Probability Theory — BV functions in a Hilbert space with respect to a Gaussian measure, by Luigi Ambrosio, Giuseppe Da Prato and Diego Pallara, communicated on 24 June 2010.

Abstract. — Functions of bounded variation in Hilbert spaces endowed with a Gaussian measure $\gamma$ are studied, mainly in connection with Ornstein-Uhlenbeck semigroups for which $\gamma$ is invariant.

Key words: Gaussian measures, BV functions, Ornstein-Uhlenbecks semigroups.

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1. Introduction

Functions of bounded variation, whose introduction in [13] was based on the heat semigroup, are by now a well-established tool in Euclidean spaces, and more generally in metric spaces endowed with a doubling measure, see e.g. [6] and the references there. Applications run from variational problems with possibly discontinuous solutions along surfaces and geometric measure theory (see [3] and the references there) to renormalized solutions of ODEs without uniqueness (see [1]). More recently, the theory has been extended to infinite dimensional settings (see [16, 17, 4, 5]), aiming to apply the theory to variational problems (see [14, 18]), infinite dimensional geometric measure theory (see [15]), ODEs (see [2] for the Sobolev case), as well as stochastic differential equations (see [11, 12]).

If the ambient space is a Hilbert space $X$ endowed with a Gaussian measure $\gamma$, then, beside the Malliavin calculus, on which the above quoted papers are based, an approach based on the infinite dimensional analysis as presented in [10] is possible. As in the case of Sobolev spaces, this approach turns out to be similar but not equivalent to the other, and a smaller class of $BV$ functions is obtained. The aim of this paper is to deepen this analysis, mainly in connection with the Ornstein-Uhlenbeck semigroup $R_t$ studied in [10] whose invariant measure is $\gamma$, which enjoys stronger regularizing properties compared to the operator $P_t$ of the Malliavin calculus. We prove that, for $u \in L^1(X, \gamma)$, the property of having measure derivatives in a weak sense (i.e., of being $BV$) is equivalent to the boundedness of a (slightly enforced) Sobolev norm of the gradient of $R_t u$. This regularity result on $R_t u$, for $u \in BV$, is used as a tool, but can be interesting on its own.

2. Notation and preliminaries

Let $X$ be a separable real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$, and let us denote by $B(X)$ the Borel $\sigma$-algebra and by $B_b(X)$ the space of...
bounded Borel functions; since $X$ is separable, $B(X)$ is generated by the cylindrical sets, that is by the sets of the form $E = \Pi_m^{-1}B$ with $B \in B(\mathbb{R}^m)$, where $\Pi_m : X \to \mathbb{R}^m$ is orthogonal (see [19, Theorem I.2.2]). The symbol $\mathcal{C}^k_b(X)$ denotes the space of $k$ times continuously Fréchet differentiable functions with bounded derivatives up to the order $k$, and the symbol $\mathcal{FC}^k_b(X)$ that of cylindrical $\mathcal{C}^k_b(X)$ functions, that is, $u \in \mathcal{FC}^k_b(X)$ if $u(x) = v(\Pi_m, x)$ for some $v \in \mathcal{C}^k_b(\mathbb{R}^m)$. We also denote by $\mathcal{M}(X, Y)$ the set of countably additive measures on $X$ with finite total variation with values in a separable Hilbert space $Y$, $\mathcal{M}(X)$ if $Y = \mathbb{R}$. We denote by $|\mu|$ the total variation measure of $\mu$, defined by

$$\tag{2.1} |\mu|(B) := \sup \left\{ \sum_{h=1}^{\infty} |\mu(B_h)| \ : \ B = \bigcup_{h=1}^{\infty} B_h \right\},$$

for every $B \in B(X)$, where the supremum runs along all the countable disjoint unions. Notice that, using the polar decomposition, there is a unit $|\mu|$-measurable vector field $\sigma : X \to Y$ such that $\mu = |\sigma| |\mu|$, and then the equality

$$|\mu|(X) = \sup \left\{ \int_X \langle \sigma, \phi \rangle d|\mu|, \ \phi \in C_b(X, Y), |\phi(x)|_Y \leq 1 \ \forall x \in X \right\}$$

