Nil graded self-similar algebras

Victor M. Petrogradsky\textsuperscript{1}, Ivan P. Shestakov\textsuperscript{2} and Efim Zelmanov\textsuperscript{3}

Abstract. In \cite{19}, \cite{24} we introduced a family of self-similar nil Lie algebras $L$ over fields of prime characteristic $p > 0$ whose properties resemble those of Grigorchuk and Gupta–Sidki groups. The Lie algebra $L$ is generated by two derivations

$$v_1 = \partial_1 + t_0^{p-1}(\partial_2 + t_1^{p-1}(\partial_3 + t_2^{p-1}(\partial_4 + t_3^{p-1}(\partial_5 + t_4^{p-1}(\partial_6 + \cdots))))).$$

$$v_2 = \partial_2 + t_1^{p-1}(\partial_3 + t_2^{p-1}(\partial_4 + t_3^{p-1}(\partial_5 + t_4^{p-1}(\partial_6 + \cdots))))$$

of the truncated polynomial ring $K[t_i, i \in \mathbb{N} | t_i^p = 0, i \in \mathbb{N}]$ in countably many variables. The associative algebra $A$ generated by $v_1, v_2$ is equipped with a natural $\mathbb{Z} \oplus \mathbb{Z}$-gradation. In this paper we show that for $p$, which is not representable as $p = m^2 + m + 1, m \in \mathbb{Z}$, the algebra $A$ is graded nil and can be represented as a sum of two locally nilpotent subalgebras. L. Bartholdi [3] and Ya. S. Krylyuk [15] proved that for $p = m^2 + m + 1$ the algebra $A$ is not graded nil. However, we show that the second family of self-similar Lie algebras introduced in [24] and their associative hulls are always $\mathbb{Z}^p$-graded, graded nil, and are sums of two locally nilpotent subalgebras.


Keywords. Modular Lie algebras, growth, nil-algebras, self-similar, Gelfand–Kirillov dimension, Lie algebras of vector fields, Grigorchuk group, Gupta–Sidki group.

1. Definitions and constructions

Let $L$ be a Lie algebra over a field $K$ of characteristic $p > 0$ and let $\text{ad} x : L \to L$, $\text{ad} x(y) = [x, y]$ for $x, y \in L$, be the adjoint map. Recall that $L$ is called a restricted Lie algebra or Lie $p$-algebra \cite{12}, \cite{26}, \cite{1} if $L$ additionally affords a unary operation $x \mapsto x^p$, $x \in L$, satisfying

i) $(\lambda x)^p = \lambda^p x^p$ for all $\lambda \in K, x \in L$;

ii) $\text{ad}(x^p) = (\text{ad} x)^p$ for all $x \in L$;

\textsuperscript{1}The first author was partially supported by grants FAPESP 05/58376-0 and RFBR-07-01-00080.

\textsuperscript{2}The second author was partially supported by grants FAPESP 05/60337-2 and CNPq 304991/2006-6.

\textsuperscript{3}The third author was partially supported by the NSF grant DMS-0758487.
iii) for all $x, y \in L$ one has

$$
(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y),
$$

(1)

where $s_i(x, y)$ is the coefficient of $Z^{i-1}$ in the polynomial $(\text{ad}(Zx + y))^{p-1}(x)$ in $L[Z]$, with $Z$ is an indeterminate. Also, $s_i(x, y)$ is a Lie polynomial in $x, y$ of degrees $i$ and $p - i$, respectively.

Suppose that $L$ is a restricted Lie algebra and $X \subset L$. Then by $\text{Lie}_p(X)$ we denote the restricted subalgebra generated by $X$. Let $H \subset L$ be a Lie subalgebra, i.e., $H$ is a vector subspace which is closed under the Lie bracket. Then by $H_p$ we denote the restricted subalgebra generated by $H$. In what follows by an associative enveloping algebra of a Lie algebra we mean the associative algebra without $1$.

We recall the notion of growth. Let $A$ be an associative (or Lie) algebra generated by a finite set $X$. Denote by $A^{(X, n)}$ the subspace of $A$ spanned by all monomials in $X$ of length not exceeding $n$. If $A$ is a restricted Lie algebra, then we define $A^{(X, n)} = \{ [x_{i_1}, \ldots, x_{i_s}]^{p^k} \mid x_{i_j} \in X, sp^k \leq n \}_K$. In either situation, one defines the growth function:

$$
\gamma_A(n) = \gamma_A(X, n) = \dim_K A^{(X, n)}, \quad n \in \mathbb{N}.
$$

The growth function clearly depends on the choice of the generating set $X$. Furthermore, it is easy to see that the exponential growth is the highest possible growth for Lie and associative algebras. The growth function $\gamma_A(n)$ is compared with the polynomial functions $n^k$, $k \in \mathbb{R}^+$, by computing the upper and lower Gelfand–Kirillov dimensions [14], namely

$$
\text{GKdim } A = \lim_{n \to \infty} \frac{\ln \gamma_A(n)}{\ln n},
$$

$$
\text{GKdim } A = \lim_{n \to \infty} \frac{\ln \gamma_A(n)}{\ln n}.
$$

This setting assumes that all elements of $X$ have the same weight equal to 1. We shall mainly use a somewhat different growth function. Namely, we consider the weight function $\text{wt } v, v \in A$, and the growth with respect to it: $\tilde{\gamma}_A(n) = \dim_K \{ y \mid y \in A, \text{wt } y \leq n \}, n \in \mathbb{N}$, where the elements of the generating set $X$ have different weights. Standard arguments [14] prove that this growth function yields the same Gelfand–Kirillov dimensions.

Now suppose that $\text{char } K = p > 0$. Denote $I = \{0, 1, 2, \ldots \}$ and $\mathbb{N}_p = \{0, 1, \ldots, p - 1\}$. Consider the truncated polynomial algebra

$$
R = K[t_i, i \in I \mid t_i^p = 0, i \in I].
$$

Let $\mathbb{N}_p^I = \{ \alpha : I \to \mathbb{N}_p \}$ be the set of functions with finitely many nonzero values. For $\alpha \in \mathbb{N}_p^I$ denote $|\alpha| = \sum_{i \in I} \alpha_i$ and $t^\alpha = \prod_{i \in I} t_i^{\alpha_i} \in R$. The set $\{ t^\alpha \mid \alpha \in \mathbb{N}_p^I \}$
is clearly a basis of $R$. Consider the ideal $R^+$ spanned by all elements $t^\alpha$, $\alpha \in \mathbb{N}_p^I$, $|\alpha| > 0$. Let $\partial_i = \frac{\partial}{\partial t_i}$, $i \in I$, denote the partial derivatives of $R$.

We introduce the so-called Lie algebra of special derivations of $R$ [22], [23], [20]:

$$ W(R) = \{ \sum_{\alpha \in \mathbb{N}_p^I} t^\alpha \sum_{j=1}^{m(\alpha)} \lambda_{\alpha,i} \frac{\partial}{\partial t_i} | \lambda_{\alpha,i} \in K, i_j \in I \}.$$  

It is essential that the sum at each $t^\alpha$, $\alpha \in \mathbb{N}_p^I$, is finite.

**Lemma 1.1** ([21]). For arbitrary complex numbers $a_i \in \mathbb{C}$, $i \in \mathbb{N}$, there exist gradations on the algebras $R$, $W(R)$ such that $\text{wt}(t_i) = -a_i$, $\text{wt}(\partial_i) = a_i$.

Denote by $\tau : R \to R$ the shift endomorphism $\tau(t_i) = t_{i+1}$, $i \in I$. Extending it by $\tau(\partial_i) = \partial_{i+1}$, $i \in I$, we get the shift endomorphism $\tau : W(R) \to W(R)$.

**2. First example**

We define the following two derivations of $R$:

$$ v_1 = \partial_1 + t_0^{p-1}(\partial_2 + t_1^{p-1}(\partial_3 + t_2^{p-1}(\partial_4 + t_3^{p-1}(\partial_5 + t_4^{p-1}(\partial_6 + \cdots))))), $$

$$ v_2 = \partial_2 + t_1^{p-1}(\partial_3 + t_2^{p-1}(\partial_4 + t_3^{p-1}(\partial_5 + t_4^{p-1}(\partial_6 + \cdots))). $$

These operators are special derivations $v_1, v_2 \in W(R)$. Observe that we can write these derivations recursively:

$$ v_1 = \partial_1 + t_0^{p-1}\tau(v_1), \quad v_2 = \tau(v_1). $$

Let $L = \text{Lie}_p(v_1, v_2) \subset W(R) \subset \text{Der } R$ be the restricted subalgebra generated by $\{v_1, v_2\}$. This algebra was introduced in [24]. In the case of characteristic $p = 2$, it coincides with the Fibonacci restricted Lie algebra introduced in [19]. Similarly, define

$$ v_i = \tau^{i-1}(v_1) = \partial_i + t_{i-1}^{p-1}(\partial_{i+1} + t_i^{p-1}(\partial_{i+2} + t_{i+1}^{p-1}(\partial_{i+3} + \cdots))), \quad (2) $$

$i = 1, 2, \ldots$. We also can write

$$ v_i = \partial_i + t_{i-1}^{p-1}v_{i+1}, \quad i = 1, 2, \ldots. \quad (3)$$

**Lemma 2.1.** Let $L = \text{Lie}_p(v_1, v_2) \subset \text{Der } R$ be the restricted Lie algebra generated by $\{v_1, v_2\}$. Then the following relations holds:

1. $[v_i, v_{i+1}] = -t_i^{p-2}v_{i+2}$ for $i = 1, 2, \ldots$;
2. $[v_i, v_{i+2}] = -t_{i-1}^{p-1}t_i^{p-2}v_{i+3}$ for $i = 1, 2, \ldots$;
Lemma 2.2. Let

\[ [v_i, v_j] = -(t_{i-1}t_i \ldots t_{j-3})^{p-1}t_{j-1}^{p-2}v_{j+1}; \]

(4) for all \( n \geq 1, j \geq 0 \) we have the action
\[
v_n(t_j) = \begin{cases} (t_{n-1}t_n \cdots t_{j-2})^{p-1}, & n < j, \\ 1, & n = j, \\ 0, & n > j; \end{cases}
\]

(5) for all \( k, n \geq 1, \)
\[
[\partial_n, v_k] = \begin{cases} -(t_{k-1}t_k \ldots t_{n-1})^{p-1}t_n^{p-2}v_{n+2}, & k < n + 1, \\ -t_n^{p-2}v_{n+2}, & k = n + 1, \\ 0, & k > n + 1; \end{cases}
\]

(6) \( v_i^p = -t_{i-1}^{p-1}v_{i+2} \) for all \( i \geq 1. \)

