The Stable Cohomotopy Ring of $G_2$

*Dedicated to Professor Hirosi Toda on his 60th birthday*

By

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§ 1. Introduction

The fact that a Lie group (generally a finite $H$-space) has a stably trivial attaching map of its top cell makes a little bit easier to determine the cohomotopy groups, especially when the space has a few cells. Actually, for $Sp(2)$ and $SU(3)$, it is easy to obtain 0-th cohomotopy groups, and moreover ring structure can also be calculated. These are carried out by G. Walker in [9]. But the more cells the space has, the more difficult the determination becomes.

In this paper we shall give the 0-th stable cohomotopy group of $G_2$, the exceptional Lie group, by means of G. Walker's method in the above mentioned paper and S. Oka's accurate study of the stable homotopy type of $G_2$ in [6]. We shall also determine the ring structure by the results of P. Eccles and G. Walker [3]. Then we shall be able to recover that $[G_2, L] = \kappa$ (see §4).

We denote the $q$-th reduced stable cohomotopy of $X$ by $\pi^q(X) (= \lim[S^n X, S^{n+q}])$. We state our main results.

**Theorem 1.1.**

\[ \pi^q(G_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4 . \]

Generators are $q^*\sigma^2$, $q^*\kappa$, $q^*\nu j'$, $\text{Ext} \varepsilon - \sigma \text{Ext} \gamma$, $\text{Ext} \varepsilon$, $\nu$, $\alpha_1, \alpha_1, \gamma$, respectively (see §3).

**Theorem 1.2.**

1) $\nu^2 = q^*\nu j' + \nu \mod 4\text{Ext} \varepsilon (= 4\sigma \text{Ext} \gamma)$. 2) $(q^*\nu j') \nu = 2\nu$. 3) $\nu^3 = 2\nu + q^*\kappa$, where $\nu = \sigma \text{Ext} \gamma + \text{Ext} \varepsilon$. Other products are trivial.

This paper is organized as follows. In Section 2, we recall the result of [6]. In Section 3, we shall prove our main Theorem 1.1.
In Section 4, we shall give above results on the ring structure and prove our application to \([G_2, L]\).

§ 2. \(\pi^0(X^3)\) and \(\pi^0(Y^{11})\)

First we recall that \(G_2\) is stably equivalent to the space \(Q \bigvee S^{14}\). For the space \(Q\), there exists a cofibration \(X^3 \rightarrow Q \rightarrow Y^{11}\), where \(X^3\) and \(Y^{11}\) are following cofibers. ([6]).

\[\begin{align*}
(2.1) & \quad M^5 \xrightarrow{\eta} S^3 \xrightarrow{i^*} X^3 \xrightarrow{i''} M^5. \\
(2.2) & \quad S^{10} \xrightarrow{\eta} M^6 \xrightarrow{i^*} Y^{11} \xrightarrow{i''} S^{11}.
\end{align*}\]

Here \(M^a\) denotes the Moore space \(S^a \cup e^{a+1}\).

From above cofibrations we obtain exact sequences as follows.

\[\begin{align*}
(2.3) & \quad 0 \rightarrow \pi^0(S^3) \xrightarrow{i^*} \pi^0(X^3) \xrightarrow{i''} \pi^0(M^5) \rightarrow 0.
\end{align*}\]

\[\begin{align*}
(2.4) & \quad \pi^0(S^{10}) \xrightarrow{\eta} \pi^0(M^6) \xrightarrow{i^*} \pi^0(Y^{11}) \xrightarrow{i''} \pi^0(S^{11}) \rightarrow \pi^0(M^8).
\end{align*}\]

**Lemma A.** In the exact sequence (2.4),

\(a). \ker \eta^* = Z_4\langle \sigma \text{Ext } \eta \rangle + Z_4\langle \text{Ext } \epsilon \rangle. \quad b). \ker j^* = Z_4\langle 2\zeta \rangle.

**Proof.** By J. Mukai [4], \(\pi^0(M^8) = Z_4\langle \text{Ext } \epsilon \rangle \bigoplus Z_4\langle \sigma \text{Ext } \eta \rangle \bigoplus Z_4\langle \mu \rho \rangle\). Now \(\eta^* (\mu \rho) = \mu \eta\) is the generator of \(\pi^0_0(\text{Toda } [8])\), where \(\rho\) is a projection map.