holds. Note that, by the Stone-Weierstrass theorem, the algebra $\mathcal{FC}^1_b(X)$ of $C^1$ cylindrical functions is dense in $C(K)$ in sup norm, since it separates points, for all compact sets $K \subset X$. Since $|\mu|$ is tight, it follows that $\mathcal{FC}^1_b(X)$ is dense in $L^1(X, |\mu|)$. Arguing componentwise, it follows that also the space $\mathcal{FC}^1_b(X, Y)$ of cylindrical functions with a finite-dimensional range is dense in $L^1(X, |\mu|, Y)$. As a consequence, $\sigma$ can be approximated in $L^1(X, |\mu|, Y)$ by a uniformly bounded sequence of functions in $\mathcal{FC}^1_b(X, Y)$, and we may restrict the supremum above to these functions only to get

$$\tag{2.2} |\mu|(X) = \sup \left\{ \int_X \langle \sigma, \phi \rangle d|\mu|, \ \phi \in \mathcal{FC}^1_b(X, Y), |\phi(x)|_Y \leq 1 \ \forall x \in X \right\}.$$  

We recall the following well-known result (see for instance [5]): given a sequence of real measures $(\mu_j)$ on $X$ and an orthonormal basis $(e_j)$, if

$$\tag{2.3} \sup_m |(\mu_1, \ldots, \mu_m)|_1(X) < \infty,$$

then the measure $\mu = \sum_j \mu_j e_j$ belongs to $\mathcal{M}(X, X)$.

Let us come to a description of the differential structure in $X$. We refer to [10] for more details and the missing proofs. By $N_{a, Q}$ we denote a non degenerate Gaussian measure on $(X, B(X))$ of mean $a$ and trace class covariance operator $Q$ (we also use the simpler notation $N_Q = N_{0, Q}$). Let us fix $\gamma = N_Q$, and let $(e_k)$ be an orthonormal basis in $X$ such that $Q e_k = \lambda_k e_k$, $\forall k \geq 1$, 

$$Q e_k = \lambda_k e_k, \quad \forall k \geq 1,$$
with \( \lambda_k \) a nonincreasing sequence of strictly positive numbers such that 
\[ \sum_k \lambda_k < \infty. \]
Set \( x_k = \langle x, e_k \rangle \) and for all \( k \geq 1, f \in C_b(X), \) define the partial derivatives
\[ D_k f(x) = \lim_{t \to 0} \frac{f(x + te_k) - f(x)}{t} \tag{2.4} \]
(provided that the limit exists) and, by linearity, the gradient operator
\[ D : \mathcal{F} C^1_b(X) \to \mathcal{F} C_b(X, X). \]
The gradient turns out to be a closable operator with respect to the topologies \( L^p(X, \gamma) \) and \( L^p(X, \gamma, X) \) for every \( p \geq 1, \) and we denote by \( W^{1,p}(X, \gamma) \) the domain of the closure in \( L^p(X, \gamma), \) endowed with the norm
\[ \|u\|_{1,p} = \left( \int_X |u(x)|^p \, d\gamma + \int_X \left( \sum_{k=1}^{\infty} |D_k u(x)|^2 \right)^{p/2} \, d\gamma \right)^{1/p}, \]
where we keep the notation \( D_k \) also for the closure of the partial derivative operator. For all \( \varphi, \psi \in C^1_b(X) \) we have
\[ \int_X \psi D_k \varphi \, d\gamma = -\int_X \varphi D_k \psi \, d\gamma + \frac{1}{\lambda_k} \int_X x_k \varphi \psi \, d\gamma. \]
and this formula, setting \( D_k^* \varphi = D_k \varphi - \frac{x_k}{\lambda_k} \varphi, \) reads
\[ \int_X \psi D_k \varphi \, d\gamma = -\int_X \varphi D_k^* \psi \, d\gamma. \tag{2.5} \]