Proof. The claims (1)–(5) are proved in [24]. The last claim for \( p = 2 \) is checked in [19], we assume that \( p \geq 3. \) We have \( v_i^p = (\partial_i + t_i^{p-1}v_i+1)^p. \) By formula (1) we obtain the sum of commutators of length \( p. \) We apply the previous claim \( [\partial_i, t_i^{p-1}v_i+1] = -t_i^{p-1}t_i^{p-2}v_i+2. \) In further commutators we cannot use \( t_i^{p-1}v_i+1 \) anymore because of the total power of \( t_i-1. \) Thus, only one term in (1) is nontrivial, namely \( s_{p-1}(x, y) = (\text{ad } x)^{p-1}(y). \) We get
\[
v_i^p = (\partial_i + t_i^{p-1}v_i+1)^p
= (\text{ad } \partial_i)^{p-1}(t_i^{p-1}v_i+1)
= (\text{ad } \partial_i)^{p-2}(-t_i^{p-1}t_i^{p-2}v_i+2)
= -t_i^{p-1}v_{i+2}. \]

\[ \text{Lemma 2.2. Let } H \text{ be the } K\text{-linear span of all elements } t_0^{\alpha_0}t_1^{\alpha_1} \ldots t_{n-2}^{\alpha_{n-2}}v_n, \text{ where } 0 \leq \alpha_i \leq p - 1, \alpha_{n-2} \leq p - 2, n \geq 1. \text{ Then } H \text{ is a restricted subalgebra of } \text{Der } R \text{ and } L \subset H. \]

Proof. Let us prove that \( H \) is a Lie subalgebra. We apply Lemma 2.1 to check that the product of two monomials of type above is expressed via such monomials again. Let \( n < m. \) Then
\[
[t_0^{\alpha_0} \ldots t_{n-2}^{\alpha_{n-2}}v_n, t_0^{\beta_0} \ldots t_{m-2}^{\beta_{m-2}}v_m]
= -t_0^{\alpha_0} \ldots t_{n-2}^{\alpha_{n-2}}(t_{n-1} \ldots t_{m-3})^{p-1}t_0^{\beta_0} \ldots t_{m-2}^{\beta_{m-2}}t_{m-1}^{p-2}v_{m+1}
+ t_0^{\alpha_0} \ldots t_{n-2}^{\alpha_{n-2}} \sum_{\beta_j \neq 0} \left( \prod_{i=0, i \neq j}^{m-2} t_i^{\beta_i} \right) \beta_j t_j^{\beta_j-1}v_n(t_j)v_m. \]
The first term is of type \(t_{m-2}^{p-2} t_{m-1}^{p-2} v_{m+1}\), as required.

By claim (4), \(v_m\) acts on all \(t_i\)'s trivially because \(m > n > n - 2 \geq i\), and no respective terms appear. It remains to consider the second term above. Similarly, \(v_n(t_j)\) is nonzero only for \(n \leq j\), namely

\[v_n(t_j) = (t_{n-1} t_n \ldots t_{j-2})^{p-1}, \quad n \leq j.\]

In this case, \(n \leq j \leq m - 2\) and the new \(t_i\)'s above have indices such that \(n - 1 < \ldots < j - 2 \leq m - 4\). We again obtain monomials of the required type. Hence, \(H \subseteq \text{Der } R\) is a Lie subalgebra. The last claim of Lemma 2.1 implies that the subalgebra \(H \subseteq \text{Der } R\) is restricted.

Let \(H_n\) denote the \(K\)-linear span of all elements \(t_{i0}^{a_0} \ldots t_{i_{m-2}}^{a_{m-2}} v_m\), where \(0 \leq \alpha_i \leq p - 1, \alpha_{m-2} \leq p - 2, m \geq n\).

**Corollary 2.3.** (1) \(H_n \triangleleft H, n \geq 1; H = H_1 \supset H_2 \supset \cdots.\)

(2) Let \(L_n = L \cap H_n\) for \(n \geq 1\). Then the factor algebras \(H_n/H_{n+2}\) and \(L_n/L_{n+2}\) are abelian with the trivial \(p\)-mapping for all \(n \geq 1\).

**Proof.** The fact that \(H_n\) are ideals and \(H_n/H_{n+2}\) are abelian follows from eq. (4) and other arguments of Lemma 2.2. In order to check that the \(p\)-mapping on \(H_n/H_{n+2}\) is trivial, we use claim (6) of Lemma 2.1 and eq. (1).

A Lie algebra \(L\) is said to be just-infinite if it is infinite-dimensional and any proper factor algebra \(L/J\) is finite dimensional.

**Lemma 2.4.** The algebra \(L\) is not just-infinite.

**Proof.** In the case \(p = 2\) all elements \(\{v_n \mid n \geq 1\}\) belong to \(L\); see \([19]\). In the case of arbitrary characteristic the situation is more complicated, nevertheless, we have \(v_{2n} \in L\) for all \(n \geq 1\); see \([24]\). Now let \(J\) be the restricted ideal of \(L\) generated by the elements

\[[v_1, v_{2n}] = -(t_0 t_1 \ldots t_{2n-3})^{p-1} t_{2n-1}^{p-2} v_{2n+1}, \quad n \geq 2.\]

Observe that they all contain the common factor \(t_0^{p-1}\), which has no chances to disappear by any further commutation. So, all elements of \(J\) have the factor \(t_0^{p-1}\). Hence, the ideal \(J\) is abelian, infinite-dimensional, and has the trivial \(p\)-mapping. Since the elements \(\{v_{2n} \mid n \geq 1\}\) are linearly independent modulo \(J\), we conclude that \(\dim L/J = \infty\).

In our constructions we are motivated by analogies with constructions of self-similar groups and algebras \([9]\) \([8]\), \([2]\). In particular, the following property is analogous to the periodicity of the Grigorchuk and Gupta–Sidki groups \([7]\), \([10]\).

**Theorem 2.5** \([19]\), \([24]\). Let \(L = \text{Lie}_p(v_1, v_2) \subseteq \text{Der } R\) be the restricted subalgebra generated by \(\{v_1, v_2\}\). Then \(L\) has a nil \(p\)-mapping.
3. The first example: the gradation

In this section we introduce a $\mathbb{Z} \oplus \mathbb{Z}$-gradation on our algebras. Suppose that all elements $v_i$ are homogeneous, $\text{wt } v_i = -\text{wt } t_i = a_i \in \mathbb{R}$, where $i = 1, 2, \ldots$, such that all terms in (3) are homogeneous. To achieve this, we assume that

$$a_i = \text{wt } v_i = \text{wt } \delta_i = (p - 1) \text{wt } t_{i-1} + \text{wt } v_{i+1} = -(p - 1)a_{i-1} + a_{i+1}.$$ 

Hence, we get the recurrence relation

$$a_{i+1} = a_i + (p - 1)a_{i-1}, \quad i \in \mathbb{N}. \quad (5)$$

This equation has the characteristic polynomial $\phi(t) = t^2 - t - (p - 1)$ with two different roots

$$\lambda = \frac{1 + \sqrt{4p - 3}}{2}, \quad \lambda_1 = \frac{1 - \sqrt{4p - 3}}{2}.$$ 

It is well known that all solutions of the recurrence relation (5) are linear combinations of the two sequences $a_i = \lambda_i^i, i \in \mathbb{N}$, and $a_i = \lambda_1^i, i \in \mathbb{N}$.

We distinguish two cases.

**Irrational:** $\lambda, \lambda_1$ are irrational, e.g., for primes $p = 2, 5, 11, 17, 19, \ldots$.

**Rational:** $\lambda, \lambda_1$ are rational (moreover, in this case $\lambda, \lambda_1$ are integers), e.g., for primes $p = 3, 7, 13, 31, 43, \ldots$. Note that in this case $\lambda \in \mathbb{Z}$ and $p = \lambda^2 - \lambda + 1$.

**Remark.** To the best of our knowledge the question if there are infinitely many such primes is open. A more general question asks whether there are infinitely many primes of the form $an^2 + bn + c$, where $a, b, c$ are relatively prime integers, $a$ positive, $a + b$ and $c$ are not both even, and $b^2 - 4ac$ is not a perfect square; see [11], p. 19.

The existence of two linearly independent weight functions yields a $\mathbb{Z} \oplus \mathbb{Z}$-gradation.

**Theorem 3.1.** Let $L = \text{Lie}_p(v_1, v_2) \subset \text{Der } R$ be the restricted subalgebra generated by $\{v_1, v_2\}$. We introduce weight and superweight functions as follows:

$$\text{wt } v_n = -\text{wt } t_n = \lambda^n, \quad n = 1, 2, \ldots, \quad \lambda = \frac{1 + \sqrt{4p - 3}}{2},$$

$$\text{swt } v_n = -\text{swt } t_n = \lambda_1^{n-2}, \quad n = 1, 2, \ldots, \quad \lambda_1 = \frac{1 - \sqrt{4p - 3}}{2}.$$ 

Then:

1. Both functions are additive on products of homogeneous elements of $L$.
2. We have the $\mathbb{Z} \oplus \mathbb{Z}$-gradation $L = \bigoplus_{a,b \geq 0} L_{a,b}$, where $L_{a,b}$ is spanned by products with $a$ factors $v_1$ and $b$ factors $v_2$. 

\(\text{(3) Let } v \in L_{a,b}, \text{ where } a, b \geq 0. \text{ Then} \)

\[
\begin{align*}
\text{wt } v &= \lambda a + \lambda^2 b, \\
\text{swt } v &= -\frac{\lambda}{p-1} a + b.
\end{align*}
\]

\textbf{Proof.} Let us introduce one more function that takes values in \(\mathbb{R}^2:\)

\[
\text{Wt}(v_i) = -\text{Wt}(t_i) = (\text{wt}(v_i), \text{swt}(v_i)), \quad i \in \mathbb{N}.
\]

Consider a monomial \(v \in L\) that is a product of \(a\) elements \(v_1\) and \(b\) elements \(v_2\). Then both weight functions are well defined on \(v\). Moreover, Wt(*) is additive on products of monomials in \(v_i\) and \(t_j\). Therefore, we get

\[
\text{Wt}(v) = a \text{ Wt}(v_1) + b \text{ Wt}(v_2).
\]

Consider another pair of integers \((a', b') \neq (a, b)\) and a monomial \(v' \in L\) that contains \(a', b'\) letters \(v_1, v_2\), respectively. By construction,

\[
\begin{align*}
\text{Wt}(v_1) &= (\lambda, \lambda_{1}^{-1}) = (\lambda, -\lambda/(p - 1)), \\
\text{Wt}(v_2) &= (\lambda^2, 1) = (\lambda + p - 1, 1).
\end{align*}
\]

Since these two vectors are linearly independent over \(\mathbb{R}\), we get \(\text{Wt}(v_1) = a' \text{ Wt}(v_1) + b' \text{ Wt}(v_2) \neq \text{Wt}(v)\) and the claimed \(\mathbb{Z} \oplus \mathbb{Z}\)-gradation \(L = \bigoplus_{a,b \geq 0} L_{a,b}\).

