Calculus of Variations — $W^{1,1}_0$ minima of noncoercive functionals, by Lucio Boccardo, Gisella Croce and Luigi Orsina, presented by Nicola Fusco on 16 June 2011.

Abstract. — We study an integral non coercive functional defined on $H^1_0(\Omega)$, proving the existence of a minimum in $W^{1,1}_0(\Omega)$.

Key words: Integral functionals, direct methods of the calculus of variations, coercivity, lower order terms.

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In this paper we study a class of integral functionals defined on $H^1_0(\Omega)$, but non coercive on the same space, so that the standard approach of the Calculus of Variations does not work. However, the functionals are coercive on $W^{1,1}_0(\Omega)$ and we will prove the existence of minima, despite the non reflexivity of $W^{1,1}_0(\Omega)$, which implies that, in general, the Direct Methods fail due to lack of compactness.

Let $J$ be the functional defined as

$$J(v) = \int_{\Omega} \frac{j(x, \nabla v)}{1 + b(x)|v|}^2 + \frac{1}{2} \int_{\Omega} |v|^2 - \int_{\Omega} f v, \quad v \in H^1_0(\Omega).$$

We assume that $\Omega$ is a bounded open set of $\mathbb{R}^N$, $N > 2$, that $j : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is such that $j(\cdot, \xi)$ is measurable on $\Omega$ for every $\xi$ in $\mathbb{R}^N$, $j(x, \cdot)$ is convex and belongs to $C^1(\mathbb{R}^N)$ for almost every $x$ in $\Omega$, and

(1) \quad \alpha|\xi|^2 \leq j(x, \xi) \leq \beta|\xi|^2,

(2) \quad |j_\xi(x, \xi)| \leq \gamma|\xi|,

for some positive $\alpha$, $\beta$ and $\gamma$, for almost every $x$ in $\Omega$, and for every $\xi$ in $\mathbb{R}^N$. We assume that $b$ is a measurable function on $\Omega$ such that

(3) \quad 0 \leq b(x) \leq B, \quad \text{for almost every } x \text{ in } \Omega,

where $B > 0$, while $f$ belongs to some Lebesgue space. For $k > 0$ and $s \in \mathbb{R}$, we define the truncature function as $T_k(s) = \max(-k, \min(s, k))$. 
In [3] the minimization in $H^1_0(\Omega)$ of the functional

$$I(v) = \int_{\Omega} \frac{j(x, \nabla v)}{\Omega} - \int_{\Omega} f v, \quad 0 < \theta < 1, \ f \in L^m(\Omega),$$

was studied. It was proved that $I(v)$ is coercive on the Sobolev space $W^{1,q}_0(\Omega)$, for some $q = q(\theta, m)$ in $(1, 2)$, and that $I(v)$ achieves its minimum on $W^{1,q}_0(\Omega)$. This approach does not work for $\theta > 1$ (see Remark 7 below). Here we will able to overcome this difficulty thanks to the presence of the lower order term $\int_{\Omega} |v|^2$, which will yield the coercivity of $J$ on $W^{1,1}_0(\Omega)$; then we will prove the existence of minima in $W^{1,1}_0(\Omega)$, even if it is a non reflexive space.

Integral functionals like $J$ or $I$ are studied in [1], in the context of the Thomas–Fermi–von Weizsäcker theory.

We are going to prove the following result.

**Theorem 1.** Let $f \in L^2(\Omega)$. Then there exists $u$ in $W^{1,1}_0(\Omega) \cap L^2(\Omega)$ minimum of $J$, that is,

$$\int_{\Omega} \frac{j(x, \nabla u)}{\Omega} + \frac{1}{2} \int_{\Omega} |u|^2 - \int_{\Omega} f u \leq \int_{\Omega} \frac{j(x, \nabla v)}{\Omega} + \frac{1}{2} \int_{\Omega} |v|^2 - \int_{\Omega} f v,$$

for every $v$ in $H^1_0(\Omega)$. Moreover $T_k(u)$ belongs to $H^1_0(\Omega)$ for every $k > 0$.