Notice that \( Q^{1/2} \) is still a compact operator on \( X, \) and define the Cameron-Martin space
\[ H = Q^{1/2} X = \{ x \in X : \exists y \in X \text{ with } x = Q^{1/2} y \} = \left\{ x \in X : \sum_{k=1}^{\infty} \frac{|x_k|^2}{\lambda_k} < \infty \right\}, \]
endowed with the orthonormal basis \( e_k = \frac{\lambda_k^{1/2}}{\sum_{k=1}^{\infty} \lambda_k^{1/2}} e_k \) relative to the norm \( |x|_H := \left( \sum_k |x_k|^2 / \lambda_k \right)^{1/2}. \) The Malliavin derivative of \( f \in C^1_b(X) \) is defined by
\[ \partial_{e_k} f(x) = \lim_{t \to 0} \frac{f(x + te_k) - f(x)}{t} \tag{2.6} \]
(provided that the limit exists) and turns out to be a closable operator as well (see [7] or apply (2.8) below) with respect to the topology \( L^p(X, \gamma) \) for every \( p \geq 1. \)
We denote by \( \nabla_H f \) the gradient and by \( \mathbb{D}^{1,p}(X, \gamma) \) the domain of its closure in \( L^p(X, \gamma), \) endowed with the obvious norm. As a consequence of the relation \( e_k = \lambda_k^{1/2} e_k \) we have also
\[ \partial_{e_k} = \lambda_k^{1/2} D_k, \tag{2.7} \]
so that $W^{1,p}(X, \gamma) \subset D^{1,p}(X, \gamma)$, since $|\nabla_H f|_H = \left(\sum_k \sqrt{\lambda_k} |D_k f|^2\right)^{1/2}$. By (2.7) and (2.5) the integration by parts formula corresponding to the Malliavin calculus reads

$$\int_X \psi \partial_k \phi \, d\gamma = -\int_X \phi \partial_k \psi \, d\gamma + \int_X \frac{1}{\sqrt{\lambda_k}} x_k \phi \psi \, d\gamma. \tag{2.8}$$

There exist infinitely many Ornstein-Uhlenbeck semigroups having $\gamma$ as invariant measure. Let us choose the one corresponding to the stochastic evolution equation

$$dX = AX \, dt + dW(t), \quad X(0) = x \in X \tag{2.9}$$

where $A := -\frac{1}{2} Q^{-1}$ is selfadjoint and

$$\langle W(t), z \rangle = \sum_{k=1}^\infty W_k(t)z_k, \quad z \in X,$$

with $(W_k)_{k \in \mathbb{N}}$ sequence of independent real Brownian motions. We have $Ae_k = -\lambda_k e_k$, where

$$\lambda_k = \frac{1}{2\lambda_k}.$$

The transition semigroup corresponding to (2.9) is given by

$$R_t f(x) = \int_X f(y) \, dN_{e^{tA}x, Q_t}(y) = \int_X f(e^{tA}x + y) \, dN_{Q_t}(y), \quad f \in B_b(X), \tag{2.10}$$

where

$$Q_t = \int_0^t e^{2sA} \, ds = -\frac{1}{2} A^{-1}(1 - e^{2tA}).$$

Therefore $N_{Q_t} \to N_Q = \gamma$ weakly as $t \to \infty$, so that $\gamma$ is invariant for $R_t$. Moreover, for every $k \geq 1$, $v \in C^1_b(X)$, from (2.10) we get

$$D_k R_t v(x) = e^{-\lambda_k t} \int_X D_k v(e^{tA}x + y) \, dN_{Q_t}(y) = e^{-\lambda_k t} R_t D_k v(x),$$

whence, since $R_t$ is symmetric, we deduce that for every $u \in L^1(X, \gamma)$ and $\varphi \in \mathcal{FC}^1_b(X)$ the equality

$$\int_X R_t u D_k^* \varphi \, d\gamma = e^{-\lambda_k t} \int_X u D_k^* R_t \varphi \, d\gamma \tag{2.11}$$
holds. In fact, if $u$ is bounded, by [10, Theorem 8.16] we know that $R_t u \in C_b^\infty (X)$ for every $t > 0$, and then for every $\varphi \in C_b^1 (X)$ we have

$$
\int_X R_t u D_k^* \varphi \, d\gamma = -\int D_k (R_t u) \varphi \, d\gamma = -e^{-2kt} \int_X R_t D_k u \varphi \, d\gamma

= -e^{-2kt} \int_X D_k u R_t \varphi \, d\gamma = e^{-2kt} \int_X u D_k^* R_t \varphi \, d\gamma.
$$

In the general case $u \in L^1 (X, \gamma)$ we use the density of $C_b^1 (X)$ in $L^1 (X, \gamma)$, as both sides in (2.11) are continuous with respect to $L^1 (X, \gamma)$ convergence in $u$.

