Calculus of Variations — A sharp isoperimetric inequality in the plane involving Hausdorff distance, by Angelo Alvino, Vincenzo Ferone, Carlo Nitsch.

Dedicated to the memory of Renato Caccioppoli

Abstract. — We show that among all the convex bounded domain in $\mathbb{R}^2$ having an assigned asymmetry index related to Hausdorff distance, there exists only one convex set (up to a similarity) which minimizes the isoperimetric deficit. We also show how to construct this set. The result can be read as a sharp improvement of the isoperimetric inequality for convex planar domain.

Key words: Isoperimetric inequality, Bonnesen-style inequality, Hausdorff distance, isoperimetric deficit.

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1. Introduction

The classical isoperimetric inequality in the plane states that, among all the subsets of $\mathbb{R}^2$ of prescribed finite measure, the disk has the smallest perimeter, namely

$$P(E) \geq (4\pi |E|)^{1/2}, \quad \text{with equality if and only if } E \text{ is a disk.}$$

Here $|E|$ and $P(E)$ denote, as usual, the measure and the perimeter of the set $E \subset \mathbb{R}^2$.

It is almost impossible to give exhaustive references concerning the isoperimetric inequality, therefore we refer the reader to some pioneering papers [2, 5, 16, 19], to the paper by De Giorgi [9] in the general framework of finite perimeter sets in $\mathbb{R}^n$, to the reviews [12, 20, 25] and to the books [7, 8].

In [3, 4] Bonnesen introduced some remarkable inequalities which imply the isoperimetric one (see also the reviews [7, 21]). For example, we recall that for bounded convex planar sets he proved that

$$P(E)^2 - 4\pi |E| \geq 4\pi d^2.$$  \hfill (1.1)

Here $d$ is the thickness of the minimal annulus containing the boundary of $E$ and we remark that the constant $4\pi$ and the exponent 2 on the right hand side are optimal. The chief tool in the proof was a symmetrization technique known as annular symmetrization. Later Bonnesen’s work led to the study of a wider class
of inequalities nowadays known as Bonnesen-style isoperimetric inequalities (see [7, 21]).
Following Osserman [7, 21] we say that a Bonnesen-style isoperimetric inequality can be written in the form
\[ P(E)^2 - 4\pi|E| \geq F(E), \]
where the function \( F \) is nonnegative, vanishes only on the disks, and somehow measures how much \( E \) deviates from a disk. There are many different kinds of functions \( F \) satisfying these properties, and each one leads to a different refinement of the standard isoperimetric inequality.

Typical situations addressed in the literature are those where the function \( F \) depends on the set \( E \) through the so-called Fraenkel asymmetry index or through the Hausdorff distance from a ball. For the first case we quote the results contained in [15, 13, 10, 1]. In [15, 1] it is considered the case of convex planar sets and the best form of the inequality is given, while in [13, 10] the \( n \)-dimensional case is addressed.

As regards the second case, it is clear that inequality (1.1) can be written in terms of Hausdorff distance from a disk (see also [14])
\[ P(E)^2 - 4\pi|E| \geq 16\pi d_H(E, C)^2, \]
where \( d_H(A, B) \) denotes the Hausdorff distance between the sets \( A, B \subset \mathbb{R}^2 \), and \( C \) is the disk halfway between the inner and the outer circle of the annulus of minimal width that contains the boundary of \( E \). A sharp estimate of this type can be found also in [11], for nearly spherical domains in \( \mathbb{R}^n \).

In this paper we are interested in those functions \( F \) whose dependence on the set \( E \) is only through the Hausdorff asymmetry index \( \delta(E) \) defined as the transitive Hausdorff distance of \( E \) from a disk \( D_R \) having the same measure,
\[ \delta(E) = \min_{x \in \mathbb{R}^2} d_H(E, D_R(x)), \]
where \( D_R(x) \) is the disk centered at \( x \), such that \( |E| = |D_R(x)| \). We provide a sharp Bonnesen–style inequality for planar convex domains \( E \) involving just \( P(E)^2 - 4\pi|E| \) and \( \delta(E) \). Obviously the trivial relation \( d \geq \delta \) already implies the following inequality
\[ P(E)^2 - 4\pi|E| \geq 4\pi\delta(E)^2. \]
However such an inequality is not sharp. Actually there exists a maximal function \( G \) such that it holds
\[ P(E)^2 - 4\pi|E| \geq G(\delta(E)). \]
The determination of the function \( G \) relies on the investigation of the shape of the optimal sets, i.e., those sets which minimize the left hand side of (1.2), for fixed \( |E| \) and \( \delta(E) \).
We show that, for any $0 \leq \delta < +\infty$, it is possible to compute $G(\delta)$. In particular, for any fixed value of $|E|$, we work out the analytic expression of the set $E$ with asymmetry index $\delta(E) = \delta$, which achieves the equality sign in \eqref{eq:1.2}. Moreover we prove that such a set is unique up to translations. Our result is based on a new symmetrization technique introduced in \cite{1}. It is closely related to the circular symmetrization, a technique which is well suited to the bidimensional framework (see also \cite{17,18}). Using this tool we show how to reshape a given planar convex set keeping, step by step, its measure and its Hausdorff asymmetry index fixed and shortening the perimeter. The procedure eventually provides the family of optimal sets.