Let \(v \in L_{a,b}, \text{ where } a, b \geq 0. \text{ Then} \)

\[
\begin{align*}
\text{wt } v &= a \text{ wt } v_1 + b \text{ wt } v_2 = a\lambda + b\lambda^2, \\
\text{swt } v &= a \text{ swt } v_1 + b \text{ swt } v_2 = a\lambda_{1}^{-1} + b = -\frac{\lambda}{p-1} a + b. \quad \Box
\end{align*}
\]

Let us introduce a new coordinate system on the plane. For a point \(A = (x, y) \in \mathbb{R}^2\) we define its new coordinates as

\[
\begin{align*}
\xi &= \text{wt}(x, y) = \lambda x + \lambda^2 y = \lambda(x + \lambda y), \\
\eta &= \text{swt}(x, y) = -\frac{\lambda}{p-1} x + y = \lambda_{1}^{-1}(x + \lambda_{1} y) \quad (x, y) \in \mathbb{R}^2. \quad (6)
\end{align*}
\]

We will refer to these coordinates as the \textit{weight} and the \textit{superweight} of the point \((x, y)\) respectively. Gradations by superweights yield \textit{triangular decompositions}.

\textbf{Corollary 3.2.} Consider the restricted Lie algebra \(L = \text{Lie}_p(v_1, v_2)\), the associative algebra \(A = \text{Alg}(v_1, v_2)\) generated by \(v_1, v_2\), the universal enveloping algebra \(U = U(L)\), and the universal restricted enveloping algebra \(u = u(L)\). Then:

\(\text{(1) All these algebras have decompositions into direct sums of three subalgebras,}\)

\[
\begin{align*}
L &= L_+ \oplus L_0 \oplus L_- , \quad A = A_+ \oplus A_0 \oplus A_-, \\
U &= U_+ \oplus U_0 \oplus U_- , \quad u = u_+ \oplus u_0 \oplus u_-.
\end{align*}
\]
where \( L_+ \), \( L_0 \), and \( L_- \) are spanned by homogeneous elements \( v \in L \) such that \( \text{swt} \, v > 0 \), \( \text{swt} \, v = 0 \), and \( \text{swt} \, v < 0 \), respectively. The decompositions of other algebras are defined similarly.

(2) In the irrational case we have \( L_0 = \{0\}, A_0 = \{0\}, U_0 = \{0\}, u_0 = \{0\} \).

**Proof.** Suppose that \( \lambda \) is irrational. Consider \( 0 \neq v \in L_{a,b} \), where \( (a, b) \in \mathbb{Z}^2 \). Suppose that \( \text{swt}(v) = -a\lambda/(p-1) + b = 0 \). If \( b \neq 0 \) then \( \lambda \in \mathbb{Q} \), a contradiction. \( \square \)

**Lemma 3.3.** In the irrational case for an arbitrary lattice point \( (a, b) \in \mathbb{Z}^2 \subset \mathbb{R}^2 \) we have

\[
|\text{wt}(a, b) \cdot \text{swt}(a, b)| \geq \frac{\lambda^2}{p-1}, \quad (a, b) \in \mathbb{Z}^2.
\]

**Proof.** Note that the polynomial \( \psi(t) = t^2 + t - (p-1) \) has the discriminant \( D = 4p - 3 \) and no rational roots. For arbitrary integers \( a, b \in \mathbb{Z} \) we have \( 0 \neq |\psi(a/b)| = |a^2 + ab - (p - 1)b^2|b^{-2} \). Hence, \( |a^2 + ab - (p - 1)b^2| \geq 1 \). By the formulas (6),

\[
|\text{wt}(a, b) \cdot \text{swt}(a, b)| = |\lambda(a + \lambda b)\lambda_1^{-1}(a + \lambda_1 b)|
\]
\[
= |\lambda \lambda_1^{-1}| |a^2 + (\lambda + \lambda_1)ab + \lambda \lambda_1 b^2|
\]
\[
= \frac{\lambda^2}{p-1}|a^2 + ab - (p - 1)b^2| \geq \frac{\lambda^2}{p-1}.
\]

**Lemma 3.4.** Suppose that \( p \geq 3 \). Let \( w = t_0^{\alpha_0} t_1^{\alpha_1} \ldots t_{n-2}^{\alpha_{n-2}} v_n, n \geq 1 \), be a monomial of the subalgebra \( H \) above, namely, \( 0 \leq \alpha_i \leq p - 1, \alpha_{n-2} \leq p - 2 \). Then

1. \( \lambda^{n-2} \leq \text{wt}(w) \leq \lambda^n; \)
2. \( |\text{swt}(w)| \leq C|\lambda_1|^{n-2} \) in case \( p \geq 5; \)
3. \( |\text{swt}(w)| \leq pn \) in case \( p = 3. \)

**Proof.** Clearly, \( \text{wt} \, w \leq \text{wt} \, v_n \leq \lambda^n \). In case \( n = 1 \) we have only one monomial \( w = v_1 \) and our estimates are valid. Let \( n \geq 2 \), we obtain the bounds

\[
\text{wt}(w) = \text{wt}(v_n) + \sum_{i=0}^{n-2} \alpha_i \text{wt} \, t_i
\]
\[
= \lambda^n - \sum_{i=0}^{n-2} \alpha_i \lambda^i
\]
\[
\geq \lambda^n - (p - 1) \sum_{i=0}^{n-2} \lambda^i + \lambda^{n-2}
\]
\[
\geq \lambda^n (1 - \frac{(p-1)\lambda^{-2}}{1-1/\lambda} + \frac{1}{\lambda^2})
\]
\[
= \lambda^n \left( \frac{\lambda^2 - 2 - (p-1) \lambda^{-2}}{\lambda^2 - \lambda} + \frac{1}{\lambda^2} \right) = \lambda^{n-2}.
\]
Claims 2 and 3. In case $p \geq 5$ we have $|\lambda_1| > 1$ and get the bound
\[
|\text{sww}(w)| = \left| \text{sww}(v_n) + \sum_{i=0}^{n-2} \alpha_i \text{sww} t_i \right| \leq |\lambda_1|^n + (p - 1) \frac{|\lambda_1|^{n-2}}{|\lambda_1| |\lambda_1|} = C |\lambda_1|^{n-2}.
\]
If $p = 3$ then $\lambda_1 = -1$; in this case we have the bound
\[
|\text{sww}(w)| = \left| \text{sww}(v_n) + \sum_{i=0}^{n-2} \alpha_i \text{sww} t_i \right| \leq 1 + (p - 1)(n - 1) \leq pn.
\]
In [24] it is shown that
\[
\text{GKdim } L = \frac{\ln p}{\ln \lambda}, \quad \text{GKdim } A \leq \frac{2 \ln p}{\ln \lambda},
\]
where $1 < \ln p / \ln \lambda < 2$. This and the theory of M. Smith [25] imply that the growth of $u(L)$ is subexponential and therefore intermediate. Let us determine the growth of $u(L)$ more precisely. We will need some definitions. Consider two series of functions $\Phi^q_\alpha(n), q = 2, 3,$ of natural argument with the parameter $\alpha \in \mathbb{R}^+$:
\[
\Phi^2_\alpha(n) = n^\alpha, \quad \Phi^3_\alpha(n) = \exp(n^\alpha/(\alpha + 1)).
\]
We compare functions $f : \mathbb{N} \to \mathbb{R}^+$ by means of the partial order: $f(n) \leq^a g(n)$ if and only if there exists $N > 0$ such that $f(n) \leq g(n), n \geq N$. Suppose that $A$ is a finitely generated algebra and $\gamma_A(n)$ is its growth function. We define the dimension of level $q, q \in \{2, 3\}$, and the lower dimension of level $q$ by
\[
\text{Dim}^q A = \inf \{\alpha \in \mathbb{R}^+ \mid \gamma_A(n) \leq^a \Phi^q_\alpha(n)\},
\]
\[
\text{Dim}^q A = \sup \{\alpha \in \mathbb{R}^+ \mid \gamma_A(n) \geq^a \Phi^q_\alpha(n)\}.
\]
The $q$-dimensions for arbitrary level $q \in \mathbb{N}$ were introduced by the first author in order to specify the subexponential growth of universal enveloping algebras [17]. They generalize the Gelfand–Kirillov dimensions. The condition $\text{Dim}^q A = \Phi^q A = \alpha$ means that the growth function $\gamma_A(n)$ behaves like $\Phi^q_\alpha(n)$. The dimensions of level 2 are exactly the upper and lower Gelfand–Kirillov dimensions [6], [14]. The dimensions of level 3 correspond to the superdimensions of [4] up to normalization (see [18]). We describe the growth of $u(L)$ in terms of $\text{Dim}^3 A$.

**Lemma 3.5.** Let $\theta = \ln p / \ln \lambda$. The growth of the restricted enveloping algebra $u(L)$ is intermediate and
\[
1 \leq \text{Dim}^3 u(L) \leq \theta.
\]

**Proof.** We have $\text{Dim}^2 L = \text{GKdim } L = \theta$. Now the claim follows from (the proof) of Proposition 1 in [18]. That proposition deals with the growth of the universal enveloping algebra, some minor changes are needed to modify the proof for the restricted enveloping algebra. \qed
4. The first example: weight structure

The weights in the case $p = 2$ were studied in [21]. In that case the weights of the algebras $L = \text{Lie}_p(v_1, v_2)$ and $A = \text{Alg}(v_1, v_2)$ lie in the strips $|\eta| < \text{const}$, while the weights of the restricted enveloping algebra $u = u(L)$ are bounded by a parabola-like curve $|\eta| \leq C \xi^\theta$, for some constant $0 < \theta < 1$.

Now we assume that $p \geq 3$. We shall show that the weights of all three algebras $L, A, u$ belong to a region bounded by a parabola-like curve as well.

![Figure 1. $p = 3$, weights of $L$.](image)

**Theorem 4.1.** Let $p \geq 3$. Consider subalgebras the $H, L$ of Der $R$. Then in terms of the new coordinates $(\xi, \eta)$, homogeneous elements of these algebras belong to the following plane regions.

1. For $p \geq 5$ we have $|\eta| \leq C \xi^\theta$, where $0 < \theta = \frac{\ln|\lambda|}{\ln \lambda} < 1$.
2. For $p = 3$ we have $|\eta| \leq B + C \ln \xi$,

for some positive constants $B, C$.