By a standard duality argument we can define a linear contraction operator $R_t^* : \mathcal{M} (X) \to L^1 (X, \gamma)$ characterized by:

$$
(2.12) \quad \int_X R_t^* \mu \varphi \, d\gamma = \int_X R_t \varphi \, d\mu, \quad \varphi \in B_b (X).
$$

To see that this is a good definition, using Hahn decomposition we may assume with no loss of generality that $\mu$ is nonnegative. Under this assumption, we notice that $(\varphi_i) \subset B_b (X)$ equibounded and $\varphi_i \uparrow \varphi$, with $\varphi \in B_b (X)$, implies

$$
\int_X R_t \varphi_i \, d\mu \uparrow \int_X R_t \varphi \, d\mu,
$$

hence Daniell’s theorem (see e.g. [8, Theorem 7.8.1]) shows that $\varphi \mapsto \int_X R_t \varphi \, d\mu$ is the restriction to $B_b (X)$ of $\varphi \mapsto \int_X \varphi \, d\mu^*$ for a suitable (unique) nonnegative $\mu^* \in \mathcal{M} (X)$. In order to show that $R_t^* \mu \ll \gamma$, take a Borel set $B$ with $\gamma (B) = 0$. Then

$$
(R_t^*) \mu (B) = \int_X \chi_B \, dR_t^* \mu = \int_X R_t \chi_B \, d\mu,
$$

but $R_t \chi_B (x) = N_{t, x} (\gamma) (B)$ and since $N_{t, x} (\gamma) \ll \gamma$ (see [12, Lemma 10.3.3]) we have $R_t \chi_B (x) = 0$ for all $x$ and the claim follows. Finally, since $R_t 1 = 1$ we obtain that $\mu^* (X) = \mu (X)$, hence $R_t^*$ is a contraction. It is also useful to notice that $R_t^*$ is contractive on vector measures as well. In fact, $R_t$ is a contraction in $C_b$, hence $|\langle R_t^* \mu, \phi \rangle| = |\langle \mu, R_t \phi \rangle| \leq |\mu|, |\phi|$ for every $\varphi \in C_b (X)$. Since for every vector measure $v$ the minimal positive measure $\sigma$ such that $|\langle v, \phi \rangle| \leq |\sigma|, |\phi|$ for all $\varphi$ is $|v|$, taking $v = R_t^* \mu$ we conclude.

3. Functions of bounded variation

In the present context it is possible to define functions of bounded variation, as it has been done, using the Malliavin derivative, in [16], [17] and [4], [5], and to relate $BV$ functions to the Ornstein-Uhlenbeck semigroup $R_t$. According to [5], in order to distinguish the two notions of $BV$ functions, we keep the notation $BV (X, \gamma)$ for the functions coming from the $\nabla H$ operator and use the notation $BV_X (X, \gamma)$ for those coming from $D$. 
**Definition 3.1.** A function \( u \in L^1(X, \gamma) \) belongs to \( BV_X(X, \gamma) \) if there exists \( v^u \in \mathcal{M}(X, X) \) such that for any \( k \geq 1 \) we have

\[
\int_X u(x) D_k \varphi(x) \, d\gamma = -\int_X \varphi(x) \, dv^u_k + \frac{1}{\lambda_k} \int_X x_k u(x) \varphi(x) \, d\gamma, \quad \varphi \in \mathcal{FC}_b^1(X),
\]

with \( v^u_k = \langle v^u, e_k \rangle_X \). If \( u \in BV_X(X, \gamma) \), we denote by \( Du \) the measure \( v^u \), and by \(|Du|\) its total variation.

According to (2.2), for \( u \in BV_X(X, \gamma) \) the total variation of \( Du \) is given by

\[
|Du|(X) = \sup \left\{ \int_X u \left[ \sum_k D_k^c \phi_k \right] \, d\gamma : \phi \in \mathcal{FC}_b^1(X, X), |\phi(x)| \leq 1 \, \forall x \in X \right\}.
\]

Obviously, if \( u \in W^{1,1}(X, \gamma) \) then \( u \in BV_X(X, \gamma) \) and \(|Du|(X) = \int_X |Du| \, d\gamma\).

Recalling that \( u \in BV(X, \gamma) \) if there is a finite measure \( D_j u = (D^k_j u)_k \in \mathcal{M}(X, X) \) such that

\[
\int_X u(x) D_k \varphi(x) \, d\gamma = -\int_X \varphi(x) \, dD^k_j u + \frac{1}{\sqrt{\lambda_k}} \int_X x_k u(x) \varphi(x) \, d\gamma, \quad \varphi \in \mathcal{FC}_b^1(X), \quad k \geq 1,
\]

it is immediate to check that \( BV_X(X, \gamma) \) is contained in \( BV(X, \gamma) \) and that

\[
D^k_j u = \lambda_k^{1/2} v^u_k, \quad \forall k \geq 1.
\]

The next proposition provides a simple criterion, analogous to the finite-dimensional one, for the verification of the \( BV_X \) property.