In [2] we studied the following elliptic boundary problem:

$$\begin{cases}
-\text{div} \left( \frac{a(x)\nabla u}{(1 + b(x)|u|)^2} \right) + u = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}$$

under the same assumptions on $\Omega$, $b$ and $f$, with $0 < \alpha \leq a(x) \leq \beta$. It is easy to see that the Euler equation of $J$, with $j(x, \xi) = \frac{1}{2} a(x)|\xi|^2$, is not equation (5). Therefore Theorem 1 cannot be deduced from [2]. Nevertheless some technical steps of the two papers (for example, the a priori estimates) are similar.

We will prove Theorem 1 by approximation. Therefore, we begin with the case of bounded data.

**Lemma 2.** If $g$ belongs to $L^\infty(\Omega)$, then there exists a minimum $w$ belonging to $H^1_0(\Omega) \cap L^\infty(\Omega)$ of the functional

$$v \in H^1_0(\Omega) \mapsto \int_{\Omega} \frac{j(x, \nabla v)}{\Omega} + \frac{1}{2} \int_{\Omega} |v|^2 - \int_{\Omega} g v.$$ 

**Proof.** Since the functional is not coercive on $H^1_0(\Omega)$, we cannot directly apply the standard techniques of the Calculus of Variations. Therefore, we
begin by approximating it. Let $M > 0$, and let $J_M$ be the functional defined as

$$J_M(v) = \int_{\Omega} \frac{j(x, \nabla v)}{[1 + b(x)|T_M(v)|]} + \frac{1}{2} \int_{\Omega} |v|^2 - \int_{\Omega} gv, \quad v \in H^1_0(\Omega).$$

Since $J_M$ is both weakly lower semicontinuous (due to the convexity of $j$ and to De Giorgi's theorem, see [4]) and coercive on $H^1_0(\Omega)$, for every $M > 0$ there exists a minimum $w_M$ of $J_M$ on $H^1_0(\Omega)$. Let $A = \|g\|_{L^\infty(\Omega)}$, let $M > A$, and consider the inequality $J_M(w_M) \leq J_M(T_A(w_M))$, which holds true since $w_M$ is a minimum of $J_M$. We have

$$\int_{\Omega} \frac{j(x, \nabla w_M)}{[1 + b(x)|T_M(w_M)|]} + \frac{1}{2} \int_{\Omega} |w_M|^2 - \int_{\Omega} gw_M$$

$$\leq \int_{\Omega} \frac{j(x, \nabla T_A(w_M))}{[1 + b(x)|T_M(T_A(w_M))|]} + \frac{1}{2} \int_{\Omega} |T_A(w_M)|^2 - \int_{\Omega} gT_A(w_M)$$

$$= \int_{\{|w_M| \leq A\}} \frac{j(x, \nabla w_M)}{[1 + b(x)|T_M(w_M)|]} + \frac{1}{2} \int_{\Omega} |T_A(w_M)|^2 - \int_{\Omega} gT_A(w_M),$$

where, in the last passage, we have used that $T_M(T_A(w_M)) = T_M(w_M)$ on the set $\{|w_M| \leq A\}$, and that $j(x, 0) = 0$. Simplifying equal terms, we thus get

$$\int_{\{|w_M| \geq M\}} \frac{j(x, \nabla w_M)}{[1 + b(x)|T_M(w_M)|]} + \frac{1}{2} \int_{\Omega} [|w_M|^2 - |T_A(w_M)|^2] \leq \int_{\Omega} g[w_M - T_A(w_M)].$$