As a corollary to our result we provide the sharp inequality

$$P(E)^2 - 4|E| \geq 16\delta(E)^2.$$ 

2. Main statement

In order to state our main result we define the class of lenses $\mathcal{Y}_\delta$ as the family of convex set $E$ satisfying the following properties:

- $E$ has measure $\pi$;
- $E$ is symmetric with respect to a straight line such that the part of it which stays on one side of the line coincides with a circular segment (the smallest part of a disk cut by a chord).

Such a class satisfies the following properties.

Proposition 2.1. For any given positive $\pi$ and any given positive $\delta$ there exists a unique set $Y_\delta \in \mathcal{Y}_\delta$ such that

$$\delta(Y_\delta) = \delta.$$

In particular, for such a set, $\delta$ will be the difference of the radii of the disk having the same measure, and of the smallest circumscribed disk. Moreover it holds

$$4\pi^2 = \lim_{\delta \to 0} \frac{P(Y_\delta)^2 - 4\pi \pi}{\delta^2} > \lim_{\delta \to +\infty} \frac{P(Y_\delta)^2 - 4\pi \pi}{\delta^2} = 16.$$ 

We are now able to state our main result.

Theorem 2.1. For every convex set $\Omega \in \mathbb{R}^2$, the set $Y_{\delta(\Omega)} \in \mathcal{Y}_{|\Omega|}$ satisfies the inequality

$$P(\Omega) \geq P(Y_{\delta(\Omega)}),$$

equality holding if and only if $\Omega = Y_{\delta(\Omega)}$, up to translations.

As a consequence we have the following corollary.
Corollary 2.1. Every convex set $\Omega \in \mathbb{R}^2$ satisfies the inequalities

$$P(\Omega)^2 - 4\pi|\Omega| \geq 16\delta(\Omega)^2,$$

(2.1) $$P(\Omega)^2 - 4\pi|\Omega| \geq \delta(\Omega)^2 (4\pi^2 - H(\delta(\Omega))) ,$$

for some positive $H(\delta) = O(\delta)$.

We postpone the proof of Proposition 2.1 until the last section where we carry over a detailed study of the class $\mathcal{Y}_\omega$.

Remark 2.1. We observe that inequality (2.1) can be obtained in a different way. Namely, one can consider a one-parameter family of sets $E_\delta$, $\delta > 0$, which smoothly converge to a disk as $\delta \to 0$, being $\delta(E_\delta) = \delta$. The computation of the second derivative of $P(E_\delta)$ with respect to $\delta$ gives:

$$\lim_{\delta \to 0} \frac{P(E_\delta)^2 - 4\pi|E_\delta|}{\delta^2} \geq 4\pi^2 .$$

3. Proof of Theorem 2.1

Let $\Omega$ be an open bounded and convex subset of $\mathbb{R}^2$, and $D$ be a circle of radius $R = (\frac{|\Omega|}{\pi})^{1/2}$ that achieves the minimal Hausdorff distance to $\Omega$, i.e.:

$$\delta(\Omega) = d_H(\Omega, D).$$

We refer to the last condition as the optimality condition for $D$ with respect to $\Omega$. From now on we shall use as the origin of the coordinate system in $\mathbb{R}^2$ the center $O$ of $D$. We also denote by $D_i$ and $D_e$ the two disks $D_{R-\delta(\Omega)}(O)$ and $D_{R+\delta(\Omega)}(O)$. It is trivial to check that $D_i \subseteq \Omega \subseteq D_e$.

Since $\Omega$ is also starshaped with respect to $O$, we shall use $\rho(\theta)$ to denote a generic Lipschitz radial function which parametrizes the boundary of $\Omega$ with respect to the angular variable $\theta$. Such a parametrization will by possibly chosen case by case. The optimality condition immediately implies

$$\max_\theta \{ R - \rho(\theta), \rho(\theta) - R \} = \delta(\Omega),$$

(3.1) roughly speaking there exists at least one point in which the boundary of $\Omega$ touches either the boundaries of $D_i$ or $D_e$.

We also observe that, if $D$ is the optimal disk, a line $l$ passing through $O$ cannot split the plane into two open halfplanes $T_i$ and $T_e$, such that all the points where the boundary of $\Omega$ eventually touches the boundary of $D_i$ belong to $T_i$, while all the points where the boundary of $\Omega$ eventually touches the boundary of $D_e$ belong to $T_e$. Indeed, this would imply that there exists a slight translation of the set $\Omega$ in the direction normal to $l$ and towards the halfplane $T_i$, such that still $D_i \subseteq \Omega \subseteq D_e$ but $\partial \Omega$ will have no intersection with both the boundaries of $D_i$.
and $D_e$ contradicting (3.1). It follows that at least one of the following four cases certainly happens.