**Proposition 3.2.** Let \( u \in L^1(X, \gamma) \) and let us assume that

\[
(3.3) \quad \mathcal{R}(u) := \sup_m \sup \left\{ \int_X \sum_{k=1}^m u D_k \varphi_k \, d\gamma : \varphi_k \in C_b^1(X), \sum_{i=1}^m \varphi_i^2 \leq 1 \right\} < \infty.
\]

Then \( u \in BV_X(X, \gamma) \) and \(|Du|(X) \leq \mathcal{R}(u)\).

**Proof.** Fix \( k \geq 1 \), set \( X_k = \{ x \in X : x = se_k, s \in \mathbb{R} \} \), \( X_k^\perp = \{ x \in X : \langle x, e_k \rangle = 0 \} \), and define

\[
V_k(u) := \sup \left\{ \int_X u \left( c_k \phi - \frac{1}{\sqrt{\lambda_k}} \phi \right) \, d\gamma : \phi \in C^1_c(X), |\phi(x)| \leq 1 \, \forall x \in X \right\},
\]

\[
V^\perp_k(u) := \sup \left\{ \int_X u \left( D_k \phi - \frac{1}{\lambda_k} \phi \right) \, d\gamma : \phi \in C^1_c(X), |\phi(x)| \leq 1 \, \forall x \in X \right\}.
\]
For \( y \in X_k^\perp \), define the function \( u_y(s) = u(y + s e_k) \), \( s \in \mathbb{R} \), and notice that \( V_k(u) = \sqrt{\lambda_k} \mathcal{Y}_k(u) \), so that by \([5, \text{Theorem 3.10}]\) we have

\[
\mathcal{V}_k(u) = \int_{X_k^\perp} \mathcal{V}(u_y) \, d\gamma^\perp(y),
\]

where \( \mathcal{V}^\perp \) denotes the 1-dimensional variation of \( u_y \) and we have used the factorization \( \gamma = \gamma_1 \otimes \gamma^\perp \) induced by the orthogonal decomposition \( X = X_k \oplus X_k^\perp \).

Since \( \mathcal{V}_k(u) \leq \mathcal{R}(u) \) we have

\[
\int_{X_k^\perp} \mathcal{V}(u_y) \, d\gamma^\perp(y) < \infty.
\]

It follows that for \( \gamma^\perp \)-a.e. \( y \in X_k^\perp \) the function \( u_y \) has bounded variation in \( \mathbb{R} \). By a Fubini argument, based on the factorization \( \gamma = \gamma_1 \otimes \gamma^\perp \), the 1-dimensional integration by parts formula yields that the measure \( D_k u \) coincides with \( D_u \otimes \gamma^\perp \), i.e.,

\[
D_k u(A) = \int_{X_k^\perp} D_u(A_y) \, d\gamma^\perp(y)
\]

(where \( A_y := \{ s : y + s e_k \in A \} \) is the \( y \)-section of a Borel set \( A \)) provides the derivative of \( u \) along \( e_k \). Notice that \( D_k u \) is well defined, since we have just proved that \( \int_{X_k^\perp} |D_u|(\mathbb{R}) \, d\gamma^\perp \) is finite.

Now, setting \( \mu_k = D_k u \), by the implication stated in (2.3) we obtain that \( |D u|(X) \leq \mathcal{R}(u) \). \( \square \)

The next theorem characterizes the BV class in terms of the semigroup \( R_t \); notice that the functions \( R_t u \), for \( u \in BV(X, \gamma) \), turn out to be slightly better than \( W^{1,1}(X, \gamma) \), since not only \( |DR_t u| \), but also \( |e^{-tA} DR_t u| \) is integrable.