Dropping the first term, which is nonnegative, we obtain

$$\frac{1}{2} \int_{\Omega} [w_M - T_A(w_M)][w_M + T_A(w_M)] \leq \int_{\Omega} g[w_M - T_A(w_M)],$$

which can be rewritten as

$$\frac{1}{2} \int_{\Omega} [w_M - T_A(w_M)][w_M + T_A(w_M) - 2g] \leq 0.$$
Since $|g| \leq A$, we have $A - 2g \geq -A$, and $-A - 2g < A$, so that
\[
0 \leq \frac{1}{2} \int_{\{w_M > A\}} [w_M - A]^2 + \frac{1}{2} \int_{\{w_M < -A\}} [w_M + A]^2 \leq 0,
\]
which then implies that $\text{meas}(\{|w_M| \geq A\}) = 0$, and so $|w_M| \leq A$ almost everywhere in $\Omega$. Recalling the definition of $A$, we thus have
\[
\|w_M\|_{L^\infty(\Omega)} \leq \|g\|_{L^\infty(\Omega)}.
\]
Since $M > \|g\|_{L^\infty(\Omega)}$, we thus have $T_M(w_M) = w_M$. Starting now from $J_M(w_M) \leq J_M(0) = 0$ we obtain, by (6),
\[
\frac{\int j(x, \nabla w_M)}{\int [1 + b(x) |w_M|^2]} + \frac{1}{2} \int |w_M|^2 \leq \int g w_M \leq \text{meas}(\Omega) \|g\|_{L^\infty(\Omega)}^2;
\]
which then implies, by (1) and (3), and dropping the nonnegative second term,
\[
\frac{x}{[1 + B \|g\|_{L^\infty(\Omega)}^2]} \int \|\nabla w_M\|^2 \leq \text{meas}(\Omega) \|g\|_{L^\infty(\Omega)}^2.
\]
Thus, $\{w_M\}$ is bounded in $H^1_0(\Omega) \cap L^\infty(\Omega)$, and so, up to subsequences, it converges to some function $w$ in $H^1_0(\Omega) \cap L^\infty(\Omega)$ weakly in $H^1_0(\Omega)$, strongly in $L^2(\Omega)$, and almost everywhere in $\Omega$. We prove now that
\[
\int \frac{j(x, \nabla w)}{[1 + b(x) |w|^2]} \leq \liminf_{M \to +\infty} \int \frac{j(x, \nabla w_M)}{[1 + b(x) |w_M|^2]}.
\]
Indeed, since $j$ is convex, we have
\[
\int \frac{j(x, \nabla w_M)}{[1 + b(x) |w_M|^2]} \geq \int \frac{j(x, \nabla w)}{[1 + b(x) |w|^2]} - \int \frac{j(x, \nabla w)}{[1 + b(x) |w_M|^2]} \cdot \nabla [w_M - w].
\]
Using assumption (1), the fact that $w$ belongs to $H^1_0(\Omega)$, the almost everywhere convergence of $w_M$ to $w$ and Lebesgue’s theorem, we have
\[
\lim_{M \to +\infty} \int \frac{j(x, \nabla w)}{[1 + b(x) |w_M|^2]} = \int \frac{j(x, \nabla w)}{[1 + b(x) |w|^2]}.
\]
Using assumption (2), the fact that $w$ belongs to $H^1_0(\Omega)$, and the almost everywhere convergence of $w_M$ to $w$, we have by Lebesgue’s theorem that
\[
\lim_{M \to +\infty} \frac{j_\xi(x, \nabla w)}{[1 + b(x) |w_M|^2]} = \frac{j_\xi(x, \nabla w)}{[1 + b(x) |w|^2]}, \quad \text{strongly in } (L^2(\Omega))^N.
\]
Since $\nabla w_M$ tends to $\nabla w$ weakly in the same space, we thus have that

$$\lim_{M \to +\infty} \int_{\Omega} \frac{j(x, \nabla w)}{[1 + b(x)|w_M|^2]} \cdot \nabla [w_M - w] = 0.$$  \tag{9}$$

Using (8) and (9), we have that (7) holds true. On the other hand, using (1) and Lebesgue’s theorem again, it is easy to see that

$$\lim_{M \to +\infty} \int_{\Omega} \frac{j(x, \nabla v)}{[1 + b(x)|T_M(v)|^2]} = \int_{\Omega} \frac{j(x, \nabla v)}{[1 + b(x)|v|^2]}, \quad \forall v \in H^1_0(\Omega).$$

Thus, starting from $J_M(w_M) \leq J_M(v)$, we can pass to the limit as $M$ tends to infinity (using also the strong convergence of $w_M$ to $w$ in $L^2(\Omega)$), to have that $w$ is a minimum.