**Proof.** Take a basis monomial $w = t_0^{\alpha_0}t_1^{\alpha_1} \ldots t_{n-2}^{\alpha_{n-2}}v_n \in H$. Consider the new coordinates $(\xi, \eta)$ of $w$. By Lemma 3.4 we have $\xi = \text{wt}(w) \geq \lambda^{n-2}$. Hence, $n \leq 2 + \ln \xi / \ln \lambda$. Consider the case $p \geq 5$, we apply the second estimate of
Lemma 3.4:

\[ |\eta| = |\text{swt}(w)| \leq C|\lambda_1|^{n-2} \leq C|\lambda_1|^{|\ln \xi|/\ln \lambda} = C\xi^{n|\lambda_1|/\ln \lambda}. \]

In the case \( p = 3 \) we use the third estimate of Lemma 3.4 to get

\[ |\eta| = |\text{swt}(w)| \leq pn \leq p(2 + \ln \xi/\ln \lambda). \]

**Theorem 4.2.** Consider the restricted enveloping algebra \( u = u(L) \). Then there exist constants \( C > 0 \) and \( 0 < \theta < 1 \) such that homogeneous elements of \( u \) belong to the plane region

\[ |\eta| \leq C\xi^\theta. \]

**Proof.** The case \( p = 2 \) was settled in [21]. First, consider the case \( p \geq 5 \).

We shall consider coordinates of homogeneous elements of the bigger algebra \( u(H) \supset u = u(L) \). Let \( \{w_i \mid i \in \mathbb{N}\} \) be the ordered basis of \( H \), which consists of the elements \( w = t_0^{\alpha_0}t_1^{\alpha_1} \cdots t_{n-2}^{\alpha_{n-2}}v_n \), where \( 0 \leq \alpha_i \leq p - 1 \) and \( 0 \leq \alpha_{n-2} \leq p - 2 \). Consider the function \( l: \{w_i \mid i \in \mathbb{N}\} \to \mathbb{N} \), \( l(t_0^{\alpha_0}t_1^{\alpha_1} \cdots t_{n-2}^{\alpha_{n-2}}v_n) = n \). By Lemma 3.4 we have the estimates

\[ \lambda^{n-2} \leq \text{wt}(w) \leq \lambda^n, \quad |\text{swt}(w)| \leq C|\lambda_1|^{n-2}, \quad n \in \mathbb{N}. \]  

Consider a standard basis element of \( u(H) \) of type \( v = v_{i_1} \cdots v_{i_s} \), where the \( v_{ij} \) enter the product in ordered way and each \( v_{ij} \) occurs at most \( p - 1 \) times. Let \( N = \max\{l(w_{ij}) \mid 1 \leq j \leq s\} \). Denote by \( \mu_k \) the number of \( w_{ij} \) such that
$k = l(w_{ij})$, for all $k = 1, \ldots, N$. Consider the new coordinates $(\xi, \eta)$, where $\xi = \text{wt}(v)$ and $\eta = \text{swt}(v)$. We apply (7) and obtain the estimates

$$
\frac{1}{\lambda^2} \sum_{k=1}^{N} \mu_k \lambda^k \leq \xi \leq \sum_{k=1}^{N} \mu_k \lambda^k, \quad |\eta| \leq \frac{C}{|\lambda_1|^2} \sum_{k=1}^{N} \mu_k |\lambda_1|^k. \quad (8)
$$

Let $\alpha = \alpha(v)$ be the number such that $|\eta| = \xi^\alpha$. Then

$$
\alpha(v) = \frac{\ln |\eta|}{\ln \xi} \leq \frac{\ln \left( \frac{C}{|\lambda_1|^2} \sum_{k=1}^{N} \mu_k |\lambda_1|^k \right)}{\ln \left( \frac{1}{\lambda^2} \sum_{k=1}^{N} \mu_k \lambda^k \right)}. \quad (9)
$$

The number of different basis elements $w_i$ such that $l(w_i) = k$ equals $p^{k-2}(p-1) < p^{k-1}$ for all $k \geq 2$. Each of them can enter $v$ at most $(p-1)$ times. Hence we get

$$
\mu_k \leq p^{k-1}(p-1) < p^k, \quad k = 1, \ldots, N. \quad (10)
$$

Let us evaluate the maximal value of $\alpha(v)$ among all $v$’s with the fixed value $\xi(v) = \xi_0$. From (8) we have $\xi_0 \leq \sum_{k=1}^{N} \mu_k \lambda^k \leq \lambda^2 \xi_0$, this estimate yields the range of values for the denominator of (9). To estimate the numerator of (9) we consider the maximum of the linear function

$$
f(x_1, \ldots, x_N) = \sum_{k=1}^{N} x_k |\lambda_1|^k, \quad 0 \leq x_k \leq p^k, \quad k = 1, \ldots, N,
$$

subject to a constraint of the form of a hyperplane $\sum_{k=1}^{N} x_k \lambda^k = A$, where the constant $A$ is such that $\xi_0 \leq A \leq \lambda^2 \xi_0$. Note that the denominator of (9) is fixed on each hyperplane. Since $|\lambda_1| < \lambda$, the maximum on each hyperplane is achieved when we assign the biggest possible values for $x_k$ with the smallest $k$’s. By (10), we have the bounds $0 \leq x_k \leq p^k, \quad k = 1, \ldots, N$. Thus, we take the point on the hyperplane $x_k = p^k, \quad k = 1, \ldots, m$, for some $m \leq N$, the appropriate value $x_{k+1} \in [0, p^{k+1})$, and $x_{k+2} = \cdots = x_N = 0$. This point yields the upper bound

$$
\alpha(v) \leq \frac{\ln \left( \frac{C}{|\lambda_1|^2} \left( \sum_{k=1}^{m} p^k |\lambda_1|^k + x_{k+1} |\lambda_1|^{k+1} \right) \right)}{\ln \left( \frac{1}{\lambda^2} \left( \sum_{k=1}^{m} (p\lambda)^k + x_{k+1} \lambda^{k+1} \right) \right)} \leq \frac{\ln \left( C_1 p^m |\lambda_1|^m \right)}{\ln \left( C_2 p^m \lambda^m \right)} = \frac{1}{m} \ln \left( C_1 + \ln (p |\lambda_1|) \right).
$$

When $\xi_0$ increases, the number $m$ increases as well. Let us choose the number $\theta$ such that $\ln(p |\lambda_1|)/\ln(p\lambda) < \theta < 1$. Then for sufficiently large $\xi$ we have $\alpha(v) \leq \theta$, hence $|\eta| \leq \xi^\theta$. By choosing an appropriate constant $C$ we get $|\eta| \leq C \xi^\theta$ for all $\xi > 0$.

It remains to consider the case $p = 3$. For elements of the Lie algebra we have the bound $|\eta| \leq B + C \ln \xi$. Recall that $\lambda_1 = -1$ in this case. Take $\lambda_2 = 3/2 < \lambda = 2.$
Let \( \{w_i \mid i \in \mathbb{N}\} \) be the ordered basis of \( H \) as above. Then we have \( |\text{swt}(w_i)| \leq \lambda_{2}^{n} \), where \( n = l(w_i) \), for \( i \geq N \), provided that the number \( N \) is sufficiently large. We find constant \( C_1 \) such that \( |\text{swt}(w_i)| \leq C_1 \lambda_{2}^{l(w_i)} \) for all \( w_i \). Now we can formally apply the arguments above.

Consider the triangular decompositions of Corollary 3.2.

**Corollary 4.3.** Let \( L, A, u \) be as above and consider the decompositions given by the superweight

\[
L = L_+ \oplus L_0 \oplus L_-, \quad A = A_+ \oplus A_0 \oplus A_-, \quad u = u_+ \oplus u_0 \oplus u_-.
\]

(1) Then the upper and lower components \( L_\pm, A_\pm, u_\pm \), are locally nilpotent.

(2) In the irrational case, the zero components above are trivial and we obtain decompositions into a direct sum of two locally nilpotent subalgebras.

**Proof.** Consider, for example, \( u_+ \). The line \( \eta = \text{swt}(x, y) = 0 \) separates the upper and lower components. By (6), this line is given by the equation \( x + \lambda_1 y = 0 \). Consider homogeneous monomials \( u_1, \ldots, u_k \in u_+ \) above this line and the subalgebra \( A = \text{Alg}(u_1, \ldots, u_k) \) generated by these elements. Let \( N \in \mathbb{N} \) and consider \( u = \sum_{j; n \geq N} \alpha_j u_{j_1} \ldots u_{j_n}, \alpha_j \in K \). Then it is geometrically clear (see Figure 3) that the respective vectors of all homogeneous components belong to the shaded angle. All components go out of the region \( |\eta| < C \xi^{\theta} \) provided that \( N \) is sufficiently large. Hence, \( A^N = 0 \).

In irrational case, we obtain some more examples of finitely generated infinite-dimensional associative algebras that are direct sums of two locally nilpotent subal-

![Figure 3](image-url)

Figure 3. \( p \geq 2 \), weights of \( u \).
gebras. Those examples were constructed in [13], [5]. Our examples, $A$ and $u$, are of polynomial and intermediate growth.

5. Second example

Now we turn to the study of the second example suggested in [24]. We keep the same notations $I$, $R$, $L$, $A$, $H$ as in the first example, but now they denote another objects.

We add some negative indices to the index set $I = \{2-p, 2-p+1, \ldots, 0, 1, \ldots\}$ and consider the truncated polynomial ring $R = K[t_i \mid i \in I]/(t_i^p \mid i \in I)$. Then we introduce the derivations

$$v_m = \partial_m + t_{m-p+1}^{p-1}(\partial_{m+1} + t_{m-p+2}^{p-1}(\partial_{m+2} + t_{m-p+3}^{p-1}(\partial_{m+3} + \ldots)))). \quad m \geq 1.$$  

As above, $\tau : R \to R$ is the endomorphism given by $\tau(t_i) = t_{i+1}, i \in I$. Observe that

$$v_m = \partial_m + t_{m-p+1}^{p-1}v_{m+1} = \tau^{m-1}(v_1), \quad m \geq 1.$$  

Now let $L = \text{Lie}_p(v_1, \ldots, v_p) \subset \text{Der} R$ denote the restricted subalgebra generated by $\{v_1, v_2, \ldots, v_p\}$. In the case of characteristic $p = 2$, this example coincides with the Fibonacci restricted Lie algebra [19]. In what follows we assume that $p \geq 3$.

We also can consider a slight modification $\hat{L} = \text{Lie}_p(\partial_1, \ldots, \partial_{p-1}, v_p) \subset \text{Der} R$.

Let us make the convention that if the upper index of a product (sum) is less than the lower index, then the product is empty. Similarly, if we list a set as $\{i, i+1, \ldots, j\}$ and $i > j$, then the set is assumed to be empty.

**Lemma 5.1.** Let $p \geq 3$. The following commutation relations hold:

1. $[v_m, v_{m+1}] = -(\prod_{j=m-p+2}^{m-1} t_j^{p-1})t_{m-p}^{p-2}v_m + p$ for $m \geq 1$;
2. for $n \geq 1$, $k \geq 2$ we have (both sets in the product below may be empty)

$$[v_n, v_{n+k}] = -\sum_{j=\max\{0, k-p+1\}}^{k-1} \left( \prod_{l \in \{1, \ldots, j\} \cup \{k+1, \ldots, j+p-1\}} t_{n-p+l}^{p-1} \right) t_{n+j+1}^{p-2}v_n + p;$$

3. for all $n, m \geq 1$

$$[\partial_n, v_m] = \begin{cases} 
-(\prod_{j=m-p+1}^{n-1} t_j^{p-1})t_{n-p}^{p-2}v_m + p, & m < n + p - 1, \\
-t_{n-p}^{p-2}v_m + p, & m = n + p - 1, \\
0, & m > n + p - 1;
\end{cases}$$

4. for all $n \geq 1$, $j \geq 0$ we have the action

$$v_m(t_j) = \begin{cases} 
\prod_{i=m-p+1}^{j-1} t_i^{p-1}, & m < j, \\
1, & m = j, \\
0, & m > j;
\end{cases}$$
(5) \( v_n^p = -(t_{n-(p-1)} \ldots t_{n-1})^{p-1}v_{n+p} \) for all \( n \geq 1 \).