**Case 1** There exists a line $l$ passing through $O$ such that $l \cap D_e$ is included in $\Omega$. 
**Case 2** There exists a line $l$ passing through $O$ such that $l \cap \Omega$ is included in $D_i$. 
**Case 3** There exist three points $P_1$, $P_2$, and $P_3$ of the boundary of $\Omega$ such that, $P_1$ and $P_2$ also belong to $\partial D_e$, $P_3$ lies inside the acute angle having vertex in $O$ and bounded by the halflines passing through $P_1$ and $P_2$, and $\text{dist}(P_3, \partial D_i) = \min_{x \in \partial \Omega} \text{dist}(x, \partial D_i)$. 
**Case 4** There exist three points $P_1$, $P_2$, and $P_3$ of the boundary of $\Omega$ such that, $P_1$ and $P_2$ also belong to $\partial D_i$, $P_3$ lies inside the acute angle having vertex in $O$ and bounded by the halflines passing through $P_1$ and $P_2$, and moreover $\text{dist}(P_3, \partial D_e) = \min_{x \in \partial \Omega} \text{dist}(x, \partial D_e)$.

In Figure 1 and Figure 2 we represent two examples of convex sets for which **Case 3** or **Case 4** happens. 

We claim that any element of the family $\mathcal{Y}_\delta$ has the property of being the unique set (up to translations) with the smallest possible perimeter among all the sets having same measure and same Hausdorff asymmetry index. In particular we shall see that for any given convex set $\Omega$ it holds $P(\Omega) \geq P(Y_{\delta}(\Omega))$, where $Y_{\delta}(\Omega)$ belongs to the family $\mathcal{Y}_{|\Omega|}$. We shall provide the proof of such an assertion for each one of the four aforementioned cases.

3.1. **Case 1.** By hypotheses there exists a line $l$ passing through $O$ intersecting $\partial D_e$ in two points $P_1$ and $P_2$ which belong to $\partial \Omega$. We want to find a convex set having the same property, but also same $\delta$ and measure as $\Omega$, and the least possible perimeter. We can restrict our attention to those sets which have two orthogonal axes of symmetry: the line $l$, and the line intersecting $l$ in $O$. In fact,
assuming that $\Omega$ does not posses such a symmetry, we denote by $\Omega^*$ the Steiner symmetric of $\Omega$ with respect to theese two axis, and from well-known properties of the Steiner symmetrizzazione we know that $\Omega^*$ is convex, $|\Omega^*| = |\Omega|$, $\Omega^* \subseteq D_e$, $|\partial \Omega^*| < |\partial \Omega|$, and moreover $P_1$ and $P_2$ belong to $\partial \Omega^*$. Using well known isoperimetric properties of the circular arcs, it is easy to prove that the unique set with the least possible perimeter is the lens $Y_{\delta(\Omega)}$ belonging to the family $\mathcal{Y}_{\Omega}$. In view of Proposition 2.1 our claim is proved.

Remark 3.1. It is important to observe that the proof of Case 1 works in hypothesis weaker than convexity, for instance the starshapedness is enough.

3.2. Case 2. By hypotheses there exist two parallel lines $l_1$ and $l_2$ tangent to $D_i$ and such that the set $\Omega$ lies in the strip $S$ between these two lines and contains $D_i$. We want to find a set having the same property, which is also included in $D_e$, which has the same measure as $\Omega$, and such that it has the least possible perimeter. This set exists and it is unique [22] (possibly up to translations) and we shall denote it by $E_r$, where $r = \delta(\Omega)^2/|\Omega|$. We observe that there exists a feasible range of values of $r$, namely $[0, r_{\text{max}}]$, such that $E_r$ actually exists. Indeed, it is easy to prove that when $\delta$ is too big with respect to $|\Omega|$ then the set given by the intersection of the strip $S$ and the disk $D_e$ is too small to contain a set having measure $|\Omega|$.

Some important properties of the set $E_r$ can be found in [22]. In particular $E_r$ is either given by the convex hull of two disks both being translations of $D_i$, in case $r$ is large enough, $E_r$ is a convex set which includes the largest possible (in terms of measure) convex hull of two balls lying in the intersection between the strip $S$ and the disk $D_e$. In the first case our claim follows as a consequence of the following result, whose proof is postponed until the last section.

![Figure 2. A convex set for which Case 4 happens.](image-url)
Proposition 3.1. If $\Omega$ is the convex hull of two ball having the same radius then

$$P(\Omega) \geq P(Y_{\delta}(\Omega))$$

whenever $|\Omega| = |Y_{\delta}(\Omega)|$.

In the second case, since the largest possible convex hull is certainly tangent to $D_\varepsilon$, $E_\varepsilon$ will certainly contain a diameter of $D_\varepsilon$ and therefore $E_\varepsilon$ can be treated as in Case 1 (see Remark 3.1).

Remark 3.2. It is important to observe that the proof of Case 2 works even if $\Omega$ is not convex provided it is starshaped and lies between two parallel lines $l_1$ and $l_2$ both tangent to the inner disk $D_i$.