Consider elements \(\eta^* (\sigma \text{Ext } \eta), \eta^* (\text{Ext } \epsilon),\) these are nothing but Toda brackets \(\{\sigma \eta, 2, \eta\}\) and \(\{\epsilon, 2, \eta\}\). We see easily these contain zero. Thus \(a\) is obvious. For \(b\), this time we need to investigate \(\{\mu, 2, \eta\}, \{\eta \sigma \eta, 2, \eta\}\) and \(\{\eta \sigma, 2, \eta\}\). \(\{\eta \sigma \eta, 2, \eta\} \supset \eta \sigma \{\eta, 2, \eta\}, \{\eta \sigma, 2, \eta\} \supset \epsilon \{\eta, 2, \eta\}\) and \(\{\eta, 2, \eta\} = \{\nu', -\nu'\}\) (5.4 [8]). These contain zero since \(\nu' = 2\nu\). We see easily that \(\{\mu, 2, \eta\}\) contains \(2\zeta\), for example by \(e\)-invariant of Adams (Theorem 11.1 in [1]). q. e. d.

We shall determine group extension in (2.3). Since \(\pi^3 = Z_4\langle \nu \rangle \bigoplus Z_4\langle \alpha_1 \rangle, \pi^0(M^8) = Z_2\langle \nu^2 \rho \rangle\), we only consider the two primary component. We obtain an equality as follows.

\[8a \eta^{* -1}(\nu) = j^* \{8a, \nu, \eta\} \mod j'(8\zeta[M^8, S^9]).\]
This equality is due to Toda (Proposition 1.9 [8], also refer Walker [9]). By the natural property of Toda brackets, \(i_\circ \{8, \nu, \eta\} = \{i_0, 8, \nu\} (\sim S^\#)\), where \(i_0: S^0 \to M^0\) is the inclusion. We obtain \(\{i_0, 8, \nu\} = (\text{Coext } \eta) \eta^g\) since \(\rho \{i_0, 8, \nu\}\) can be easily seen to be \(4\nu = \rho (\text{Coext } \eta) \eta^g\) and by Theorem 3.2. [4] (Coext \(\eta) \eta^g\) is a generator of \([S^1, M^0]\) = \(Z_2\). On the other hand, as \((\text{Coext } \eta) \eta^g (\sim \text{Ext } \eta) = \gamma^2\eta^g = 0\) by vi) of Proposition 2.1 [5], \(\{i_0, 8, \nu\} (\sim \eta) = 0\). Finally, \(i_0\) induces a monomorphism \(i_0: [M^g, S^0] \to [M^g, M^0]\) again by [4, Theorem 3.1 and Theorem 3.3]. So \(\{8, \nu, \eta\} = 0\). Thus (2.3) is split. We summarize our result as follows.

**Proposition 2.5.** \(\pi^0(\mathcal{X}^3) = Z_6 \langle \nu \rangle \oplus Z_3 \langle \alpha_1 \rangle \oplus Z_2 \langle \nu \rho \eta^g \rangle \), where \(i^{*} (\nu) = \nu, i^{*} (\alpha_1) = \alpha_1\).

Analogously, we obtain the following.

**Proposition 2.6.** \(\pi^0(\mathcal{Y}^{11}) = Z_4 \langle \text{Ext } \epsilon - \sigma \text{Ext } \eta \rangle \oplus Z_3 \langle \text{Ext } \eta \rangle \oplus Z_3 \langle \alpha_1 \rangle \oplus Z_7 \langle \alpha_1 \gamma \rangle \).

**Proof.** \([4\epsilon, \text{Ext } \epsilon, \eta]\) \(4\epsilon = \{2\epsilon, 2\text{Ext } \epsilon, \eta\}\) \(4\epsilon = \{2\epsilon, \sigma \text{Ext } \eta, \eta\}\) \(4\epsilon = \{2\epsilon, \epsilon \eta, \eta\}\) \(4\epsilon = 4\zeta\) since \(\{2\epsilon, \epsilon \eta, \eta\} = \zeta + 2\pi_3[8, (9.4)]\). Therefore \(\{4\epsilon, \text{Ext } \epsilon, \eta\}\) contains the element \(\zeta\). Thus the extension of \(\text{Ext } \epsilon\), we denote it by \(\overline{\text{Ext } \epsilon}\), is the element of order 8. Similarly \(\sigma \text{Ext } \eta\) has the order 8. Finally, by (2.4) and Lemma A we obtain our proposition.

