Rings of Fractions of $B(H)$

By

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§ 1. Introduction

In this paper we discuss the following question: What are rings of fractions of $B(H)$, the algebra of all bounded linear operators on a separable, infinite dimensional, Hilbert space $H$? We recall the definition of a ring of fractions of a (generally non-commutative) ring according to [4].

Definition. A subset $S$ of a ring $A$ with a unit 1 is called a (right) denominator set if $S$ satisfies the following conditions:

1. If $s, t \in S$, then $st \in S$ and $1 \in S$.
2. If $s \in S$ and $a \in A$, then there exist $t \in S$ and $b \in A$ such that $sb = at$.
3. If $sa = 0$ with $s \in S$, then $at = 0$ for some $t \in S$.
4. $S$ does not contain 0. (to avoid triviality).

Definition. The ring $A[S^{-1}]$ of fractions of a ring $A$ with respect to a (right) denominator set $S$ is defined by $A[S^{-1}] = (A \times S)/\sim$, where $\sim$ is the equivalence relation on $A \times S$ defined as $(a, s) \sim (b, t)$ if there exist $c, d \in A$ such that $ac = bd$ and $sc = td \in S$. We define addition and multiplication of $(a, s)\sim$, $(b, t)\sim \in (A \times S)/\sim$ in the obvious way:

$$(a, s)\sim + (b, t)\sim = (ac + bd, u)\sim$$

for some $c \in A$, $u$ and $d \in S$ with $u = sc = td$,

$$(a, s)\sim \cdot (b, t)\sim = (ac, tu)\sim$$

for some $c \in A$ and $u \in S$ with $sc = bu$.

Moreover if $A$ has a scalar (complex number) multiple, then also does $A[S^{-1}]$. Then $\varphi(a) = (a, 1)\sim$ defines a homomorphism $\varphi : A \to (A \times S)/\sim = A[S^{-1}]$.
Our main theorem asserts that any ring of fractions \( B(H)[S^{-1}] \) is isomorphic to \( B(H) \) or the quotient ring \( B(H)/J \) of \( B(H) \) by the ideal \( J \) of finite rank operators. The next problem is the existence of such a denominator set \( S \). It clear that \( B(H)[S^{-1}] = B(H) \) if we take \( S = \{1\} \). We shall show that there exist at least countably infinite many different denominator sets \( S \) such that \( B(H)[S^{-1}] \) are isomorphic to \( B(H)/J \).

§ 2. Main Theorem

An operator \( x \in B(H) \) is a Fredholm operator if \( \text{ran } x \) is closed, \( \dim \ker x \) is finite and \( \dim \ker x^* \) is finite, where \( \text{ran } x \) is the range of \( x \) and \( \ker x \) is the kernel of \( x \). The collection of Fredholm operators is denoted by \( F \). The ind is the function from \( F \) to the integers \( \mathbb{Z} \) defined by \( \text{ind } x = \dim \ker x - \dim \ker x^* \). This function enjoys the following property: For \( x, y \in F \), \( \text{ind } xy = \text{ind } x + \text{ind } y \), \( \text{ind } x^* = - \text{ind } x \), \( \text{ind } 1 = 0 \). Put \( F_0 = \{ x \in F \mid \text{ind } x = 0 \} \). Then \( F \) and \( F_0 \) satisfy (S0). Moreover \( F \) and \( F_0 \) are invariant under compact perturbations ([1]). If \( x \) and \( y \in B(H) \) satisfy \( xy = x, yx = y, (xy)^* = xy \) and \( (yx)^* = yx \), then \( y \) is called a Moore-Penrose inverse of \( x \) and \( y \) is denoted by \( x^t \). A Moore-Penrose inverse \( x^t \) does not always exist but it is unique if it exists. It is known that \( x^t \) exists if and only if \( \text{ran } x \) is closed ([3]). In particular if \( x \) is in \( F \), then \( x \) has \( x^t \).

We need the following Theorem in [2; Theorem 3.6]:

**Theorem F-W.** Let \( S \) be in \( B(H) \). If \( \text{ran } s \) is not closed, then there exists a unitary \( u \in B(H) \) such that \( \text{ran } s \cap \text{ran } us = \{0\} \).