3.3. Case 3. By hypotheses there exists a parametrization of the radial function $\rho(\theta)$ such that $\rho(0) = \rho(\theta_1) = R + \delta(\Omega)$ for some $\theta_1 < \pi$ and moreover there exists $0 < \theta_2 < \theta_1$ such that $\rho(\theta_2) = \min_{\theta} \rho(\theta)$. We reshape the set $\Omega$ as follows: first of all we replace the restriction $\rho_1$ of the radial function $\rho$ to the domain $[0, \theta_1]$ with its symmetric increasing rearrangement see [17], namely $\rho_{1\#}(\theta - \theta_1/2)$; the same is done with the complementary part $\rho_2$, restriction of $\rho$ to the domain $[\theta_1, 2\pi]$, which we replace by $\rho_{2\#}(\theta - (\theta_1 + 2\pi)/2)$. The new radial function that we denote by $\hat{\rho}$ describes the boundary of a star shaped set having same measure as $\Omega$, but in view of the properties of rearrangements [6, 17, 23, 24], also a shorter perimeter.

We can assume that $\rho(\theta_1/2 + \pi) = \min_{\theta \in [\theta_1, 2\pi]} \hat{\rho} > \min_{\theta \in [0, \theta_1]} \hat{\rho} = \rho(\theta_1/2)$ otherwise, if $\rho(\theta_1/2) = \rho(\theta_1/2 + \pi)$ we replace the restriction of $\rho$ to the set $[\theta_1/2, \theta_1/2 + \pi]$ with its symmetric decreasing rearrangement, and the restriction of $\hat{\rho}$ to $[\theta_1/2 + \pi, \theta_1 + 2\pi]$ with its symmetric decreasing rearrangement [17]. The resulting function will describe the boundary of a starshaped set having a perimeter shorter then $P(\Omega)$ and containing a diameter of $D_\varepsilon$, and the proof will continue as in Case 1.

Assuming that $\rho(\theta_1/2) < \rho(\theta_1/2 + \pi)$ we have $0 = |\{\theta \in [\theta_1, 2\pi] : \hat{\rho} < \min \hat{\rho} + \sigma\}| < |\{\theta \in [0, \theta_1] : \rho < \min \rho + \sigma\}|$ for some $\sigma > 0$ small enough. On the other hand we can also assume that $|\{\theta \in [\theta_1, 2\pi] : \hat{\rho} < \max \rho\}| > |\{\theta \in [0, \theta_1] : \rho < \max \rho\}|$ otherwise the starshaped set described by $\hat{\rho}$ can be treated as in Case 1, see Remark 3.1. Therefore by continuity there exist $\min \rho < t < \max \rho$ such that

$$|\{\theta \in [\theta_1, 2\pi] : \hat{\rho} < t\}| \leq |\{\theta \in [0, \theta_1] : \rho < t\}|$$

and

$$|\{\theta \in [\theta_1, 2\pi] : \hat{\rho} \leq t\}| \geq |\{\theta \in [0, \theta_1] : \rho \leq t\}|.$$

As a consequence there exists $0 < \bar{\theta} < \theta_1/2$ such that

$$\hat{\rho}(\theta_1/2 - \bar{\theta}) = \hat{\rho}((\theta_1 - 2\pi)/2 + \bar{\theta}) = t.$$
and by simmetry
\[ \hat{\rho}(\theta/2 + \tilde{\theta}) = \hat{\rho}((\theta + 2\pi)/2 - \tilde{\theta}) = t \]

We consider now \( \hat{\rho}_1 \) and \( \hat{\rho}_2 \) restriction of \( \hat{\rho} \) to \([0, \theta_1] \) and \([\theta_1, 2\pi] \). We replace \( \hat{\rho}_1 \) with its symmetric decreasing rearrangement namely \( \hat{\rho}_1^*(\theta - (\theta_1 - \pi)/2) \), and \( \hat{\rho}_2 \) by \( \hat{\rho}_2^*(\theta - (\theta_1 + \pi)/2) \). We obtain in this way a radial function which describes the boundary of a starshaped set having shorter perimeter, which can be treated as in Case 1.

3.4. Case 4. By hypotheses there exists a parametrization of the radial function \( \rho(\theta) \) such that \( \rho(0) = \rho(\theta_1) = R - \delta(\Omega) \) for some \( \theta_1 < \pi \) and moreover there exists \( 0 < \theta_2 < \theta_1 \) such that \( \rho(\theta_2) = \max_{\theta \in [0, \theta_1]} \rho(\theta) \). We reshape the set \( \Omega \) as follows: first of all we replace the restriction \( \rho_1 \) of the radial function \( \rho \) to the domain \([0, \theta_1] \) with its symmetric decreasing rearrangement, namely \( \rho_1^*(\theta - \theta_1/2) \); the same is done with the complementary part \( \rho_2 \), restriction of \( \rho \) to the domain \([\theta_1, 2\pi] \), which we replace by \( \rho_2^*(\theta - (\theta_1 + \pi)/2) \). The new radial function that we denote by \( \check{\rho} \) describes the boundary of a star shaped set having same measure as \( \Omega \), but in view of the properties of the symmetric decreasing rearrangements, also a shorter perimeter.