§ 3. The Determination of \(\pi^0(G_2)\)

Let \(\phi\) be a map given in [6]. Then there exists the cofibration as follows.

\[(3.1) \quad \mathcal{X}^3 \xrightarrow{i} \mathcal{Q} \xrightarrow{j} \mathcal{Y}^{11} \xrightarrow{\phi} \sum \mathcal{X}^3 (= \mathcal{X}^4).\]

Because first we see that \(\pi^1(\mathcal{Y}^{11})\) is easily seen to be zero and \(\pi^1(\mathcal{X}^3)\) contains only elements of order 2, on the other hand \(\phi\) is equal to \(2 (\Sigma \epsilon') \sigma j^g\) by [6, Theorem 4.12]. Then it is not hard to show that the following is exact.

\[(3.2) \quad 0 \xrightarrow{\pi^0(\mathcal{X}^3)} \pi^0(\mathcal{Q}) \xrightarrow{\pi^0(\mathcal{Y}^{11})} 0.\]

We have to determine this group extension. First we consider the
2-component. As in Section 2, we need to know Toda brackets \[ \{2 \nu, 2 \nu' \phi', \Sigma^{-1} \phi\} \] and \[ \{2 \nu, 2 \nu' \phi, \Sigma^{-1} \phi\} \supset \{2 \nu, 2 \nu' \phi, \Sigma^{-1} \phi\} \] contains zero since \( \phi = 2(\Sigma i') a j' \) and \( j' \) is order 2. Thus the \( Z_2 \)-summand splits. We claim that \( \{2 \nu, 2 \nu' \phi, \Sigma^{-1} \phi\} = 0 \) since without indeterminacy we obtain the equality: \( \{2 \nu, 2 \nu' \phi, \Sigma^{-1} \phi\} = \{2 \nu, 2 \nu' \phi, \Sigma^{-1} \phi\} \). Therefore \( Z_2 \)-summand also splits. As at the prime 3 \( (\Sigma^2 \cup \epsilon) \), we only have to consider the Toda bracket \( \{3 \alpha, \alpha_3, 2 \alpha_3\} \). By Theorem 11.4 [1], we see that its \( e_c \)-invariant, \( e_c \{3 \alpha, \alpha_3, 2 \alpha_3\} = -\delta(4, 6)/3 \mod Z \) and \( (1/3)Z \). As we may take \( \delta(4, 6) = 2 \cdot 5 \cdot 23 \cdot 7 \), our invariant is nontrivial. Thus we obtain a nontrivial extension on the 3-primary part. Now we complete the proof.

§ 4. The Ring Structure (Proof of Theorem 1.2)

To prove Theorem 1.2, we use the results of [3] and the spectral sequence of Atiyah-Hirzebruch associated to the filtration \( F^q(X) \), \( F^q(X) = \ker[\pi^q(X) \to \pi^q(X^{q-1})] \), \( X^{q-1} \) is a \((q-1)\)-skeleton of \( X \). Thus \( v, \alpha \in F^3 \), \( v \nu' j' \in F^6 \), \( \Ext \varepsilon, \sigma \Ext \eta \in F^8 \), \( 4 \Ext \varepsilon = 4 \sigma \Ext \eta = j'(\zeta) \), \( \alpha_1, \gamma \in F^{11} \), \( q^*(\alpha^2) \), \( q^*(\varepsilon) \in F^{14} \), where \( F^m = F^m(G_2) \). It is easy to see that all products except \( v^2 \), \( \alpha_1^2 \), \( v \nu' j' \), \( (v \nu' j')^2 \), \( (v \nu' j') \nu \), \( (v \nu' j') \nu \), \( v \nu \) and \( (v \nu' j') x \) \( (x = \Ext \varepsilon \) or \( \sigma \Ext \eta ) \), \( v^3 \), \( v^4 \), \( v \cdot j'(\zeta) \) are zero for filtration reasons.

In the Atiyah-Hirzebruch spectral sequence, \( E^{2, j}_{2} = H^j(G_2; \pi^j) \to \pi^{-j}(G_2) \).