**Proof.** Let us check the first claim:

\[
[v_m, v_{m+1}] = [\partial_m + t_{m-p+1}^{p-1}v_{m+1}, v_{m+1}] = [\partial_m, v_{m+1}]
\]

\[= [\partial_m, \partial_{m+1} + t_{m-p+2}^{p-1}(\partial_{m+2} + \ldots \ldots + t_{m-p}^{p-1}(\partial_{m+p-1} + t_{m}^{p-1}v_{m+p}) \ldots )]
\]

\[= -t_{m-p+2}^{p-1} \ldots t_{m-1}^{p-1}t_{m}^{p-2}v_{m+p}.
\]

To prove the claim (3), observe that the product is nontrivial only for \( m \leq n + p - 1 \). In this case we get

\[ [\partial_n, v_m] = [\partial_n, \partial_m + t_{m-p+1}^{p-1}(\partial_{m+p-1} + \ldots + t_{m-1}^{p-1}(\partial_{m+p-1} + t_{m}^{p-1}v_{m+p}) \ldots )]
\]

\[= -\left( \prod_{j=m-p+1}^{n-1} t_j^{p-1} \right)t_n^{p-2}v_{n+p}.
\]

Now we prove claim (2). Let \( k \geq 2 \). We have

\[ [v_n, v_{n+k}] = [\partial_n + t_{n-p+1}^{p-1}(\partial_{n+p} + \ldots \ldots + t_{n+k-p-1}^{p-1}(\partial_{n+k-1} + t_{n+k-p}^{p-1}v_{n+k}) \ldots ), v_{n+k}]
\]

\[= [\partial_n + t_{n-p}^{p-1}(\partial_{n+p} + \ldots + t_{n+k-p-1}^{p-1}(\partial_{n+k-1} + t_{n+k-p}^{p-1}v_{n+k}) \ldots ), v_{n+k}]
\]

\[= \sum_{j=0}^{k-1} \left( \prod_{l=1}^{j} t_{n-p+l}^{p-1} \right)[\partial_{n+j}, v_{n+k}]
\]

\[= \sum_{j=\max\{0,k-p+1\}}^{k-1} \left( \prod_{l=1}^{j} t_{n-p+l}^{p-1} \right)[\partial_{n+j}, \partial_{n+k} + t_{n+k-p}^{p-1}(\ldots + t_{n+j}^{p-1}v_{n+j+p})]
\]

\[= \sum_{j=\max\{0,k-p+1\}}^{k-1} \left( \prod_{l=1}^{j} t_{n-p+l}^{p-1} \right) t_{n+j}^{p-2}v_{n+j+p}.
\]

Claim 4 is proved as follows:

\[ v_m(t_j) = (\partial_m + t_{m-p+1}^{p-1}(\partial_{j+p} + \ldots \ldots + t_{j}^{p-1}(\partial_{j+p} + \ldots \ldots )))(t_j) = \prod_{i=m-p+1}^{j-p} t_i^{p-1}.
\]

Consider the last claim.

\[ v_n^p = ((\partial_n + t_{n+1-p}^{p-1}v_{n+1}) + t_{n+1-p}^{p-1}t_{n+2-p}^{p-1}v_{n+2})^p = (x + y)^p.
\]
We have
\[
\text{ad } x(y) = \text{ad}(\partial_n + t_{n+1-p}^{p-1} \partial_{n+1})(t_{n+1-p}^{p-1} t_{n+2-p}^{p-1} v_{n+2})
\]
\[
= t_{n+1-p}^{p-1} t_{n+2-p}^{p-1} \text{ad}(\partial)(v_{n+2})
\]
\[
= -t_{n-(p-1)}^{p-1} t_{n-(p-2)}^{p-1} (t_{n-(p-3)} \ldots t_{n-1})^{p-1} t_{n}^{p-2} v_{n+p}
\]
\[
= -(t_{n-(p-1)} \ldots t_{n-1})^{p-1} t_{n}^{p-2} v_{n+p},
\]
where the factor \((t_{n-(p-1)} t_{n-(p-2)})^{p-1}\) is present for all \(p \geq 3\). We consider all Lie polynomials in \(x\) and \(y\). We observe that a further multiplication by \(y\) is zero due to the total power of the letter \(t_{n+1-p}\), which cannot be killed by derivations involved. Thus, only one term in (1) is nontrivial, namely \(s_{p-1}(x, y) = (\text{ad } x)^{p-1}(y)\). We get
\[
v_n^p = (x + y)^p
\]
\[
= (\text{ad } x)^{p-1}(y)
\]
\[
= (\text{ad } x)^{p-2}([x, y])
\]
\[
= (\text{ad}(\partial_n + t_{n-(p-1)}^{p-1} \partial_{n+1}))^{p-2}(-t_{n-(p-1)} \ldots t_{n-1})^{p-1} t_{n}^{p-2} v_{n+p}
\]
\[
= -(t_{n-(p-1)} \ldots t_{n-1})^{p-1} t_{n}^{p-2} v_{n+p}.
\]

\[\square\]

**Lemma 5.2.** Let \(H\) be the \(K\)-linear span of the set
\[\{t_{2-p}^{\alpha_2} \ldots t_{n-p}^{\alpha_n-1} t_{n-p}^{\alpha_{n-p}} v_n \mid 0 \leq \alpha_i \leq p-1, \alpha_{n-p} \leq p-2, n \geq 1\}.
\]
Then \(H\) is a restricted subalgebra of \(\text{Der } R\) and \(L \subset H\).

**Proof.** We use Lemma 5.1 and proceed as in Lemma 2.2. \(\square\)

## 6. Second example: weights

As above we will define a gradation on the Lie algebra \(L\) by presenting it as a direct sum of weight spaces, all weights being complex numbers. We assume that \(w(\partial_i) = -w(t_i) = a_i \in \mathbb{C}\) for all \(i \in \mathbb{N}\). Let us choose numbers \(a_i\) so that all terms in the expression of \(v_i\) are homogeneous. We obtain \(a_i = \text{wt } \partial_i = \text{wt } \partial_{i+1} + (p-1) \text{wt } t_{i-p+1} = a_{i+1} - (p-1)a_{i-p+1}\), which implies the recurrence relation
\[
a_i = a_{i-1} + (p-1)a_{i-p} \quad i \geq p.
\]

Let us study its characteristic polynomial \(\phi(x) = x^p - x^{p-1} - (p-1)\).

**Lemma 6.1.** Consider a prime \(p \geq 3\) and the polynomial \(\phi(x) = x^p - x^{p-1} - (p-1)\). Then:
(1) \( \phi(x) \) is irreducible and has distinct roots.
(2) The equation \( \phi(x) = 0 \) has a unique real root \( \lambda_0 \).
(3) Let \( \lambda_1, \ldots, \lambda_{p-1} \) be the remaining complex roots. Then \( 1 < |\lambda_i| < \lambda_0 < 2 \) for all \( i = 1, \ldots, p - 1 \).
(4) Let \( |\lambda_i| = |\lambda_j| \). Then \( i = j \), or \( \lambda_i = \overline{\lambda}_j \).

Proof. Let us prove that \( \phi(x) \) is irreducible modulo \( p \). Making the substitution \( x = 1/y \), we see that it is sufficient to prove that \( g(y) = y^p - y + 1 \) is irreducible over \( \mathbb{Z}_p \). Let \( a \) be a root of \( g \), then so is \( a + 1 \) and hence \( a + i, i = 0, \ldots, p - 1 \), are all roots of \( g(x) \). If \( m(x) \) is the minimal polynomial of \( a \) then \( m_i(x) = m(x - i) \) is the minimal polynomial of the root \( a + i \). Therefore, the minimal polynomials of all roots of \( g(y) \) have the same degree \( k \), and hence any polynomial that has a common root with \( g(y) \) has an irreducible factor of degree \( k \). If \( g(y) \) is not irreducible then it is a product of several polynomials of degree \( k \), which implies that \( k \) is a proper factor of \( p = \deg g \), a contradiction.

Since the equation is over \( \mathbb{Q} \), the roots are distinct.

We consider the derivative \( \phi'(x) = x^{p-2}(px - (p - 1)) \). It has two roots \( x_0 = 0 \) and \( x_1 = 1 - 1/p \). Observe that \( \phi(0) = \phi(1) = -(p - 1) \). Also, \( \phi(2) = 2^{p-1} - (p - 1) > 0 \). We conclude that \( \phi(x) = 0 \) has a unique real root \( \lambda_0 \), moreover \( 1 < \lambda_0 < 2 \).

Now let \( \lambda_1 \) be a root. Recall that \( \lambda_1 \notin \mathbb{R} \). Suppose that \( |\lambda_1| \geq \lambda_0 \). We have two equalities \( \lambda_0^p - \lambda_0^{p-1} = p - 1 \) and \( \lambda_1^p - \lambda_1^{p-1} = p - 1 \), the latter can be depicted as a non-degenerate triangle in \( \mathbb{C} \). By the triangle inequality, \( p - 1 > |\lambda_1|^p - |\lambda_1|^{p-1} \).

Consider the function \( f(x) = x^p - x^{p-1}, x \in \mathbb{R} \). Using the derivative \( f'(x) = x^{p-2}(px - (p - 1)) \) we see that \( f(x) \) is increasing for \( x > 1 - 1/p \). We obtain
\[
p - 1 = \lambda_0^p - \lambda_0^{p-1} = f(\lambda_0) \leq f(|\lambda_1|) = |\lambda_1|^p - |\lambda_1|^{p-1} < p - 1.
\]
This contradiction proves that \( |\lambda_1| < \lambda_0 \). Suppose that \( |\lambda_1| \leq 1 \) for a root \( \lambda_1 \notin \mathbb{R} \). Then \( p - 1 = \lambda_1^p - \lambda_1^{p-1} = |\lambda_1|^p - |\lambda_1|^{p-1} < 2 \), a contradiction. Thus \( |\lambda_1| > 1 \) and the third claim is proved.