As stated before, we prove Theorem 1 by approximation. More in detail, if $f_n = T_n(f)$ then Lemma 2 with $g = f_n$ implies that there exists a minimum $u_n$ in $H^1_0(\Omega) \cap L^\infty(\Omega)$ of the functional

$$J_n(v) = \int_{\Omega} \frac{j(x, \nabla v)}{[1 + b(x)|v|^2]} + \frac{1}{2} \int_{\Omega} |v|^2 - \int_{\Omega} f_n v, \quad v \in H^1_0(\Omega).$$

In the following lemma we prove some uniform estimates on $u_n$.

**Lemma 3.** Let $u_n$ in $H^1_0(\Omega) \cap L^\infty(\Omega)$ be a minimum of $J_n$. Then

1. $$\int_{\Omega} \frac{|
abla u_n|^2}{(1 + b(x)|u_n|)^2} \leq \frac{1}{2\alpha} \int_{\Omega} |f|^2;$$  \tag{10}
2. $$\int_{\Omega} |
abla T_k(u_n)|^2 \leq \frac{(1 + Bk)^2}{2\alpha} \int_{\Omega} |f|^2;$$  \tag{11}
3. $$\int_{\Omega} |u_n|^2 \leq 4 \int_{\Omega} |f|^2;$$  \tag{12}
4. $$\int_{\Omega} |
abla u_n| \leq \left[ \frac{1}{2\alpha} \int_{\Omega} |f|^2 \right]^{1/2} \left( \text{meas}(\Omega)^{1/2} + 2B \left[ \int_{\Omega} |f|^2 \right]^{1/2} \right);$$  \tag{13}
5. $$\int_{\Omega} |G_k(u_n)|^2 \leq 4 \int_{\{ |u_n| \geq k \}} |f|^2,$$  \tag{14}

where $G_k(s) = s - T_k(s)$ for $k \geq 0$ and $s$ in $\mathbb{R}$.

