalues $\lambda_j^e$ and $\lambda_j^o$, $j = 1, 2, \ldots$, arranged in descending order. Remembering that the top eigenvalue of $K$ is non-degenerate and its eigenfunction is positive a.e., one easily concludes that $\lambda_1^e > \lambda_1^o$. Are there similar inequalities for the pairs $\lambda_j^e, \lambda_j^o$ with $j > 1$?

7. Appendix. Boundedness of integral and pseudo-differential operators

In this Appendix, for the reader’s convenience we remind (without proofs) simple tests of boundedness for integral and pseudo-differential operators acting on $L^2(\mathbb{R}^d)$, $d \geq 1$. Consider the integral operator

$$(K u)(x) = \int_{\mathbb{R}^d} K(x, y) u(y) dy,$$

with the kernel $K(x, y)$, and the pseudo-differential operator

$$(\text{Op}(a) u)(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(x-y) \cdot \xi} a(x, \xi) u(y) dy \xi,$$

with the symbol $a(x, \xi)$.

The following classical result is known as the Schur Test and it can be found, even in a more general form, in [4], Theorem 5.2.

**Proposition 15.** Suppose that the kernel $K$ satisfies the conditions

$$M_1 = \sup_x \int_{\mathbb{R}^d} |K(x, y)| dy < \infty, \quad M_2 = \sup_y \int_{\mathbb{R}^d} |K(x, y)| dx < \infty.$$

Then the operator (36) is bounded on $L^2(\mathbb{R}^d)$ and $\|K\| \leq \sqrt{M_1 M_2}$.

For pseudo-differential operators on $L^2(\mathbb{R}^d)$ we use the test of boundedness found by H. O. Cordes in [2], Theorem $B'_1$.

**Proposition 16.** Let $a(x, \xi), x, \xi \in \mathbb{R}^d, d \geq 1$, be a function such that its distributional derivatives of the form $\nabla_x^n \nabla_\xi^m a$ are $L^\infty$-functions for all $0 \leq n, m \leq r$, where

$$r = \left\lfloor \frac{d}{2} \right\rfloor + 1.$$
Then the operator (37) is bounded on $L^2(\mathbb{R}^d)$ and

$$\| \text{Op}(a) \| \leq C \max_{0 \leq n, m \leq r} \| \nabla_x^n \nabla_\xi^m a \|_{L^\infty},$$

with a constant $C$ depending only on $d$.

It is important for us that for $d = 1$ the above test requires the boundedness of derivatives $\partial_x^n \partial_\xi^m a$ with $n, m \in \{0, 1\}$ only. This result is extended to arbitrary dimensions by M. Ruzhansky and M. Sugimoto, see [13], Corollary 2.4. Recall that the classical Calderón-Vaillancourt theorem needs more derivatives with respect to each variable, see [2] and [13] for discussion. A short prove of Proposition 16 was given by I. L. Hwang in [5], Theorem 2 (see also [8], Lemma 2.3.2 for a somewhat simplified version).

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