Calculus of Variations — On a Sobolev-type inequality, by Angelo Alvino.

Dedicated to the memory of Renato Caccioppoli

Abstract. — A new proof of the classical Sobolev inequality in \( \mathbb{R}^n \) with the best constant is given. The result follows from an intermediate inequality which connects in a sharp way the \( L^p \) norm of the gradient of a function \( u \) to \( L^{p^*} \) and \( L^{p^*} \)-weak norms of \( u \), where \( p \in ]1, n[ \) and \( p^* = \frac{np}{n-p} \) is the Sobolev exponent.

Key words: Sobolev inequality, Isoperimetric inequalities, one-dimensional Calculus of Variations.

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1. Introduction

The celebrated Sobolev inequality states that

\[
S(n, p) \| u \|_{L^{p^*}} \leq \| \nabla u \|_{L^p},
\]

where \( u \) is a sufficiently smooth function, defined in \( \mathbb{R}^n \), \( \nabla u \) is the gradient of \( u \), \( p \in ]1, n[ \), \( p^* = \frac{np}{n-p} \).

The optimal value of \( S(n, p) \) in (1) is

\[
\pi^{1/n} n^{1/p} (n - p)^{(p-1)/p} (p - 1)^{1/(n-(p-1)/p)} p^{-1/n} \left[ \frac{\Gamma\left(\frac{n}{p}\right) \Gamma\left(n - \frac{2}{p}\right)}{\Gamma(n) \Gamma\left(\frac{2}{p}\right)} \right]^{1/n}.
\]

This means that (2) is the infimum of the functional

\[
F(u) = \frac{\| \nabla u \|_{L^p}}{\| u \|_{L^{p^*}}};
\]

it is actually attained (see [1], [8] and, also, [2]) when

\[
u(x) = \frac{h}{\left[1 + k |x|^{(p-1)(n-p)/p}\right]},
\]

where \( h, k \) are positive constants.

The proof proceeds in two steps. The first one consists of a symmetrization procedure: \( u \) is replaced by its rearrangement \( u^\# \) which is a spherically
symmetric function and decreases with respect to $|x|$. Moreover $u, u^\#$ have the same distribution function, hence they have the same $L^p$ norm. On the other side, the $L^p$ norm of the gradient decreases as a consequence of the following Pólya Principle

\[ \int_{\mathbb{R}^n} |\nabla u^\#|^p \, dx \leq \int_{\mathbb{R}^n} |\nabla u|^p \, dx. \]  

(4)

In conclusion $F(u) \geq F(u^\#)$; so only radial functions compete in reaching the best constant in (1).

We stress the central role of (4) and recall that it follows from a combined use of the Hölder inequality and the classical isoperimetric inequality

\[ P(E) \geq n^{(n-1)/n} \omega_n^{1/n} \min\{|E|, |\mathbb{R}^n \setminus E|\}^{(n-1)/n}; \]

here $|E|$ is the Lebesgue measure of a Caccioppoli set $E$, $P(E)$ is the perimeter of $E$ in the sense of De Giorgi [5],

\[ \omega_n = \frac{n \pi^{n/2}}{\Gamma(1 + \frac{n}{2})} \]

is the measure of the unitary $(n - 1)$-dimensional sphere.

The problem thus becomes a classical question of one-dimensional Calculus of Variation with constraints. It can be dealt with turning it into a Lagrange Problem whose extremals are available. These form a Mayer field; introducing the Weierstrass excess function leads to the result.

As for the second step our proof appeals to simpler tools for free functionals of the Calculus of Variations. A more general Sobolev-type inequality, involving the norm of $u$ in a Marcinkiewicz space, is established. The classical Sobolev inequality (1), with the optimal value (2) of the constant, easily follows.

2. Main result

Let $a > 0$ and consider the following one-parameter family of extremals (3)

\[ u_e(x) = u_e(|x|) = \frac{e^{(n-p)/p}}{[1 + (ae|x|)^{p/(p-1)}]^{(n-p)/p}}. \]

(5)

These functions have the same $L^p^\ast$ norm

\[ \|u_e\|_{L^p^\ast} = a^{-n} 2\pi^{n/2} \left( \frac{p - 1}{p} \right) \frac{\Gamma(\frac{n}{p}) \Gamma(n - \frac{n}{p})}{\Gamma(\frac{2}{p}) \Gamma(n)}. \]

Moreover they all solve the nonlinear partial differential equation

\[ -\Delta_p u_e = n \left( \frac{n - p}{p - 1} \right)^{p-1} a^p u_e^{p^\ast - 1}, \]
which is the Euler-Lagrange equation of the functional
\[
J(u) = \frac{1}{p} \int_{\mathbb{R}^n} |\nabla u|^p \, dx - \frac{1}{p} \frac{(n-p)^p}{(p-1)^{p-1}} a^p \int_{\mathbb{R}^n} |u|^p \, dx,
\]
or
\[
(6) \quad J(u) = \frac{\omega_n}{p} \int_0^\infty |u'|^p r^{n-1} \, dr - \frac{\omega_n}{p} \frac{(n-p)^p}{(p-1)^{p-1}} a^p \int_0^\infty |u|^p r^{n-1} \, dr
\]
if \( u \) is a radial function.