We can assume that \( \rho(\theta_1/2 + \pi) = \max_{\theta \in [0, 2\pi]} \hat{\rho} < \max_{\theta \in [0, \theta_1]} \hat{\rho} = \rho(\theta_1/2) \) otherwise, if \( \rho(\theta_1/2) = \rho(\theta_1/2 + \pi) \) we replace the restriction of \( \rho \) to the set \([\theta_1/2, \theta_1/2 + \pi] \) with its symmetric increasing rearrangement, and the restriction of \( \hat{\rho} \) to \([\theta_1/2 + \pi, \theta_1 + 2\pi] \) with its symmetric increasing rearrangement. The resulting function \( \check{\rho} \) will describe the boundary of a starshaped set \( \check{\Omega} \), such that \( P(\check{\Omega}) \leq P(\Omega) \), and touching the boundary of \( \Omega \) in two points symmetric with respect to the origin \( O \). In this case the proof can continue as in Case 2, see Remark 3.2, provided that the set lies between two parallel lines tangent to \( \Omega \). The last condition can be obtained arguing as in [1], indeed the convexity of the set \( \Omega \) and the fact that \( \Omega \) implies that \( \rho \) is an absolutely continuous function which almost everywhere satisfies (see [26])

\[
(3.2) \quad |\rho'(\theta)| \leq \frac{\rho(\theta)}{R - \delta(\Omega)} \sqrt{\rho(\theta)^2 - (R - \delta(\Omega))^2} \quad \text{a.e. } \theta \in [0, 2\pi]
\]

By well known properties of rearrangements (see [18, 23, 24]) the function \( \check{\rho} \) is also an absolutely continuous function which satisfies (3.2) and since the same inequality holds with equality sign for the radial functions describing any line tangent to \( \Omega \) the set \( \check{\Omega} \) lies in a strip between two of such parallel lines.

Assuming that \( \rho(\theta_1/2) > \rho(\theta_1/2 + \pi) \) we have \( 0 = \{|\theta \in [\theta_1, 2\pi] : \hat{\rho} > \min \hat{\rho} - \sigma\} < \{|\theta \in [0, \theta_1] : \hat{\rho} > \min \hat{\rho} - \sigma\} \) for some \( \sigma > 0 \) small enough. On the other hand we can also assume that \( \{|\theta \in [\theta_1, 2\pi] : \hat{\rho} > \min \hat{\rho}\} > \{|\theta \in [0, \theta_1] : \hat{\rho} > \min \hat{\rho}\} \) otherwise the starshaped set described by \( \check{\rho} \) can be treated as in Case 2.

Therefore by continuity there exist \( \min \rho < t < \max \rho \) such that

\[
|\{\theta \in [0, 2\pi] : \hat{\rho} > t\}| \leq |\{\theta \in [0, \theta_1] : \hat{\rho} > t\}|
\]
and

\[ |\{ \theta \in [\theta_1, 2\pi] : \hat{\rho} \geq t \}| \geq |\{ \theta \in [0, \theta_1] : \hat{\rho} \geq t \}|. \]

As a consequence there exists \(0 < \bar{\theta} < \theta_1/2\) such that

\[ \hat{\rho}(\theta_1/2 - \bar{\theta}) = \hat{\rho}((\theta_1 - 2\pi)/2 + \bar{\theta}) = t \]

and by simmetry

\[ \hat{\rho}(\theta_1/2 + \bar{\theta}) = \hat{\rho}((\theta_1 + 2\pi)/2 - \bar{\theta}) = t \]

We consider now \(\hat{\rho}_1\) and \(\hat{\rho}_2\) restriction of \(\hat{\rho}\) to \([\theta_1 - 2\pi)/2 + \bar{\theta}, \theta_1/2 - \bar{\theta}]\) and \([\theta_1/2 + \bar{\theta}, (\theta_1 + 2\pi)/2 - \bar{\theta}]\). We replace \(\hat{\rho}_1\) with its symmetric increasing rearrangement namely \(\hat{\rho}_{1#}(\theta - (\theta_1 - \pi)/2)\), and \(\hat{\rho}_2\) by \(\hat{\rho}_{2#}(\theta - (\theta_1 + \pi)/2)\). We obtain in this way a radial function which describes the boundary of a starshaped set which, arguing as before, lies in a strip bounded by two parallel lines tangent to \(D_i\), and can be treated as in Case 2.

4. Two classes of convex sets

In this section we study the ratio

\[
\left( \frac{P(E)^2 - 4\pi|E|}{\delta(E)^2} \right),
\]

when the set \(E\) belongs to two different classes, namely, we will consider the classes of “stadia” and of “lenses”.

Class \(\mathcal{X}_a\) (stadia)

The class \(\mathcal{X}_a\) contains any convex set \(E\) satisfying the following properties:

- \(E\) has measure \(a\);
- \(E\) is the union of a rectangle with two half disks having the diameter which coincides with two opposite sides.