**Proof.** The minimality of $u_n$ implies that $J_n(u_n) \leq J_n(0)$, that is,

$$\int_{\Omega} \frac{j(x, \nabla u_n)}{[1 + b(x)|u_n|]} \cdot \nabla [w_M - w] \leq \int_{\Omega} f_n u_n.$$  \tag{15}
Using (1) on the left hand side, and Young’s inequality on the right hand side gives
\[
\alpha \int_\Omega \frac{ |\nabla u_n|^2}{[1 + b(x)|u_n|]^2} + \frac{1}{2} \int_\Omega u_n^2 \leq \frac{1}{2} \int_\Omega u_n^2 + \frac{1}{2} \int_\Omega f_n^2, \]
which then implies (10). Let now \( k \geq 0 \). The above estimate, and (3), give
\[
\frac{1}{(1 + Bk)^2} \int_\Omega |\nabla T_k(u_n)|^2 \leq \int_{\{|u_n| \leq k\}} \frac{ |\nabla u_n|^2}{[1 + b(x)|u_n|]^2} \leq \frac{1}{2 \alpha} \int_\Omega |f|^2, \]
and therefore (11) is proved. On the other hand, dropping the first positive term in (15) and using Hölder’s inequality on the right hand side, we have
\[
\frac{1}{2} \int_\Omega |u_n|^2 \leq \int_\Omega |f_n u_n| \leq \left[ \int_\Omega |f_n|^2 \right]^{1/2} \left[ \int_\Omega |u_n|^2 \right]^{1/2},
\]
that is, (12) holds. Hölder’s inequality, assumption (3), and estimates (10) and (12) give (13):
\[
\int_\Omega |\nabla u_n| \leq \left[ \int_\Omega \frac{ |\nabla u_n|^2}{[1 + b(x)|u_n|]^2} \right]^{1/2} \left[ \int_\Omega [1 + b(x)|u_n|]^2 \right]^{1/2}
\leq \left[ \frac{1}{2 \alpha} \int_\Omega |f|^2 \right]^{1/2} \left( \text{meas}(\Omega)^{1/2} + 2B \left[ \int_\Omega |f|^2 \right]^{1/2} \right).
\]
We are left with estimate (14). Since \( J_n(u_n) \leq J_n(T_k(u_n)) \) we have
\[
\frac{1}{2} \int_\Omega \frac{j(x, \nabla u_n)}{[1 + b(x)|u_n|]^2} + \frac{1}{2} \int_\Omega |u_n|^2 - \int_\Omega f_n u_n 
\leq \frac{1}{2} \int_\Omega \frac{j(x, \nabla T_k(u_n))}{[1 + b(x)|T_k(u_n)|]^2} + \frac{1}{2} \int_\Omega |T_k(u_n)|^2 - \int_\Omega f_n T_k(u_n).
\]
Recalling the definition of \( G_k(s) \), and using that \(|s|^2 - |T_k(s)|^2 \geq |G_k(s)|^2 \), the last inequality implies
\[
\frac{1}{2} \int_\Omega \frac{j(x, \nabla G_k(u_n))}{[1 + b(x)|u_n|]^2} + \frac{1}{2} \int_\Omega |G_k(u_n)|^2 \leq \int_\Omega f_n G_k(u_n).
\]
Dropping the first term of the left hand side and using Hölder’s inequality on the right one, we obtain
\[
\frac{1}{2} \int_\Omega |G_k(u_n)|^2 \leq \left[ \int_{\{|u_n| \geq k\}} |f|^2 \right]^{1/2} \left[ \int_\Omega |G_k(u_n)|^2 \right]^{1/2},
\]
that is, (14) holds. \( \square \)
Let $u_n$ in $H^1_0(\Omega) \cap L^\infty(\Omega)$ be a minimum of $J_n$. Then there exists a subsequence, still denoted by $\{u_n\}$, and a function $u$ in $W^{1,1}_0(\Omega) \cap L^2(\Omega)$, with $T_k(u)$ in $H^1_0(\Omega)$ for every $k > 0$, such that $u_n$ converges to $u$ almost everywhere in $\Omega$, strongly in $L^2(\Omega)$ and weakly in $W^{1,1}_0(\Omega)$, and $T_k(u_n)$ converges to $T_k(u)$ weakly in $H^1_0(\Omega)$. Moreover,

$$\lim_{n \to +\infty} \nabla u_n \frac{\nabla u_n}{1 + b(x)|u_n|} = \nabla u \frac{\nabla u}{1 + b(x)|u|} \text{ weakly in } (L^2(\Omega))^N.$$  

PROOF. By (13), the sequence $u_n$ is bounded in $W^{1,1}_0(\Omega)$. Therefore, it is relatively compact in $L^1(\Omega)$. Hence, up to subsequences still denoted by $u_n$, there exists $u$ in $L^1(\Omega)$ such that $u_n$ almost everywhere converges to $u$. From Fatou’s lemma applied to (12) we then deduce that $u$ belongs to $L^2(\Omega)$.

We are going to prove that $u_n$ strongly converges to $u$ in $L^2(\Omega)$. Let $E$ be a measurable subset of $\Omega$; then by (14) we have

$$\int_E |u_n|^2 \leq 2 \int_E |T_k(u_n)|^2 + 2 \int_E |G_k(u_n)|^2 \leq 2k^2 \text{meas}(E) + 2 \int_\Omega |G_k(u_n)|^2 \leq 2k^2 \text{meas}(E) + 8 \int_{\{|u_n| \geq k\}} |f|^2.$$  

Since $u_n$ is bounded in $L^2(\Omega)$ by (12), we can choose $k$ large enough so that the second integral is small, uniformly with respect to $n$; once $k$ is chosen, we can choose the measure of $E$ small enough such that the first term is small. Thus, the sequence $\{u_n^2\}$ is equiintegrable and so, by Vitali’s theorem, $u_n$ strongly converges to $u$ in $L^2(\Omega)$.