The curve
\[
(7) \quad y = \frac{(p-1)^{(n-p)(p-1)/p^2}}{p^{(n-p)/p}} (ar)^{-(n-p)/p} = \gamma_a(r), \quad r > 0,
\]
envelopes the graphs \( y = u_\varepsilon(r) \); these cover the region of the first quadrant which lies below the curve (7) and will be called \( T \).

If \( v \) is a non negative, sufficiently smooth, compactly supported, radial function let
\[
\|v\|_{p^*, \infty} = \sup_{r > 0} [r^{n/p^*} v(r)]
\]
be its norm in the Marcinkiewicz space of the functions weakly \( L^{p^*} \). If we choose
\[
(8) \quad a = \frac{(p-1)^{(p-1)/p}}{p} \frac{1}{\|v\|_{p^*(n-p), \infty}^{p/(n-p)}},
\]
the minimum value such that \( v(r) \leq \gamma_a(r) \), for all \( r \) positive, the envelope (7) becomes
\[
y = \|v\|_{p^*, \infty} r^{-(n-p)/p} = \gamma(r).
\]

Each graph \( y = u_\varepsilon(r) \) touches the envelope at a point which splits it into two curves \( C_1(\varepsilon), C_2(\varepsilon) \). These two families of curves are the trajectories of two different fields of extremals of the functional (6), and both defined in the same set \( T \).

We denote by \( (1, q_1(r, y)) \) the former and by \( (1, q_2(r, y)) \) the latter. As usual, \( q_1(r, y) \) is the slope of the extremal of the first family passing through \((r, y)\); \( q_2(r, y) \) has an analogous meaning. The envelope also touches the graph of \( v \) at least in a point \( P = (x, \gamma(x)) \) which splits it into two arcs \( \Gamma_1, \Gamma_2 \). Moreover, we simply denote by \( C_1, C_2 \), respectively, the arcs of the families \( C_1(\varepsilon), C_2(\varepsilon) \) passing through \( P \).

In Figure 1 (2) the graphs of the envelope \( y = \gamma(r), \Gamma_1 (\Gamma_2), C_1 (C_2) \) are sketched, together with some further arcs of extremals.
Setting

\[ f(r, v, v') = \frac{\omega_p}{p} r^{n-1} \left[ |v'|^p - \frac{(n - p)^p}{(p - 1)^{p-1}} q^p |v|^p \right] \]

gives

\[ J(v) = \int_0^\alpha f(r, v, v') \, dr + \int_{\alpha}^{\infty} f(r, v, v') \, dr = J_1(v) + J_2(v). \]

We begin by estimating \( J_1(v) \) from below; to this aim we refer to the first field of extremals.

Since \( f \) is convex with respect to the last variable, we get

\[ \mathcal{E}(r, v, v', q_1) = f(r, v, v') - f(r, v, q_1) - (v' - q_1) f_{v'}(r, v, q_1) \geq 0, \]

where \( \mathcal{E} \) is the well-known Weierstrass excess function. Therefore

\[ J_1(v) \geq \int_0^\alpha \left[ f(r, v, q_1) + (v' - q_1) f_{v'}(r, v, q_1) \right] \, dr. \]
Since the 1-form

\[ \frac{1}{2} f(r, v, q_1) \left[ q_1 f_v'(r, v, q_1) - q_1 f_{u_v}(r, v, q_1) \right] \, dr + f_{u_v}(r, v, q_1) \, dv \]

is exact, the integral on the right-hand side of (9) equals the line integral of (10) along a segment of the vertical axis, which is null, plus the integral line along the curve \( C_1 \) (see Figure 1). The latter is

\[ J_1(u_v) = \int_0^x f(r, u_v, u_v') \, dr. \]

Thus, we have

\[ J_1(v) \geq J_1(u_v). \]

A similar procedure applies to \( J_2(v) \). We integrate the exact 1-form

\[ \frac{1}{2} f(r, v, q_2) \left[ q_2 f_v'(r, v, q_2) - q_2 f_{u_v}(r, v, q_2) \right] \, dr + f_{u_v}(r, v, q_2) \, dv \]

along the closed path delineated in Figure 2. A simple asymptotic argument allows us to claim that the line integral of (12) along the vertical segment \( S_\beta \) is infinitesimal when \( \beta \) goes to infinity. Therefore

\[ J_2(v) = \int_x^\infty f(r, v, v') \, dr \geq J_2(u_v) = \int_x^\infty f(r, u_v, u_v') \, dr. \]
Collecting (11) and (13) gives \( J(v) \geq J(u_e) \). Hence, computing \( J(u_e) \) leads to

\[
\int_{\mathbb{R}^n} |\nabla v|^p \, dx \geq \alpha_p \left[ \frac{(n-p)^p}{(p-1)^{p-1}} \|v\|_{p^*}^p + \alpha^{p-n} 2\pi^{n/2} \frac{(n-p)^{p-1}}{(p-1)^{p-2}} \Gamma \left( \frac{n}{p} \right) \Gamma \left( \frac{n}{p} - \frac{n}{2} \right) \Gamma \left( \frac{n}{2} \right) \right].
\]

If we recall the value (8) of \( \alpha \), by density arguments, we have the following result.