As we will see, any element in the class \(\mathcal{X}_a\) can be identified by its Hausdorff asymmetry index \(\delta\). Indeed, if we denote by \(a\) and \(b\) (see Figure 3) the measures

\[
\begin{align*}
\text{Figure 3. A set in } \mathcal{X}_a.
\end{align*}
\]
of the sides, being $b$ the diameter of the two half disks joined to the rectangle, it is very easy to compute perimeter, measure and Hausdorff asymmetry index of $E \in \mathcal{X}_\alpha$:

\begin{align}
(4.2) \quad P(E) &= 2a + \pi b, \\
(4.3) \quad |E| &= ab + \frac{\pi}{4}b^2, \\
(4.4) \quad \delta(E) &= \max \left\{ \sqrt{\frac{x}{\pi}} - \frac{b}{2}, \frac{a+b}{2} - \sqrt{\frac{x}{\pi}} \right\}.
\end{align}

Actually, equality (4.3) states a constraint from which it is possible to write $a$ in terms of $b$ and $\alpha$

\begin{align}
(4.5) \quad a &= \frac{\alpha}{b} - \frac{\pi}{4}b.
\end{align}

The above relation states that, for fixed $\alpha$, $a$ is a decreasing function of $b$ for $0 < b \leq 2\sqrt{\frac{2}{\pi}}$. Using (4.4) and (4.5), for fixed $\alpha$, $\delta(E)$ can be written as a decreasing function of $b$, for $0 < b \leq 2\sqrt{\frac{2}{\pi}}$,

\begin{align}
(4.6) \quad \delta(E) &= \begin{cases}
\sqrt{\frac{x}{\pi}} - \frac{b}{2} & \text{if } \frac{2\sqrt{\frac{\pi x}{8-\pi}}}{} \\
\frac{\alpha}{2b} + \left(\frac{1}{2} - \frac{\pi}{8}\right)b - \sqrt{\frac{x}{\pi}} & \text{if } 0 < b \leq \frac{2\sqrt{\frac{\pi x}{8-\pi}}}{8-\pi}.
\end{cases}
\end{align}

This means that we can parametrize the sets in $\mathcal{X}_\alpha$ in terms of the Hausdorff asymmetry index, that is, we will denote by $X_\delta$ the set such that $X_\delta \in \mathcal{X}_\alpha$ and $\delta(X_\delta) = \delta$.

Using (4.2), (4.5) and (4.6), the ratio (4.1) for the set $X_\delta$ can be calculated in terms of $\delta$:

\begin{align}
(4.7) \quad \frac{P(X_\delta)^2 - 4\pi|X_\delta|}{\delta^2} &= \begin{cases}
(\pi \frac{2\sqrt{\frac{\pi x}{8-\pi}} - \pi \delta}{\sqrt{\frac{\pi x}{8-\pi}}})^2 & \text{if } 0 \leq \delta \leq \frac{4 - \pi}{8 - \pi} \sqrt{\frac{x}{\pi}}, \\
(\frac{4\alpha - \pi g(\delta)}{2\delta g(\delta)})^2 & \text{if } \delta > \frac{4 - \pi}{8 - \pi} \sqrt{\frac{x}{\pi}},
\end{cases}
\end{align}

where, taking into account (4.6), $g(t)$ is the decreasing function, for $t > \frac{4 - \pi}{8 - \pi} \sqrt{\frac{x}{\pi}}$,

\begin{align}
(4.8) \quad g(t) &= \frac{4(\sqrt{\pi t} + \sqrt{x}) - 2\sqrt{4(\sqrt{\pi t} + \sqrt{x})^2 - \pi \alpha(4 - \pi)}}{\sqrt{\pi(4 - \pi)}}.
\end{align}
Using (4.7) and (4.8), it is easy to prove that, for a fixed $a$, one has

\[
\frac{P(X_d)^2 - 4 \pi a}{\delta^2} \quad \text{is increasing w.r.t. } \delta \text{ if } 0 \leq \delta \leq 2 \frac{4 - \pi}{8 - \pi} \sqrt{\frac{a}{\pi}},
\]

(4.9)

\[
\frac{P(X_d)^2 - 4 \pi a}{\delta^2} \quad \text{is decreasing w.r.t. } \delta \text{ if } \delta > 2 \frac{4 - \pi}{8 - \pi} \sqrt{\frac{a}{\pi}}.
\]

Furthermore, it is possible to evaluate the behaviour of the ratio (4.7) when $d$ goes to zero and when $d$ diverges, that is, when the stadium tends to a disk or to a line. We have:

\[
\lim_{\delta \to 0} \frac{P(X_d)^2 - 4 \pi a}{\delta^2} = 4 \pi^2,
\]

(4.10)

\[
\lim_{\delta \to +\infty} \frac{P(X_d)^2 - 4 \pi a}{\delta^2} = 16.
\]

**Class $\mathcal{Y}_a$ (lenses)**

The class $\mathcal{Y}_a$ contains any convex set $E$ satisfying the following properties:

- $E$ has measure $a$;
- $E$ is symmetric with respect to a straight line such that the part of it which stays on one side of the line coincides with a circular segment (the smallest part of a disk cut by a chord).