We shall show that a denominator is automatically a Fredholm operator.

**Theorem 1.** If a subset \( S \subset B(H) \) is a denominator set of \( B(H) \), then \( S \) is contained in the set \( F \) of Fredholm operators.

**Proof.** Let \( s \in S \). Assume that \( \text{ran } s \) is not closed. Then by Theorem F-W, there exists a unitary \( u \) such that \( \text{ran } s \cap \text{ran } us = \{0\} \). The condition \( (S1) \) implies that there exist \( t \in S \) and \( b \in B(H) \) such that \( sb = (us)t \). Then

\[
\text{ran } ust = \text{ran } sb = \text{ran } sb \cap \text{ran } ust \subset \text{ran } s \cap \text{ran } us = \{0\}.
\]

Therefore \( ust = 0 \). Then \( S \) contains \( st = 0 \). This contradicts to \( (S3) \). Hence \( \text{ran } s \) is closed. Next assume that \( \dim \ker s^* = +\infty \). Then there exists a unitary \( u \) such that \( \text{ran } u \cap \text{ran } us = \{0\} \), since \( \dim (\text{ran } s)^\perp = \dim \ker s^* = +\infty \). By the same argument of the proceeding paragraph, \( S \) contains \( 0 \). This is a contradiction. Therefore \( \dim \ker s^* < +\infty \). Next we shall show that \( \dim \ker s < +\infty \). Since \( \text{ran } s \)
is closed, $s^\dagger$ exists. Put $a = 1 - s^\dagger s$, then $sa = 0$. By (S2) there exists $t \in S$ such that $at = 0$. Since $a = a^*$, $t^*a = 0$, that is, $\text{ran } a \subseteq \ker t^*$. Then $\text{dim ran } a \leq \text{dim ker } t^* < +\infty$, because $t \in S$. Thus $\text{dim ker } s = \text{dim ran } a < +\infty$. Therefore $s \in S$ is a Fredholm operator.

Consider the canonical homomorphism $\varphi : B(H) \rightarrow B(H)[S^{-1}]$ defined by $\varphi(x) = (x, 1)^\sim$.

**Lemma 2.** The canonical map $\varphi : B(H) \rightarrow B(H)[S^{-1}]$ is onto.

**Proof.** Take $(a, s) \in B(H)[S^{-1}]$. Then $s^\dagger$ exists by Theorem 1. Put $z = 1 - s^\dagger s$. Since $sz = 0$, there exists $c \in S$ such that $zc = 0$ by (S2). Then $c = s^\dagger sc$. Put $x = as^\dagger$ and $d = sc$. Then

$$ac = as^\dagger sc = as^\dagger d \in B(H) \quad \text{and} \quad sc = ld \in S.$$  

This shows that $(a, s) \sim (as^\dagger, 1)$. Then $\varphi(x) = (as^\dagger, 1)^\sim = (a, s)^\sim$. Thus $\varphi$ is onto.

The following main theorem gives the possible rings of fractions of $B(H)$ completely:

**Theorem 3.** Let $S$ be a denominator set of $B(H)$. If $S$ contains a non-invertible operator, then the ring $B(H)[S^{-1}]$ of fractions is isomorphic to the quotient ring $B(H)/J$ of $B(H)$ by the ideal $J$ of finite rank operators. If $S$ does not, then $B(H)[S^{-1}]$ is isomorphic to $B(H)$.