Now we to prove that $u_n$ weakly converges to $u$ in $W^{1,1}_0(\Omega)$. Let $E$ be a measurable subset of $\Omega$. By Hölder’s inequality, assumption (3), and (10), one has, for $i \in \{1, \ldots, N\}$,

$$\int_E \left| \frac{\partial u_n}{\partial x_i} \right| \leq \int_E |\nabla u_n| \leq \left[ \int_E \frac{|\nabla u_n|^2}{1 + b(x)|u_n|} \right]^{1/2} \left[ \int_E \frac{|f|^2}{1 + b(x)|u_n|} \right]^{1/2} \leq \left[ \frac{1}{2\pi} \int_\Omega |f|^2 \right]^{1/2} \left[ \int_E \frac{|f|^2}{1 + B|u_n|} \right]^{1/2}.$$  

Since the sequence $\{u_n\}$ is compact in $L^2(\Omega)$, this estimate implies that the sequence $\{\frac{\partial u_n}{\partial x_i}\}$ is equiintegrable. Thus, by Dunford–Pettis theorem, and up to subsequences, there exists $Y_i$ in $L^1(\Omega)$ such that $\frac{\partial u_n}{\partial x_i}$ weakly converges to $Y_i$ in $L^1(\Omega)$. Since $\frac{\partial u_n}{\partial x_i}$ is the distributional partial derivative of $u_n$, we have, for every $n$ in $\mathbb{N}$,

$$\int_\Omega \frac{\partial u_n}{\partial x_i} \varphi = - \int_\Omega u_n \frac{\partial \varphi}{\partial x_i}, \quad \forall \varphi \in C_0^\infty(\Omega).$$
We now pass to the limit in the above identities, using that \( \hat{c}_i u_n \) weakly converges to \( Y_i \) in \( L^1(\Omega) \), and that \( u_n \) strongly converges to \( u \) in \( L^2(\Omega) \): we obtain

\[
\int_\Omega Y_i \phi = -\int_\Omega u \frac{\hat{c}_i \phi}{\hat{c}_l}, \quad \forall \phi \in C^\infty_0(\Omega).
\]

This implies that \( Y_i = \frac{\hat{c}_i \phi}{\hat{c}_l} \), and this result is true for every \( i \). Since \( Y_i \) belongs to \( L^1(\Omega) \) for every \( i \), \( u \) belongs to \( W^{1,1}_0(\Omega) \), as desired.

Since by (11) it follows that the sequence \( \{ T_k(u_n) \} \) is bounded in \( H^1_0(\Omega) \), and since \( u_n \) tends to \( u \) almost everywhere in \( \Omega \), then \( T_k(u_n) \) weakly converges to \( T_k(u) \) in \( H^1_0(\Omega) \), and \( T_k(u) \) belongs to \( H^1_0(\Omega) \) for every \( k \geq 0 \).

Finally, we prove (17). Let \( \Phi \) be a fixed function in \( (L^\infty(\Omega))^N \). Since \( u_n \) almost everywhere converges to \( u \) in \( \Omega \), we have

\[
\lim_{n \to +\infty} \frac{\Phi}{1 + b(x)|u_n|} = \frac{\Phi}{1 + b(x)|u|} \quad \text{almost everywhere in} \ \Omega.
\]

By Egorov’s theorem, the convergence is therefore quasi uniform; i.e., for every \( \delta > 0 \) there exists a subset \( E_\delta \) of \( \Omega \), with \( \text{meas}(E_\delta) < \delta \), such that

\[
\lim_{n \to +\infty} \frac{\Phi}{1 + b(x)|u_n|} = \frac{\Phi}{1 + b(x)|u|} \quad \text{uniformly in} \ \Omega \setminus E_\delta.
\]

We now have

\[
\left| \int_\Omega \frac{\nabla u_n}{1 + b(x)|u_n|} \cdot \Phi - \int_\Omega \frac{\nabla u}{1 + b(x)|u|} \cdot \Phi \right| \\
\leq \left| \int_{\Omega \setminus E_\delta} \nabla u_n \cdot \frac{\Phi}{1 + b(x)|u_n|} - \int_{\Omega \setminus E_\delta} \nabla u \cdot \frac{\Phi}{1 + b(x)|u|} \right| \\
+ \| \Phi \|_{L^\infty(\Omega)} \int_{E_\delta} [ |\nabla u_n| + |\nabla u| ].
\]