As we will see, any element in the class $\mathcal{Y}_a$ can be identified by its Hausdorff asymmetry index $\delta$. To show this fact we fix the reference axes $(x, y)$ in such a way that $x$-axis coincides with the above mentioned line of symmetry and we describe the set by using two parameters $r > 0$ and $\theta \in [0, \pi/2]$ (see Figure 4) which are the radius of the disk and half of the angle subtended by the chord. We have:

\[
E = \{(x, y) \in \mathbb{R}^2 : |x| \leq r \sin \theta, |y| \leq \sqrt{r^2 - x^2} - r \cos \theta\}.
\]

![Figure 4. Half of a set in $\mathcal{Y}_a$.](image)
It is very easy to compute perimeter, measure and Hausdorff asymmetry index of $E$:

\begin{align}
(4.11) \quad P(E) &= 4r\theta, \\
(4.12) \quad |E| &= 2r^2(\theta - \sin \theta \cos \theta), \\
(4.13) \quad \delta(E) &= \max\{r \sin \theta - \sqrt{x/\pi}, \sqrt{x/\pi} - r(1 - \cos \theta)\},
\end{align}

Actually, equality (4.12) states a constraint from which it is possible to write $r$ in terms of $\theta$ and $x$

\begin{equation}
(4.14) \quad r = \frac{x}{\sqrt{2(\theta - \sin \theta \cos \theta)}}.
\end{equation}

The above relation states that, for fixed $x$, $r$ is a decreasing function of $\theta$, for $0 < \theta \leq \pi/2$. Using (4.13) and (4.14), for fixed $x$, $\delta(E)$ can be written as a decreasing function of $\theta$, for $0 < \theta \leq \pi/2$. Indeed, it is possible to prove that, if $r$ is given by (4.14), then,

\begin{equation}
(4.15) \quad r \sin \theta - \sqrt{x/\pi} \geq \sqrt{x/\pi} - r(1 - \cos \theta), \quad 0 < \theta \leq \pi/2,
\end{equation}

that is,

\[
\sqrt{\frac{x}{2(\theta - \sin \theta \cos \theta)}}(1 + \sin \theta - \cos \theta) \geq 2 \sqrt{\frac{x}{\pi}}, \quad 0 < \theta \leq \pi/2.
\]

If we square the above inequality, it is equivalent to

\begin{equation}
(4.16) \quad \pi(1 + \sin \theta - \cos \theta) - 4\theta + (4 - \pi) \sin \theta \cos \theta \geq 0, \quad 0 < \theta \leq \pi/2.
\end{equation}

The above inequality can be proven computing the derivative of the function $h(\theta) = \pi(1 + \sin \theta - \cos \theta) - 4\theta + (4 - \pi) \sin \theta \cos \theta$ which appears on the left hand side of (4.16),

\[
h'(\theta) = \pi(\cos \theta + \sin \theta) - 4 + (4 - \pi)(\cos^2 \theta - \sin^2 \theta) = \frac{2\tan \frac{\theta}{2}}{(1 + \tan^2 \frac{\theta}{2})^2} \left(\pi + (3\pi - 16) \tan \frac{\theta}{2} + \pi \tan^2 \frac{\theta}{2} - \pi \tan^3 \frac{\theta}{2}\right).
\]

The observation that the polynomial $\pi + (3\pi - 16)t + \pi t^2 - \pi t^3$ has a negative derivative for $t \in [0, 1]$ gives the desired result (4.16). This means that (4.15) holds true and, taking into account (4.13) and (4.14), we can finally write $\delta(E)$ as a decreasing function of $\theta$:

\begin{equation}
(4.17) \quad \delta(E) = \sqrt{\frac{x \sin^2 \theta}{2(\theta - \sin \theta \cos \theta)}} - \sqrt{\frac{x}{\pi}}, \quad 0 < \theta \leq \pi/2.
\end{equation}
The above relation states that we can determine $\theta$ as a function of $\delta(E)$, that is,

$$\theta = f(\delta(E)), \quad 0 \leq \delta(E) < +\infty,$$

where $f(t)$ is the inverse function of the one given in (4.17), which applies the interval $[0, +\infty[ into ]0, \pi/2].$

This means that we can parametrize the sets in $\mathcal{Y}_\alpha$ in terms of the Hausdorff asymmetry index, that is, we will denote by $Y_\delta$ the set such that $Y_\delta \in \mathcal{Y}_\alpha$ and $\delta(Y_\delta) = \delta.$

For the set $Y_\delta$, using (4.11), (4.14) and (4.18), the ratio (4.1) can be calculated in terms of $\delta$:

$$\frac{P(Y_\delta)^2 - 4\pi|Y_\delta|}{\delta^2} = 4\pi \frac{2f(\delta)^2 - \pi(f(\delta) - \sin f(\delta) \cos f(\delta))}{\delta^2(f(\delta) - \sin f(\delta) \cos f(\delta))},$$

where $f(t)$ is the decreasing function defined in (4.18).