\( v \in E^{3, 3}_{2} \) converges to \( v \). By the multiplicative properties, \( v^2 \in E^{3, 3}_{2} \) converges to \( v \nu' j' \), \( v^3 \in E^{3, 3}_{2} \) converges to \( v^3 \). Since \( v \nu' j'(\zeta) \) has the filtration 14 and corresponds to \( \nu j' = 0 \), it is trivial. Also relations \( \nu \sigma = \nu \varepsilon = 0 \) give the results \( (v \nu' j') x = 0 \), \( (x = \Ext \varepsilon \) or \( \sigma \Ext \eta ) \). On the other hand, the element \( v^2 \) is equal to \( v \nu' j' \) at filtration 6, \( v^3 \) and \( (v \nu' j') \nu \) corresponds to \( 2 \nu \) at \( F^9 \) since \( v^3 = \nu^2 a + \nu \varepsilon \) which is \( 2(\sigma \Ext \eta + \Ext \varepsilon) \) in \( \pi^9(M^9) \). In \( \pi^9(SU(3)) \) it has been proved that \( v^2 = v \), thus by the natural inclusion we obtain that \( v^2 = v \nu' j' + v + t \), where \( t \) is an element of higher filtration. As \( G_2 \) is stably self dual, we can apply Proposition 3.1 of [3]. Using this proposition, a composition \( S^{14} \to G_2 \otimes S^0 \otimes S^0 \to S^0 \otimes S^0 = S^0 \) is the Toda bracket \( \{v, \phi, v^*\} \), where \( d \) is a duality map and \( v^* \) means the dual of \( v \). The bracket \( \{v, \phi, v^*\} \)
contains zero since $2 \{v, i'\sigma S^{-1}j', \sigma^*\} = 0$ $(\sigma_{11}^2(S^0) = (2)^2)$. Thus the restriction of $t$ to the top cell $(=S^0)$ is trivial. This is 1). Similarly, $(\nu^2p j')v \equiv 2\nu$ mod $j'(\xi)$ since $\{v, \phi, (\nu^2p j')^*\}$ also contains zero. Moreover the element $(\nu^2p j')v$ can not involve $j'(\xi)$ by the $c$-invariant argument. Namely, we define $c$-invariant on $[Q, S^0]$ and $[Y^{11}, S^0]$ in terms of the Chern character as in [6], so that we obtain the following commutative diagram.

\[ e_c: [Q, S^0] \longrightarrow Q/2Z \oplus Q/\frac{1}{2}Z \]

\[ e_c: [Y^{11}, S^0] \longrightarrow Q/\frac{1}{2}Z, \]

in which vertical arrows are monic. On $[Y^{11}, S^0]$, $e_c(j'(\xi)) = 1/4$ mod $(1/2)Z$, thus $e_c$ of $j'(\xi)$ on $[Q, S^0]$ is also nontrivial. Since we can easily see that $e_c((\nu^2p j')v) = c(2\nu) = 0$, we obtain our result.

Part 3). As $\nu^3 = (\nu^2p j' + \nu) = 2\nu + v\nu$ by 1) and 2), we have to determine $\nu v$. Since this element has the filtration 14, we can use the similar method as above to obtain that at the top cell $\nu v$ is equal to the bracket $[v, \text{Ext}\phi, \text{Coext}\phi]$ which is $\kappa$ by [8] p. 96. Namely $\nu^4$ and $(\nu^2p j')^2$ are also seen to be trivial.

(Odd prime case). It is well known that at the prime 3, $G_2$ is equivalent to $(S^3 / e^{11}) \cup e^{14}$. We obtain the following homotopy commutative diagram.

\[
\begin{array}{ccc}
C & \overset{d}{\longrightarrow} & C \wedge e^{11} \\
\downarrow & & \downarrow \\
S^{11} & \overset{g}{\longrightarrow} & C \wedge S^0 \\
\end{array}
\]

where $d$ is the diagonal map, $C = S^9 \cup e^{11}$, $g$ is a representative of the restriction of $\alpha_1$ to $C$. Obviously, there exists $\tilde{d}$ which makes this diagram commutative. We observe that $\pi_{11}^0(C \wedge C) = 0$, thus the top rows of the diagram are trivial. Therefore $\alpha_1^2$ is contained in $F^{12}(G_2)$. Since $\pi_{11}^0(S^6) = 0$ we can conclude that $\alpha_1^2 = 0$.

Let $[G_2, L]$ be a stable homotopy element obtained by applying the Pontryagin–Thom construction to the left invariant framing $L$ of
By [7], [10], it has been shown that \([G_2, L] = \kappa\). Also in [2], this fact is stated without the full proof. Combining our theorem above with the method in [2], we can easily obtain the result.

**Corollary 4.1.** ([7], [10] and [2]). \([G_2, L] = \kappa\).

**Proof.** \(q^*[G_2, L] = J_R(J_R - 2)\) by [2, (5.4) Theorem (a)], where \(J_R\) is the Hopf construction of 7-dimensional representation of \(G_2\). As it is seen by the natural inclusion \(SU(3) \to G_2\) that \(J_R = \pm 9 + t\), \(t\) an element of higher filtration. Thus \(q^*[G_2, L] = 2\varphi_2 \pm \varphi_3 = q^*\kappa\) by our theorem above.

**References**