**Theorem 3.3.** Let \( u \in L^1(X, \gamma) \). Then, \( u \in BV_X(X, \gamma) \) if and only if \( R_t u \in W^{1,1}(X, \gamma) \), \( |e^{-tA} DR_t u| \in L^1(X, \gamma) \) for all \( t > 0 \) and

\[
\liminf_{t \to 0} \int_X |e^{-tA} DR_t u| \, d\gamma < \infty.
\]

Moreover, if \( u \in BV_X(X, \gamma) \) we have \( DR_t u = e^{-tA} R_t^* Du \),

\[
\int_X |e^{-tA} DR_t u| \, d\gamma \leq |Du|(X), \quad \forall t > 0
\]

and

\[
\lim_{t \to 0} \int_X |e^{-tA} DR_t u| \, d\gamma = |Du|(X).
\]
Let \( u \in BV_X(X, \gamma) \). We use (2.11) to deduce

\[
\int_X R_t u D_k^* \varphi \, d\gamma = -e^{-\lambda_k t} \int_X R_t \varphi \, dD_k u \quad \forall \varphi \in FC^1_b(X), \ t > 0.
\]

According to (2.12), this implies that

\[
D_k R_t u = e^{-\lambda_k t} R_t^* D_k u \in L^1(X, \gamma).
\]

Therefore, as \( R_t^* \) is a contractive semigroup also on vector measures,

\[
\int_X |e^{-tA} D_r u| \, d\gamma = \int_X |R_t^* Du| \, d\gamma \leq |Du|(X)
\]

for every \( t > 0 \) and (3.5) follows.

Conversely, let us assume that \( R_t u \in W^{1,1}(X, \gamma) \) for all \( t > 0 \) and that the \( \liminf \) in (3.4) is finite. We shall denote by \( \Pi_m : X \to \mathbb{R}^m \) the canonical projection on the first \( m \) coordinates and we shall actually prove that \( u \in BV_X(X, \gamma) \) and

(3.7) \[ |Du|(X) \leq \sup_{m} \liminf_{t \downarrow 0} \int_X |\Pi_m Du| \, d\gamma \]

under the only assumption that the right hand side of (3.7) is finite. Indeed, fix an integer \( m \) and notice that an integration by parts gives

\[
\sup \left\{ \int_X \sum_{k=1}^m R_t u D_k^* \varphi_k \, d\gamma : \varphi_k \in C^1_b(X), \ \sum_{i=1}^m \varphi_k^2 \leq 1 \right\} \leq \int_X |\Pi_m Du| \, d\gamma,
\]

so that passing to the limit as \( t \downarrow 0 \) and taking the supremum over \( m \) we obtain

\[
\mathcal{R}(u) \leq \sup_{m} \liminf_{t \downarrow 0} \int_X |\Pi_m Du| \, d\gamma,
\]

with \( \mathcal{R} \) defined as in (3.3). Therefore we obtain the inequality (3.7) by Proposition 3.2. Finally (3.6) follows combining (3.5) with (3.7).

Remark 3.4. (1) Notice that the inclusion \( BV_X(X, \gamma) \subset BV(X, \gamma) \) allows us to exploit the results in [5] in order to prove one implication in the above theorem, while the other one uses the strong regularizing properties of the semigroup \( R_t \). Anyway, we have tried to keep the use of the results in the above quoted paper to a minimum, and in fact only Theorem 3.10 in [5] has been used in the proof of Proposition 3.2. It is most likely possible to give a proof completely independent from [5], but some of the arguments therein should be rephrased and proved again, basically along the same lines.

(2) The argument used in the proof of the theorem shows that \( D_k R_t u \in L^1(X, \gamma) \) for all \( t > 0, \ k \geq 1 \) and finiteness of the right hand side of (3.7) suffices to conclude that \( u \in BV_X(X, \gamma) \). Furthermore, combining (3.5) and (3.7) we obtain that \( \int_X |DR_t u| \, d\gamma \to |Du|(X) \) as \( t \downarrow 0 \), as well.
By the same argument as [5] one can use (2) to conclude that the measures $e^{-tA}DR_{t\gamma}$ are equi-tight as $t \downarrow 0$; hence, they converge (componentwise) to $Du$ not only on $\mathcal{FC}_b^1(X)$ but also on $C^0_b(X)$.

We recall also that both Sobolev and $BV$ spaces in the present context are compactly embedded into the corresponding Lebesgue spaces. The following statement is proved in [5, Theorem 5.3], see also [9] for the case $1 < p < \infty$.

**Theorem 3.5.** For every $p \geq 1$, the embedding of $W^{1,p}(X,\gamma)$ into $L^p(X,\gamma)$ is compact. The embedding of $BV_X(X,\gamma)$ into $L^1(X,\gamma)$ is also compact.

**References**


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