**Theorem 2.1.** If \( v \) belongs to the Sobolev space \( W^{1,p}(\mathbb{R}^n) \) and \( p \in ]1,n[ \), then

\[
\|v\|_{p^*,\infty}^p |\nabla v|^p \geq A(n,p)\|v\|_{p^*}^p + B(n,p)\|v\|_{p^*,\infty}^p,
\]

where

\[
A(n,p) = \left( \frac{n-p}{p} \right)^p
\]

and

\[
B(n,p) = 2\pi^{n/2} \frac{(n-p)^{p-1}}{(p-1)^{n-1-p}} \frac{\Gamma \left( \frac{n}{p} \right) \Gamma \left( \frac{n}{p} - \frac{n}{2} \right) \Gamma \left( \frac{n}{2} \right)}{\Gamma(n)\Gamma \left( \frac{n}{2} \right)}.
\]

**Remark 2.1.** Handling with a sole extremal field leads to trivial outcomes. Namely it is not possible to assemble the graphs of \( v \) and of an extremal, and make a closed path along which calculate the integral of an exact 1-form as above. This becomes possible if one thinks of the extremal fields as a unique field defined on a surface, a sort of cylinder, squashed onto \( T \). In some sense we deal with a sheet with two pages: when an extremal touches the envelope it passes from one page to another. Therefore, the extremals can be viewed as closed paths which describe a complete ring. The same happens to the graph of \( v \) when it touches the envelope. In some sense the graphs of \( v \) and of each extremal are in the same homotopy class.

**Remark 2.2.** Recently the problem of the optimality of the Sobolev constant has been tackled by different tools (see [4]). Instead of a symmetrization procedure and the Pólya inequality (4), mass transport methods and a subtle result by Brenier [3] are used. Both methods have deep, but different, geometric flavours.

### 3. The Sobolev Inequality

Inequality (14) can be viewed as a generalization of the Sobolev inequality. Namely (1) can be deduced from (14) dividing by \( \|v\|_{p^*,\infty}^p \) and minimizing the right-hand side with respect to \( \|v\|_{p^*,\infty} \).

We can also argue in a different way. For instance, if \( p = 2 \) and \( n = 3 \), (14) becomes

\[
|\nabla v|^2 \geq \frac{1}{4} \frac{\|v\|^6}{\|v\|_{6,\infty}^4} + \pi^2 \|v\|_{6,\infty}^2.
\]
By Young inequality we get
\[
\|\nabla v\|_2^2 \geq 3 \left( \frac{\pi^2 - \sigma^2}{4} \right)^{2/3} \|v\|_6^2 + \sigma^2 \|v\|_{6, \infty}^2
\]
for any $\sigma \in [0, \pi]$. If $\sigma = 0$ we obtain the Sobolev inequality, whereas, if $\sigma = \pi$, we have
\[
\|\nabla v\|_2 \geq \pi \|v\|_{6, \infty}.
\]
However the value of the constant in (16) is not sharp, as the following result shows.

**Theorem 3.1.** Let $u \in W^{1,2}(\mathbb{R}^n)$. Then
\[
(n - 2)\omega_n \|u\|_{2n/(n-2), \infty}^2 \leq \|\nabla u\|_{L^2}^2.
\]
It is obviously sufficient to deal with spherically decreasing and spherically symmetric functions. For the sake of simplicity we assume
\[
\sup_{r > 0} (r^{(n-2)/2} u(r)) = r_0^{(n-2)/2} u(r_0) = 1
\]
for a suitable $r_0 > 0$. Among all functions satisfying (18) the one with the lowest energy is
\[
w(r) = \begin{cases} 
  r_0^{-(n-2)/2} & \text{if } r \leq r_0 \\
  r_0^{(n-2)/2} r^{2-n} & \text{if } r > r_0
\end{cases}
\]
The energy of $w$ is $(n - 2)\omega_n$, then we get (17). Moreover the constant is sharp.

**Remark 3.1.** As for (15), if $S < 3(\pi^2/4)^{2/3}$, one could ask for the best constant $C(S)$ such that
\[
\|\nabla v\|_2^2 \geq S\|v\|_6^2 + C(S)\|v\|_{6, \infty}^2.
\]
Analogous question can be set when we remove any restriction on $p$ and $n$.

**References**


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