**Proof.** By Lemma 2, $B(H)[S^{-1}]$ is isomorphic to $B(H)/\ker \varphi$. We note that

$$\ker \varphi = \{x \in B(H) \mid xc = 0 \text{ for some } c \in S\}.$$  

If $S$ does not contain non-invertible elements, then $\ker \varphi = \{0\}$, so $B(H)[S^{-1}]$ is isomorphic to $B(H)$. Now suppose that $S$ contains a non-invertible operator $s$. Then $s^\dagger s \not= 1$ or $ss^\dagger \not= 1$. If $ss^\dagger \not= 1$, then $x = 1 - ss^\dagger \not= 0$ and $x \in \ker \varphi$, because $xs = s - ss^\dagger s = 0$ and $s \in S$. If $s^\dagger s \not= 1$, put $x = 1 - s^\dagger s$. Since $sx = 0$, $xt = 0$ for some $t \in S$ by (S2). Thus $x \not= 0$ and $x \in \ker \varphi$. In any case we have that $\ker \varphi \not= \{0\}$. Next we shall show that $\ker \varphi \subseteq J$. Let $x \in \ker \varphi$. By $(\ast)$ there exists $c \in S$ such that $xc = 0$. Since $c^*x^* = 0$, ran $x^* \subseteq \ker c^*$. By Theorem 1, $c$ is a Fredholm operator and $\dim \ker c^* < +\infty$. Hence $x^*$ is a finite rank operator, so $x \in J$. Since $J$ is a non-trivial minimal two-sided ideal of $B(H)$, $\ker \varphi = J$. Therefore if $S$ contains a non-invertible element, then $B(H)[S^{-1}]$ is isomorphic to $B(H)/J$. 
§ 3. Examples of Denominator Sets

In this section we shall give some examples of a denominator set $S$ such that $B(H)[S^{-1}]$ is isomorphic to $B(H)/J$. In fact there exist at least countably infinite many denominator sets with this property, although we have not yet determined all of them.

**Theorem 4.** If $S$ is a semigroup such that $F_0 \subset S \subset F$, then $S$ is a denominator set. In particular $F_0$ and $F$ are denominator sets.

**Proof.** It is clear that $S$ satisfies (S0) and (S3). We shall show that $S$ satisfies (S1). Take $s \in S$ and $a \in B(H)$. Since $s \in F$, $s^t$ exists. Then $1 - ss^t \in J$, because $\dim \ker (1 - ss^t) = \dim \ker s < + \infty$. Put $c = (1 - ss^t)a$. Then $c$ is also in $J$, so $\ker c$ is closed and $c^t$ exists. Then $c^t$ is in $J$. Put $t = 1 - c^t$. Since $t$ is a compact perturbation of 1, $t \in F_0 \subset S$. Put $b = st$. Then

$$st - sb = (1 - ss^t)at = (1 - ss^t)a(1 - c^t) = c(1 - c^t) = 0.$$ 

So $sb = at$. Thus $S$ satisfies (S1). Next we shall show that $S$ satisfies (S2). Take $s \in S$ and $a \in B(H)$ such that $sa = 0$. Since $\ker a \subset \ker s$, $a$ is in $J$. Consider a polar decomposition $a = u |a|$. We may assume that $u$ is a unitary. Put $t = u^* s^t s$. Then $\ind t = \ind u^* - \ind s + \ind s = 0$. Hence $t \in F_0 \subset S$. And $at = u |a| u^* s^t s = u(sa) s^t s = u(sa) s = 0$. Thus $S$ satisfies (S2).

Finally we shall give two kinds of examples of denominator sets of $B(H)$ which do not contain $F_0$. Let $K$ be a separable, infinite dimensional, Hilbert space and $n$ be a positive integer. Put $H = K \oplus \cdots \oplus K$ ($n$ times). Then $B(H)$ can be identified with the set $M_n(B(K))$ of $n \times n$ matrices whose entries are in $B(K)$. Let $S$ be a denominator set of $B(K)$. Define $S_n$ and $S^n \subset B(H)$ by

$$S_n = \left\{ \begin{pmatrix} s & 0 \\ s & \cdots & \cdots \\ 0 & \cdots & s \end{pmatrix} \in B(H) \mid s \in S \right\}$$

$$S^n = \left\{ \begin{pmatrix} s_1 & 0 & \cdots & 0 \\ s_2 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & s_n \end{pmatrix} \in B(H) \mid s_1, \cdots, s_n \in S \right\}.$$ 

By [4; page 61, Exercises 4], $S_n$ is a denominator set of $B(H)$. Similarly we can show that $S^n$ is also a denominator set of $B(H)$. Therefore we get the following:
**Theorem 5.** There exist countably infinite many denominator sets $S$ of $B(H)$ such that $B(H)[S^{-1}]$ are isomorphic to $B(H)/J$.

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**References**