To prove claim (4), let \( \lambda_1, \lambda_2 \) be two different complex roots of our equation such that \( |\lambda_1| = |\lambda_2| \). Consider the triangle in \( \mathbb{C} \) given by \( p - 1 = \lambda_1^p - \lambda_1^{p-1} \), where \( p - 1, \lambda_1^p \) start from the origin. Consider all triangles on the plane with the same side \( p - 1 \), with the other sides of lengths \( |\lambda_1|^p, |\lambda_1|^{p-1} \) and with the longest side starting from the origin. There are only two such triangles. They correspond to \( \lambda_1 \) and \( \lambda_2 \). We have two possibilities. a) \( \lambda_1^p = \lambda_2^p, \lambda_1^{p-1} = \lambda_2^{p-1} \), and so \( \lambda_1 = \lambda_2 \).

\[
\text{b) } \lambda_1^p = \lambda_2^p, \lambda_1^{p-1} = \lambda_2^{p-1}, \text{ and we get } \lambda_1 = \overline{\lambda}_2. \]

Denote \( s = (p - 1)/2 \). For simplicity, order the roots so that \( \lambda_{i+s} = \overline{\lambda}_i \) for \( i = 1, \ldots, s \). We introduce the \( p \) weight functions
\[
\text{wt}_j(\partial_n) = \lambda_j^n, \quad n \in \mathbb{N}, \quad j = 0, \ldots, p - 1.
\]
By Lemma 1.1, these weight functions define a gradation on the subalgebra \( H \subseteq \text{Der} R \) defined above. For a homogeneous element \( v \in H \) let
\[
Wt(v) = (\text{wt}_0 v, \text{wt}_1 v, \ldots, \text{wt}_{p-1} v), \quad v \in H.
\]

**Theorem 6.2.** Let \( L = \text{Lie}_p(v_1, \ldots, v_p) \subset H \subseteq \text{Der} R \) be the restricted subalgebras defined above. Then:

1. The weight functions are additive on products of homogeneous elements of \( H \) and \( L \).
2. We have the \( \mathbb{Z}^p \)-gradation
\[
L = \bigoplus_{a_1, \ldots, a_p \geq 0} L_{a_1, \ldots, a_p},
\]
where \( L_{a_1, \ldots, a_p} \) is spanned by products with \( a_i \) factors \( v_i, i = 1, \ldots, p \).
3. Let \( v \in L_{a_1, \ldots, a_p}, \) where \( a_i \geq 0 \). Then
\[
\text{wt}_j v = \sum_{k=1}^{p} a_k \lambda_j^k, \quad j = 0, 1, \ldots, p - 1.
\]

**Proof.** The additivity follows from Lemma 1.1 and our construction.

Also, by our construction all components of \( v_n, n \in \mathbb{N} \), have the same weights, namely, \( \text{wt}_j (v_n) = \text{wt}_j \partial_n = \lambda_j^n, \) \( j = 0, 1, \ldots, p - 1, n \in \mathbb{N} \). Let \( v \in L \) be a monomial that contains \( a_i \) factors \( v_i \) for \( i = 1, \ldots, p \). From additivity of the weight functions we get
\[
(\text{wt}_0 v, \ldots, \text{wt}_{p-1} v) = Wt v
= \sum_{k=1}^{p} a_k Wt(v_k)
= \sum_{k=1}^{p} a_k (\lambda_0^k, \ldots, \lambda_{p-1}^k)
= (\sum_{k=1}^{p} a_k \lambda_0^k, \ldots, \sum_{k=1}^{p} a_k \lambda_{p-1}^k).
\]
The vectors \( Wt(v_k) = (\lambda_0^k, \ldots, \lambda_{p-1}^k), k = 1, \ldots, p \), are linearly independent by Vandermonde’s argument. Thus we get the claimed \( \mathbb{Z}^p \)-grading and the third claim as well.

This example also has a nil \( p \)-mapping.

**Theorem 6.3.** Let \( L = \text{Lie}_p(v_1, v_2, \ldots, v_p) \subset \text{Der} R \) be the restricted Lie algebra as above. Then \( L \) has a nil \( p \)-mapping.

**Proof.** We refer the reader to the arguments in [24], where it was proved that the \( p \)-mapping is nil for a class of restricted Lie algebras.

The first and second claims are obvious.

Proof.

Now we want to introduce new coordinates in $\mathbb{R}^p$. Let $\tilde{x} = (x_1, \ldots, x_p) \in \mathbb{R}^p$ and set

\[
\begin{align*}
\xi_0(\tilde{x}) &= x_1 \lambda_0^1 + x_2 \lambda_0^2 + \cdots + x_p \lambda_0^p, \\
\xi_1(\tilde{x}) &= x_1 \lambda_1^1 + x_2 \lambda_1^2 + \cdots + x_p \lambda_1^p, \\
& \quad \vdots \\
\xi_{p-1}(\tilde{x}) &= x_1 \lambda_{p-1}^1 + x_2 \lambda_{p-1}^2 + \cdots + x_p \lambda_{p-1}^p.
\end{align*}
\]

Since $\xi_j(\tilde{x}), \xi_{j+s}(\tilde{x})$ are conjugate complex numbers for $j = 1, \ldots, s$, we get real coordinates $(\eta_0, \eta_1, \ldots, \eta_{p-1}) \in \mathbb{R}^p$ as follows (recall that $s = (p - 1)/2$). Let $\tilde{x} = (x_1, \ldots, x_p) \in \mathbb{R}^p$ and define

\[
\eta_k(\tilde{x}) = \begin{cases} 
  x_1 \lambda_0^1 + x_2 \lambda_0^2 + \cdots + x_p \lambda_0^p, & k = 0, \\
  \text{Re}(x_1 + x_2 \lambda_k^1 + \cdots + x_p \lambda_k^{p-1}), & k = 1, \ldots, s, \\
  \text{Im}(x_1 + x_2 \lambda_k^1 + \cdots + x_p \lambda_k^{p-1}), & k = s + 1, \ldots, p - 1.
\end{cases}
\]

We also consider these functions on homogeneous elements $v \in L$. Suppose that $v \in L_{a_1, \ldots, a_p}$. Then we take $\tilde{x} = (a_1, \ldots, a_p) \in \mathbb{R}^p$ and define

\[
\xi_j(v) = \xi_j(\tilde{x}), \quad \eta_j(v) = \eta_j(\tilde{x}), \quad j = 0, \ldots, p - 1.
\]

Lemma 7.1. The introduced weight functions have the following properties:

1. Let $v \in L_{a_1, \ldots, a_p}$. Then

\[
\begin{align*}
\xi_j(v) &= \text{wt}_j v, & j = 0, \ldots, p - 1, \\
\eta_0(v) &= \xi_0(v) = \text{wt}_0 v, \\
\eta_j(v) &= \text{Re}((\text{wt}_j(v))/\lambda_j), & j = 1, \ldots, s, \\
\eta_j(v) &= \text{Im}((\text{wt}_j(v))/\lambda_j), & j = s + 1, \ldots, p - 1.
\end{align*}
\]

2. These functions are additive on products of homogeneous elements of $L$.

3. Consider a lattice point $\tilde{0} \neq \tilde{x} = (n_1, \ldots, n_p) \in \mathbb{Z}^p \subset \mathbb{R}^p$. Then $\eta_j(\tilde{x}) \neq 0$ for all $j = 1, \ldots, s$.

4. Denote $\tilde{x} = (n_1, \ldots, n_p) \in \mathbb{Z}^p \subset \mathbb{R}^p$, and let $\eta_j(\tilde{x}) = 0$ for some $j \in \{s + 1, \ldots, p - 1\}$. Then $\tilde{x} = (n_1, 0, \ldots, 0)$.

Proof. The first and second claims are obvious.

Let us prove the third claim. Fix $\tilde{0} \neq \tilde{x} = (n_1, \ldots, n_p) \in \mathbb{Z}^p$ and $j \in \{1, \ldots, s\}$. Suppose that

\[
\eta_j(\tilde{x}) = \text{Re}(n_1 + n_2 \lambda_j + \cdots + n_p \lambda_j^{p-1}) = 0.
\]
We have the field extension $\mathbb{Q} \subset \mathbb{Q}(\lambda_j)$. Denote $r = n_1 + n_2\lambda_j + \cdots + n_p\lambda_j^{p-1}$. Suppose that $r \neq 0$. From (11) it follows that $r = iq$, where $q \in \mathbb{R}$. Consider $r^2 \in \mathbb{R} \cap \mathbb{Q}(\lambda_j) \neq \mathbb{Q}(\lambda_j)$. Since $|\mathbb{Q}(\lambda_j) : \mathbb{Q}| = p$ is a prime, we obtain $r^2 \in \mathbb{Q}$. Then $|\mathbb{Q}(r) : \mathbb{Q}| = 2$ divides $p$, a contradiction. Therefore, $r = n_1 + n_2\lambda_j + \cdots + n_p\lambda_j^{p-1} = 0$, which is a contradiction to the fact that $\lambda_j$ satisfies an irreducible polynomial of degree $p$.

We now turn to claim (4). Fix $\tilde{x} = (n_1, \ldots, n_p) \in \mathbb{Z}^p$ and $j \in \{s+1, \ldots, p-1\}$. Suppose that

$$\eta_j(\tilde{x}) = \text{Im}(n_1 + n_2\lambda_j + \cdots + n_p\lambda_j^{p-1}) = 0.$$ 

Denote $r = n_1 + n_2\lambda_j + \cdots + n_p\lambda_j^{p-1}$. Then $r \in \mathbb{R} \cap \mathbb{Q}(\lambda_j) \neq \mathbb{Q}(\lambda_j)$. Since $|\mathbb{Q}(\lambda_j) : \mathbb{Q}| = p$ is a prime, we get $r \in \mathbb{Q}$. We obtain $(n_1 - r) + n_2\lambda_j + \cdots + n_p\lambda_j^{p-1} = 0$, which is possible only in the case $n_1 = r, n_2 = \cdots = n_p = 0$. □

Now we get triangular decompositions where the zero component is always trivial.

**Corollary 7.2.** Let $L = \text{Lie}_p(v_1, \ldots, v_p)$, let $A = \text{Alg}(v_1, \ldots, v_p)$ be the restricted Lie algebra and associative algebra generated by $\{v_1, \ldots, v_p\}$. Let $U = U(L)$, $u = u(L)$ be the universal enveloping algebra and the restricted enveloping algebra. Then all these algebras have decompositions into direct sums of two subalgebras as follows:

$$L = L_+ \oplus L_-, \quad A = A_+ \oplus A_-, \quad U = U_+ \oplus U_-, \quad u = u_+ \oplus u_-.$$

**Proof.** Fix $j \in \{1, \ldots, s\}$ and set, for example,

$$L_+ = \{v \in L \mid \eta_j(v) > 0\}, \quad L_- = \{v \in L \mid \eta_j(v) < 0\}. \quad \square$$

Observe that the weight functions $\eta_j, j \in \{s+1, \ldots, p-1\}$, also yield triangular decompositions, but in this case the components $L_0$ and $A_0$ are nontrivial and finite dimensional. Indeed, consider $L_0 = \{v \in L \mid \eta_j(v) = 0\}$. By claim (4) of Lemma 7.1, $L_0$ is spanned by products of the element $v_1$ only. Since $v_1^{p^2} = 0$, we conclude that $L_0 = \langle v_1, v_1^p \rangle$, similarly, $A_0 = \langle v_1^j \mid 1 \leq j < p^2 \rangle$ is of dimension at most $p^2$.