Using the equiintegrability of \( |\nabla u_n| \) proved above, and the fact that \( |\nabla u| \) belongs to \( L^1(\Omega) \), we can choose \( \delta \) such that the second term of the right hand side is arbitrarily small, uniformly with respect to \( n \), and then use (18) to choose \( n \) large enough so that the first term is arbitrarily small. Hence, we have proved that

\[
\lim_{n \to +\infty} \frac{\nabla u_n}{1 + b(x)|u_n|} = \frac{\nabla u}{1 + b(x)|u|} \quad \text{weakly in} \ (L^1(\Omega))^N.
\]

On the other hand, from (10) it follows that the sequence \( \frac{\nabla u_n}{1 + b(x)|u_n|} \) is bounded in \( (L^2(\Omega))^N \), so that it weakly converges to some function \( \sigma \) in the same space. Since (19) holds, we have that \( \sigma = \frac{\nabla u}{1 + b(x)|u|} \), and (17) is proved. \( \square \)
Remark 5. The fact that we need to prove (17) is one of the main differences with the paper [2].

Proof of Theorem 1. Let $u_n$ be as in Lemma 4. The minimality of $u_n$ implies that

$$
\int_\Omega \frac{j(x, \nabla u_n)}{1 + b(x) |u_n|}^2 + \frac{1}{2} \int_\Omega |u_n|^2 - \int_\Omega f_n u_n \\
\leq \int_\Omega \frac{j(x, \nabla v)}{1 + b(x) |v|}^2 + \frac{1}{2} \int_\Omega |v|^2 - \int_\Omega f_n v
$$

for every $v$ in $H^1_0(\Omega)$. The result will then follow by passing to the limit in the previous inequality. The right hand side of (20) is easy to handle since $f_n$ converges to $f$ in $L^2(\Omega)$. Let us study the limit of the left hand side of (20). The convexity of $j$ implies that

$$
\int_\Omega \frac{j(x, \nabla u_n)}{1 + b(x) |u_n|}^2 \geq \int_\Omega \frac{j(x, \nabla T_k(u))}{1 + b(x) |u_n|}^2 \\
- \int_\Omega \frac{j_\varepsilon(x, \nabla T_k(u))}{1 + b(x) |u_n|} \cdot \left( \frac{\nabla u_n}{1 + b(x) |u_n|} - \frac{\nabla T_k(u)}{1 + b(x) |u_n|} \right).
$$

By (17), assumptions (1) and (2), and Lebesgue’s theorem, we have

$$
\liminf_{n \to +\infty} \int_\Omega \frac{j(x, \nabla u_n)}{1 + b(x) |u_n|}^2 \geq \int_\Omega \frac{j(x, \nabla T_k(u))}{1 + b(x) |u_n|}^2 \\
- \int_\Omega \frac{j_\varepsilon(x, \nabla T_k(u))}{1 + b(x) |u_n|} \cdot \frac{\nabla u - T_k(u)}{1 + b(x) |u_n|},
$$

that is, since $j_\varepsilon(x, \nabla T_k(u)) \cdot (\nabla u - T_k(u)) = 0$,

$$
\int_\Omega \frac{j(x, \nabla T_k(u))}{1 + b(x) |u_n|}^2 \leq \liminf_{n \to +\infty} \int_\Omega \frac{j(x, \nabla u_n)}{1 + b(x) |u_n|}^2.
$$

Letting $k$ tend to infinity, and using Levi’s theorem, we obtain

$$
\int_\Omega \frac{j(x, \nabla u)}{1 + b(x) |u|}^2 \leq \liminf_{n \to +\infty} \int_\Omega \frac{j(x, \nabla u_n)}{1 + b(x) |u_n|}^2.
$$