It is possible to prove that, for a fixed $\alpha$,

$$\frac{P(Y_\delta)^2 - 4\pi\alpha}{\delta^2} \quad \text{is decreasing w.r.t.} \quad \delta \quad \text{in } [0, +\infty[.$$

In order to prove the above statement we first put $\varphi = 2f(\delta)$ in (4.19) obtaining

$$\frac{P(Y_\delta)^2 - 4\pi\alpha}{\delta^2} = 8\pi \Phi(\varphi),$$

where

$$\Phi(\varphi) = \frac{\varphi^2 - \pi(\varphi - \sin \varphi)}{(\sqrt{\pi(1 - \cos \varphi)} - \sqrt{2(\varphi - \sin \varphi)})^2}, \quad \varphi \in ]0, \pi].$$

Then we show that $\Phi(\varphi)$ is an increasing function of $\varphi.$ Indeed, we have

$$\Phi'(\varphi) = \frac{\Phi_1(\varphi)\Phi_2(\varphi)}{\Phi_3(\varphi)\sqrt{\varphi - \sin \varphi}},$$

where

$$\Phi_1(\varphi) = 2\sqrt{1 - \cos \varphi} - \varphi \sqrt{1 + \cos \varphi},$$

$$\Phi_2(\varphi) = \varphi \sqrt{2(1 + \cos \varphi)} - (\pi - \varphi) \sqrt{\pi(\varphi - \sin \varphi)},$$

$$\Phi_3(\varphi) = (\sqrt{\pi(1 - \cos \varphi)} - \sqrt{2(\varphi - \sin \varphi)})^3.$$
As regards $\Phi_2(\varphi)$, we observe that

\begin{equation}
\Phi_2(\varphi) \geq \varphi \sqrt{2(1 + u(\varphi))} - (\pi - \varphi) \sqrt{\pi(\pi - v(\varphi))}, \quad \varphi \in [0, \pi],
\end{equation}

where

\begin{equation}
u(\varphi) = \begin{cases}
1 - \frac{\varphi^2}{2} & \text{if } 0 \leq \varphi \leq \frac{2}{5}\pi, \\
\frac{10}{\pi} \left(\frac{\pi}{2} - \varphi\right) \cos\left(\frac{2}{5}\pi\right) & \text{if } \frac{2}{5}\pi < \varphi \leq \frac{\pi}{2}, \\
\frac{\pi}{2} - \varphi & \text{if } \frac{\pi}{2} < \varphi \leq \frac{3}{5}\pi, \\
-1 + \frac{(\pi - \varphi)^2}{6} - \frac{(\pi - \varphi)^4}{24} & \text{if } \frac{3}{5}\pi < \varphi \leq \pi,
\end{cases}
\end{equation}

\begin{equation}v(\varphi) = \begin{cases}
\varphi - \frac{\varphi^3}{6} & \text{if } 0 \leq \varphi \leq \frac{2}{5}\pi, \\
1 + \frac{10}{\pi} \left(\varphi - \frac{\pi}{2}\right) \left(1 - \sin\left(\frac{2}{5}\pi\right)\right) & \text{if } \frac{2}{5}\pi < \varphi \leq \frac{\pi}{2}, \\
1 + \frac{10}{\pi} \left(\frac{\pi}{2} - \varphi\right) \left(1 - \sin\left(\frac{2}{5}\pi\right)\right) & \text{if } \frac{\pi}{2} < \varphi \leq \frac{3}{5}\pi, \\
\pi - \varphi - \frac{(\pi - \varphi)^3}{6} & \text{if } \frac{3}{5}\pi < \varphi \leq \pi.
\end{cases}
\end{equation}

Using (4.25) and (4.26) in (4.24) we have $\Phi_2(\varphi) \geq 0$ in $[0, \pi]$. This inequality, together with (4.17), (4.18), (4.21), (4.22) and (4.23), gives (4.20).

Furthermore, it is possible to evaluate the behaviour of the ratio (4.19) when $\delta$ goes to zero and when $\delta$ diverges, that is, when the lens tends to a disk or to a line. We have:

\begin{equation}
\lim_{\delta \to 0} \frac{P(X_\delta)^2 - 4\pi \alpha}{\delta^2} = \lim_{\varphi \to \pi} 8\pi \Phi(\varphi) = 4\pi^2,
\end{equation}

\begin{equation}
\lim_{\delta \to +\infty} \frac{P(X_\delta)^2 - 4\pi \alpha}{\delta^2} = \lim_{\varphi \to 0} 8\pi \Phi(\varphi) = 16.
\end{equation}

We conclude this section observing that properties (4.20), (4.28) proven above immediately imply Proposition 2.1. Moreover, a simple argument allows us to prove Proposition 3.1.

**Proof of Proposition 3.1.** We claim that for each $0 \leq \delta < +\infty$, if $X_\delta \in \mathcal{X}$ and $Y_\delta \in \mathcal{M}_x$, we have:

\begin{equation}
P(X_\delta) \geq P(Y_\delta).
\end{equation}
In view of (4.9), (4.10), (4.20) and (4.27) the assertion follows for

\[ 0 \leq \delta \leq \frac{4 - \pi}{8 - \pi} \sqrt{\frac{2}{\pi}}. \]

In the case

\[ \delta > \frac{4 - \pi}{8 - \pi} \sqrt{\frac{2}{\pi}}, \]

inequality (4.29) is a consequence of the fact that circular arcs minimize perimeter.

\[ \square \]

References


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