**Lemma 7.3.** Let $v = t_{2-p}^{\alpha_2-p} \cdots t_{n-p}^{\alpha_{n-p}} v_n \in H, n \in \mathbb{N}$, be as in Lemma 5.2. Then

1. $\lambda_0^{n-p} \leq \text{wt}_0 v \leq \lambda_0^n$,
2. $|\text{wt}_j v| \leq C|\lambda_j|^n$ for all $j = 1, \ldots, p-1$, where $C$ is some constant;
3. $|\eta_j(v)| \leq C|\lambda_j|^n$ for all $j = 1, \ldots, p-1$. 


Proof. The upper bound $w_{t_0}^n v \leq \lambda_0^n$ is obvious. We check the lower bound. Recall that $\alpha_{n-p} \leq p - 2$. Then

$$w_{t_0}(v) = \lambda_0^n - \sum_{i=2-p}^{n-p} \alpha_i \lambda_0^i$$

$$\geq \lambda_0^n - (p - 1) \sum_{i=2-p}^{n-p} \lambda_0^i + \lambda_0^{n-p}$$

$$\geq \lambda_0^n - (p - 1) \frac{\lambda_0^{n-p}}{1 - 1/\lambda_0} + \lambda_0^{n-p}$$

$$= \frac{\lambda_0^{n-p} (\lambda_0^p - \lambda_0^{p-1} - (p - 1))}{1 - 1/\lambda_0} + \lambda_0^{n-p} = \lambda_0^{n-p}.$$

Similarly,

$$|w_t(v)| = |\lambda_j^n - \sum_{i=2-p}^{n-p} \alpha_i \lambda_j^i|$$

$$\leq |\lambda_j|^n + (p - 1) \sum_{i=2-p}^{n-p} |\lambda_j|^i$$

$$\leq |\lambda_j|^n + (p - 1) \frac{|\lambda_j|^{n-p}}{1 - 1/|\lambda_j|} \leq C |\lambda_j^n|.$$

The third claim follows by the previous lemma. \qed

Now we are going to show that the weights of all three algebras $L$, $A$, $u$ again belong to a paraboloid-like region of $\mathbb{R}^p$ stretched along the axis $\eta_0$.

**Theorem 7.4.** Let $p \geq 3$ and $L = \text{Lie}_p(v_1, \ldots, v_p)$, $H$ be the subalgebras of $\text{Der} R$ as above. Then the new coordinates $(\eta_0, \eta_1, \ldots, \eta_{p-1})$ of homogeneous elements of these algebras belong to the following region of $\mathbb{R}^p$:

$$|\eta_j| \leq C \eta_0^\theta_j, \quad \theta_j = \frac{\ln |\lambda_j|}{\ln \lambda_0} < 1, \quad j = 1, \ldots, p - 1,$$

where $C$ is a positive constant.

Proof. Take a basic monomial $w = t_2^{a_2-p} \ldots t_n^{a_n-2} v_n \in H$ and consider its new coordinates $(\eta_0, \eta_1, \ldots, \eta_{p-1})$. By Lemma 7.3, we have $\eta_0 = w_{t_0}(w) \geq \lambda_0^{n-p}$. Hence, $n \leq p + \ln \eta_0 / \ln \lambda_0$. We apply the third estimate of Lemma 7.3

$$|\eta_j| \leq C |\lambda_j|^n \leq C |\lambda_j|^{p + \ln \eta_0 / \ln \lambda_0} = \tilde{C} \eta_0^{|\lambda_j| / \ln \lambda_0}, \quad j = 1, \ldots, p - 1.$$
**Theorem 7.5.** Let \( p \geq 3 \). Consider \( A = \text{Alg}(v_1, \ldots, v_p) \) and \( u = u(L) \). Then the new coordinates of homogeneous elements of these algebras also belong to the following region of \( \mathbb{R}^p \):

\[
|\eta_j| \leq C \eta_0^\theta, \quad j = 1, \ldots, p - 1,
\]

for some constants \( C > 0 \) and \( 0 < \theta < 1 \).

**Proof.** Let \( v = t_2^{\alpha_2-p} \ldots t_n^{\alpha_n-p} v_n \in H, n \in \mathbb{N} \). By Lemma 7.3 we have bounds similar to (7),

\[
\lambda_0^{n-p} \leq \text{wt}_0 v \leq \lambda_0^n, \quad |\eta_j(v)| \leq C |\lambda_j|^n, \quad |\lambda_j| < \lambda_0, \quad j = 1, \ldots, p - 1.
\]

It remains to repeat the arguments of Theorem 4.2. \( \Box \)

**Corollary 7.6.** Consider the triangular decompositions of Corollary 7.2,

\[
L = L_+ \oplus L_-, \quad A = A_+ \oplus A_-, \quad u = u_+ \oplus u_-.
\]

Then all the components \( L_\pm, A_\pm, \) and \( u_\pm \) are locally nilpotent subalgebras.

**Proof.** The arguments of Corollary 4.3 apply. \( \Box \)

**8. Second example: growth**

In this section we study the growth of the algebras that appear in the second example. In particular we check that \( L \) is infinite-dimensional.

**Theorem 8.1.** Let \( L = \text{Lie}_p(v_1, \ldots, v_p) \), and let \( \lambda_0 \) be the root of the characteristic polynomial above. Then \( \text{GKdim} \ L \leq \ln p / \ln \lambda_0 \).

**Proof.** We use the embedding of Lemma 5.2. Fix a number \( m \). Consider a homogeneous element \( g \in L \subset H \) such that \( \text{wt}_0(g) \leq m \). Then it is a sum of monomials \( v = t_2^{\alpha_2-p} \ldots t_n^{\alpha_n-p} v_n \), where \( 0 \leq \alpha_i \leq p - 1 \) and \( \alpha_{n-p} \leq p - 2 \). By Lemma 7.3, \( m \geq \text{wt}_0(g) \geq \lambda_0^{n-p} \). Hence, \( n \leq n_0 = p + [\ln m / \ln \lambda_0] \).

We estimate the number of monomials \( v \) of weight not exceeding \( m \) and obtain the bound

\[
\tilde{\gamma}_L(m) \leq \sum_{n=1}^{n_0} p^{n-1} \leq \frac{p^{n_0-1}}{1-1/p} \leq \frac{p^{p-1+\ln m / \ln \lambda_0}}{1-1/p} \approx C_0 m^{\ln p / \ln \lambda_0} \cdot \Box
\]

**Corollary 8.2.** Let \( L = \text{Lie}_p(v_1, \ldots, v_p), \lambda_0 \) as above and \( \theta = \ln p / \ln \lambda_0 \). Then the growth of the restricted enveloping algebra \( u(L) \) is intermediate and

\[
1 \leq \text{Dim}^3 u(L) \leq \theta.
\]
Theorem 8.3. Let $A = \text{Alg}(v_1, \ldots, v_p)$. Then $\text{GKdim } A \leq 2 \ln p / \ln \lambda_0$, where $\lambda_0$ is as above.

Proof. We embed our algebra into a bigger associative subalgebra $A \subset \text{Alg}(H) \subset \text{End}(R)$, where $H$ was defined in Lemma 5.2. We claim that elements of $\text{Alg}(H)$ can be expressed as linear combinations of the monomials

$$w = t_2^{\alpha_2 - p} \cdots t_n^{\alpha_n - p} v_1^{\beta_1} \cdots v_n^{\beta_n}, \quad 0 \leq \alpha_i, \beta_i \leq p - 1, \beta_n \geq 1, n \in \mathbb{N},$$

where in case $\beta_n = 1$ we additionally assume that $\alpha_{n-p} \leq p - 2$.

Indeed, let us consider a product $w = u_1 \cdots u_s$ of basis monomials $u_i = t_2^{\gamma_2 - p} \cdots t_{M_i}^{\gamma_{M_i} - p} v_{m_i} \in H$, where $\gamma_{m_i} \leq p - 2$, $i = 1, \ldots, s$. Consider the largest index $M(w) = \max\{m_i | i = 1, \ldots, s\}$. Then the highest $t_i$ is $t_{M-p}$. Our product satisfies the following property VT$_\text{max}$: if the highest $v_M$ is unique in the product, then the highest variable $t_{M-p}$ has the total occurrence at most $p - 2$. We straighten the product to the form (12). Let us check that VT$_\text{max}$ is kept under the process. We perform the following transformations.

Case 1. $v_n v_m = v_m v_n + [v_n, v_m]$ if $n > m$. Consider the terms of the product $[v_n, v_m]$, see Lemma 5.1, claims (1), (2). If we get a new highest $v_M$, we obtain the highest $t_{M-p}$ in degree $p - 2$ as well, the property VT$_\text{max}$ is kept. If we get one more term $v_M$, then there is nothing to check. If we obtain $v_j$ such that $j < M$, then we get at most $t_{j-p}$, and the total degree of the highest $t_{M-p}$ is not changed, as required.

Case 2. $v_n^p$ is expressed as in claim (5) of Lemma 5.1. We can only get a new highest $v_{M'}$ with no occurrence of $t_{M'-p}$ at all.

Case 3. The remaining operation is $v_n t_i = t_i v_n + v_n(t_i)$. Observe the second term. This operation cannot kill the highest $v_M$ since $i \leq M - p < M$. Also, $t_i$ is replaced by a product of smaller $t_j$s only. Thus, VT$_\text{max}$ is kept.

Finally, we arrive at a monomial of type (12), the property VT$_\text{max}$ means that in the case $\beta_n = 1$ we have $\alpha_{n-p} \leq p - 2$. Thus, $\text{Alg}(H)$ is spanned by the claimed monomials.

Let us estimate the weight of a monomial (12). In case $\beta_n = 1$, we use the fact that $\alpha_{n-p} \leq p - 2$ and obtain, as in Lemma 7.3, the estimate

$$\text{wt}_0(w) \geq \lambda_0^n - \sum_{i=2-p}^{n-p} \alpha_i \lambda_0^i - (p - 1) \sum_{i=2-p}^{n-p} \lambda_0^i + \lambda_0^{n-p} \geq \lambda_0^n - (p - 1) \frac{\lambda_0^{n-p}}{1 - 1/\lambda_0} + \lambda_0^{n-p} = \lambda_0^{n-p}.$$
In the case $\beta_n > 1$ we have
\[
\text{wt}_0(w) \geq 2\lambda_0^n - (p - 1) \sum_{i=2-p}^{n-p} \lambda_0^i \geq \lambda_0^n - (p - 1) \sum_{i=2-p}^{n-p} \lambda_0^i + \lambda_0^{n-p} \geq \lambda_0^{n-p}.
\]

Fix a number $m$. Consider all monomials $w$ of type (12) such that $\text{wt}_0(w) \leq m$. Both cases above yield the estimate $m \geq \text{wt}_0(w) \geq \lambda_0^{n-p}$. Then $n \leq n_0 = p + [\ln m/\ln \lambda_0]$.