Inequality (21) and Lemma 4 imply that

$$
\liminf_{n \to +\infty} \int_\Omega \frac{j(x, \nabla u_n)}{1 + b(x) |u_n|}^2 + \frac{1}{2} \int_\Omega |u_n|^2 - \int_\Omega f_n u_n \\
\geq \int_\Omega \frac{j(x, \nabla u)}{1 + b(x) |u|}^2 + \frac{1}{2} \int_\Omega |u|^2 - \int_\Omega f u.
$$
Thus, for every $v$ in $H^1_0(\Omega)$,
\[
\int_{\Omega} \frac{j(x, \nabla u)}{|1 + b(x)|u|^2} \, dx + \frac{1}{2} \int_{\Omega} |u|^2 - \int_{\Omega} fu \leq \int_{\Omega} \frac{j(x, \nabla v)}{|1 + b(x)|v|^2} \, dx + \frac{1}{2} \int_{\Omega} |v|^2 - \int_{\Omega} fv,
\]
so that $u$ is a minimum of $J$; its regularity has been proved in Lemma 4. 

**Remark 6.** If we suppose that the coefficient $b(x)$ satisfies the stronger assumption
\[ 0 < A \leq b(x) \leq B, \quad \text{almost everywhere in } \Omega, \]
it is possible to prove that $J(u) \leq J(w)$ not only for every $w$ in $H^1_0(\Omega)$, but also for the test functions $w$ such that
\[
\begin{cases}
T_k(w) \text{ belongs to } H^1_0(\Omega) \text{ for every } k > 0, \\
\log(1 + A|w|) \text{ belongs to } H^1_0(\Omega), \\
w \text{ belongs to } L^2(\Omega).
\end{cases}
\]
Indeed, if $w$ is as in (22), we can use $T_k(w)$ as test function in (4) and we have
\[
J(u) \leq J(T_k(w)) = \int_{\Omega} \frac{j(x, \nabla T_k(w))}{|1 + b(x)|T_k(w)|^2} \, dx + \frac{1}{2} \int_{\Omega} |T_k(w)|^2 - \int_{\Omega} fT_k(w).
\]
In the right hand side is possible to pass to the limit, as $k$ tends to infinity, so that we have $J(u) \leq J(w)$, for every test function $w$ as in (22).

**Remark 7.** We explicitly point out the differences, concerning the coercivity, between the functionals studied in [3] and the functionals studied in this paper. Indeed, let $0 < \rho < \frac{N-2}{2}$, and consider the sequence of functions
\[ v_n = \exp \left[ T_n \left( \frac{1}{|x|^{\rho}} - 1 \right) \right] - 1, \]
defined in $\Omega = B_1(0)$. Then
\[ \log(1 + |v_n|) = T_n \left( \frac{1}{|x|^{\rho}} - 1 \right), \]
is bounded in $H^1_0(\Omega)$ (since the function $v(x) = \frac{1}{|x|^{\rho}} - 1$ belongs to $H^1_0(\Omega)$ by the assumptions on $\rho$), but, by Levi’s theorem,
\[
\lim_{n \to +\infty} \int_{\Omega} |\nabla v_n| = \rho \int_{\Omega} \frac{\exp \left[ \frac{1}{|x|^{\rho}} - 1 \right]}{|x|^{\rho+1}} = +\infty.
\]
Hence, the functional
\[ v \in H^1_0(\Omega) \mapsto \int_{\Omega} \frac{|
abla v|^2}{(1 + |v|)^2} = \int_{\Omega} |\nabla \log(1 + |v|)|^2, \]
which is of the type studied in [3], is non coercive on $W^{1,1}_0(\Omega)$. On the other hand, recalling (16), we have

$$\int_\Omega |\nabla v| = \int_\Omega \frac{|\nabla v|}{1 + |v|} (1 + |v|) \leq \frac{1}{2} \int_\Omega \frac{|\nabla v|^2}{(1 + |v|)^2} + \frac{1}{2} \int_\Omega (1 + |v|)^2.$$ 

Thus, the functional

$$v \in H^1_0(\Omega) \mapsto \int_\Omega \frac{|\nabla v|^2}{(1 + |v|)^2} + \int_\Omega |v|^2,$$

which is of the type studied here, is coercive on $W^{1,1}_0(\Omega)$.

**References**


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