Now we can estimate the number of monomials $w$ of weight not exceeding $m$ and obtain the bound
\[
\tilde{\gamma}_A(m) \leq \sum_{n=1}^{n_0} p^{2n-1} \leq p^{2n_0-1} \frac{1}{1-1/p^2} \leq p^{2\ln m/\ln \lambda_0 + 2p - 1} \frac{1}{1-1/p^2} \approx C_0 m^{2\ln p/\ln \lambda_0}.
\]

Let us prove the following commutation relation.

**Lemma 8.4.** For all $n \geq 1$ we have
\[
(\text{ad } v_n)^{p-1}(v_{n+p-1}) = -v_{n+p} - t_n(t_{n-p+1})^{p-1} \cdot t_{n+1}^{p-2} v_{n+p+1} - t_n(t_{n-p+1} t_{n-p+2} \cdot t_{n+1})^{p-1} \cdot t_{n+2}^{p-2} v_{n+p+2} \ldots - t_n(t_{n-p+1} \ldots t_{n-2} \cdot t_{n+1} \ldots t_{n+p-3})^{p-1} \cdot t_{n+p-1}^{p-2} v_{n+p-2} - 2t_n(t_{n-p+1} \ldots t_{n-1} \cdot t_{n+1} \ldots t_{n+p-2})^{p-1} \cdot t_{n+p-1}^{p-2} v_{n+p-2}.
\]

**Proof.** In claim (2) of Lemma 5.1 we take $k = p - 1$
\[
[v_n, v_{n+p-1}] = -\sum_{j=0}^{p-2} \left( \prod_{l \in \{1, \ldots, j\} \setminus \{p, \ldots, p+j-1\}} t_{n+p-l}^{p-1} \right) t_{n+j}^{p-2} v_{n+j+p} \quad (13)
\]
\[
= -t_n^{p-2} v_{n+p} - (t_{n-p+1} \cdot t_n)^{p-1} \cdot t_{n+1}^{p-2} v_{n+p+1} - (t_{n-p+1} t_{n-p+2} \cdot t_{n+1})^{p-1} \cdot t_{n+2}^{p-2} v_{n+p+2} \ldots - (t_{n-p+1} t_{n-p+2} \ldots t_{n-2} \cdot t_{n+1} \ldots t_{n+p-3})^{p-1} \cdot t_{n+p-1}^{p-2} v_{n+p-2}.
\]

Let us further commute this expression with $v_n$. Recall that $v_n$ acts trivially on $t_{n-p+1}, \ldots, t_{n-2}$. By Lemma 5.1 all elements $v_n(t_j)$, where $j \geq n + 1$, contain the factor $t_{n-p+1}^{p-1}$ and we get zero due to the other factor $t_{n-p+1}^{p-1}$. The same argument
applies to \([v_n, v_j]\), where \(j \geq n + p + 1\). Thus, we get a nontrivial action only in the cases \(v_n(t_n) = 1\) and \([v_n, v_{n+p}]\). Therefore when commuted with \(v_n\) all terms in the sum above except the first one change only the power of \(t_n\). Considering the first term we take into account that

\[
\begin{align*}
[v_n, v_{n+p}] &= -(n-p+1)^{p-1} \cdot t_{n+1}^{p-2} v_{n+p+1} \\
&\quad - (n-p+1n-p+2 \cdot t_{n+1})^{p-1} \cdot t_{n+2}^{p-2} v_{n+p+2} \\
&\quad - (n-p+1n-p+2n-p+3 \cdot t_{n+1}t_{n+2})^{p-1} \cdot t_{n+3}^{p-2} v_{n+p+3} \\
&\quad \vdots \\
&\quad - (n-p+1n-p+2 \cdots n-1 \cdot t_{n+1}t_{n+2} \cdots t_{n+p-2})^{p-1} \cdot t_{n+p-1}^{p-2} v_{n+p-1+2p-1}.
\end{align*}
\]

Each time when commuting the first term of (14) with \(v_n\), these summands add to the existing ones. As a result, there exist some scalars \(B_{s,j}\) for \(s = 1, \ldots, p-1, j = 1, \ldots, p-1\) such that

\[
(\text{ad } v_n)^s(v_{n+p-1}) = (-1)(-2) \cdots (-s) t_n^{p-s} v_{n+p} + B_{s,1} t_n^{p-s} t_{n+1}^{p-1} t_{n+2}^{p-2} v_{n+p+1} + B_{s,2} t_n^{p-s} t_{n+1}t_{n+2}^{p-1} t_{n+3}^{p-2} v_{n+p+2} + \cdots + B_{s,p-2} t_n^{p-s} t_{n+1}t_{n+2} \cdots t_{n+p-3}^{p-1} t_{n+p-2}^{p-2} v_{n+p-1+p-2} + B_{s,p-1} t_n^{p-s} t_{n+1}t_{n+2} \cdots t_{n+p-2}^{p-1} t_{n+p-1}^{p-2} v_{n+p-1+p-1}.
\]

We have the recurrence relations \(B_{s+1,j} = -sB_{s,j} - (-1)^s s!\), \(s \geq 1\) for all \(j = 1, \ldots, p-1\) and the original conditions \(B_{1,1} = B_{1,2} = \cdots = B_{1,p-2} = -1\) and \(B_{1,p-1} = 0\). We check that for all \(j = 1, \ldots, p-2\) we get \(B_{s,j} = (-1)^s s!\), \(s \geq 1\); in particular, \(B_{p-1,j} = -1\). For \(j = p-1\) we have \(B_{s,p-1} = (-1)^s (s-1)(s-1)!\), \(s \geq 1\), in particular \(B_{p-1,p-1} = -2\). \(\square\)

Let us introduce the following convenient notations. Let \(v = \sum_{i \geq m} a_i v_i \in H\), where \(a_i \in R\). Then we write \(v = O(v_m)\). Also suppose that \(r_1, \ldots, r_s \in R\). Then denote by \(O((r_1, \ldots, r_s)v_m)\) an element \(h \in H\) of the form

\[
h = \sum_{i=1}^{s} r_i g_i, \quad g_i = O(v_m).
\]

**Lemma 8.5.** For all \(m \geq 1\) we have \([H, O(v_m)] = O(v_m)\).

**Proof.** Follows from the commutation relations of Lemma 5.1. \(\square\)
**Lemma 8.6.** Let \( L = \text{Lie}_p(v_1, \ldots, v_p) \). Then there exist homogeneous elements of the form

\[
\tilde{v}_n = v_n + O((t_{2-p}^{p-1}, \ldots, t_{n-2p+1}^{p-1})v_{n+1}) \in L, \quad n = 1, 2, \ldots
\]

**Proof.** We begin with \( \tilde{v}_1 = v_1, \ldots, \tilde{v}_p = v_p \). Assume that all elements \( \tilde{v}_i \), with \( i \leq n + p - 1 \), are defined. By assumption we have elements

\[
\tilde{v}_n = v_n + O((t_{2-p}^{p-1}, \ldots, t_{n-2p+1}^{p-1})v_{n+1}) \in L,
\]

\[
\tilde{v}_{n+p-1} = v_{n+p-1} + O((t_{2-p}^{p-1}, \ldots, t_{n-p}^{p-1})v_{n+p}) \in L.
\]

Consider \([\tilde{v}_n, \tilde{v}_{n+p-1}] \). We use (13) and the commutation relations of Lemma 5.1 to get

\[
[v_n, v_{n+p-1}] = -t_n^{p-2}v_{n+p} + t_n^{p-1}O(v_{n+p+1}),
\]

\[
[v_n, O((t_{2-p}^{p-1}, \ldots, t_{n-p}^{p-1})v_{n+p})] = O((t_{2-p}^{p-1}, \ldots, t_{n-p}^{p-1})v_{n+p}).
\]

\[
[v_{n+p-1}, O((t_{2-p}^{p-1}, \ldots, t_{n-2p+1}^{p-1})v_{n+1})] = O((t_{2-p}^{p-1}, \ldots, t_{n-2p+1}^{p-1})v_{n+p+1}).
\]

\[
[O((t_{2-p}^{p-1}, \ldots, t_{n-p}^{p-1})v_{n+1}), O((t_{2-p}^{p-1}, \ldots, t_{n-p}^{p-1})v_{n+p})] = O((t_{2-p}^{p-1}, \ldots, t_{n-p}^{p-1})v_{n+p+1}).
\]

Let us explain the third relation. The action \( v_{n+p-1}(t_j) \) for some \( v_m \) inside \( O(v_{n+1}) \) can appear only for \( m \geq j + p \geq n + 2p - 1 \). On the other hand, by Lemma 5.1 \([v_{n+p-1}, v_{n+1}] = O(v_{n+p+1}) \). The second and forth equations are obtained by similar arguments. Thus,

\[
[\tilde{v}_n, \tilde{v}_{n+p-1}] = -t_n^{p-2}v_{n+p} + t_n^{p-1}O(v_{n+p+1}) + O((t_{2-p}^{p-1}, \ldots, t_{n-p}^{p-1})v_{n+p}).
\]

We repeat this process and observe that our additional factors cannot disappear:

\[
(\text{ad} \tilde{v}_n)^{p-1}(\tilde{v}_{n+p-1}) = -v_{n+p} + t_{n-p+1}^{p-1}O(v_{n+p+1}) + O((t_{2-p}^{p-1}, \ldots, t_{n-p}^{p-1})v_{n+p}).
\]

The last term can contain a summand with \( v_{n+p} \), it is of the form \( t_{i}^{p-1}r v_{n+p} \), where \( r \in R \) and \( 2-p \leq i \leq n-p \). But, by construction, the element (15) is homogeneous. Then

\[
\text{wt}_0(v_{n+p}) = \text{wt}_0(t_i^{p-1}r v_{n+p})
\]

\[
= \text{wt}_0(v_{n+p}) + \text{wt}_0(t_i^{p-1}r)
\]

\[
\leq \text{wt}_0(v_{n+p}) - (p-1)\lambda_0^i,
\]

a contradiction. Therefore, the last term (15) contains only \( v_m \) with \( m \geq n + p + 1 \). Then we set

\[
\tilde{v}_{n+p} = -(\text{ad} \tilde{v}_n)^{p-1}(\tilde{v}_{n+p-1}) = v_{n+p} + O((t_{2-p}^{p-1}, \ldots, t_{n-p+1}^{p-1})v_{n+p+1}),
\]

and the induction step is proved. \( \square \)

**Corollary 8.7.** The Lie algebra \( L = \text{Lie}_p(v_1, \ldots, v_p) \) is infinite-dimensional.
References


Received August 17, 2009; revised June 5, 2010

V. M. Petrogradsky, Faculty of Mathematics and Computer Science, Ulyanovsk State University, Lev Tolstoy 42, Ulyanovsk, 432970, Russia
E-mail: petrogradsky@rambler.ru

I. P. Shestakov, Instituto de Matemática e Estatística, Universidade de São Paulo, Caixa postal 66281, CEP 05315-970, São Paulo, Brazil
E-mail: shestak@ime.usp.br

E. Zelmanov, Department of Mathematics, University of California, San Diego, La Jolla, CA 92093-0112, U.S.A.
E-mail: ezelmano@math.ucsd.edu