Some Cyclic Group Actions on a Homotopy Sphere and the Parallelizability of its Orbit Spaces

By

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§ 1. Introduction

In this paper, we introduce a way to define a free cyclic group action on a homotopy sphere and examine the stable parallelizability of its orbit spaces. J. Ewing et al [3] answered the stable parallelizability problem for the classical lens space, that is, the orbit space of the standard sphere under a linear cyclic group action.

Let $w_1, w_2, \ldots, w_{n+1}$ be positive rational numbers. A polynomial $f(z_1, z_2, \ldots, z_{n+1})$ is called a weighted homogeneous polynomial of type $(w_1, w_2, \ldots, w_{n+1})$ if it can be expressed as a linear combination of monomials

$$z_1^i z_2^j \cdots z_{n+1}^k$$

for which $\sum_{j=1}^{n+1} i_j / w_j = 1$. This is equivalent to the requirement that

$$f(e^{c/w_1} z_1, e^{c/w_2} z_2, \ldots, e^{c/w_{n+1}} z_{n+1}) = e^c f(z_1, z_2, \ldots, z_{n+1})$$

for every complex number $c$.

Throughout this paper, we assume that all weighted homogeneous polynomials have an isolated critical point at the origin. For example, the Brieskorn polynomial

$$f(z_1, z_2, \ldots, z_{n+1}) = z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+1}^{a_{n+1}},$$

all $a_i \geq 2$, is a weighted homogeneous polynomial of weights $w = (a_i, \ldots, a_{n+1})$. 

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Set $\Sigma_w = f^{-1}(0) \cap S^{2n+1}$, and consider the Milnor fibering $g: S^{2n+1} \rightarrow \Sigma_w$ defined by

$$g(z_1, \ldots, z_{n+1}) = f(z_1, \ldots, z_{n+1}) / |f(z_1, \ldots, z_{n+1})|,$$

then each fiber $F_t = g^{-1}(e^t)$ is a smooth parallelizable $2n$-dimensional manifold with the homotopy type of a bouquet of $n$-spheres. We can obtain $S^{2n+1} - \Sigma_w$ from $F \times [0, 1]$ by identifying $F \times 0$ and $F \times 1$ by a homeomorphism $h: F \rightarrow F$ called the characteristic map. Denote the characteristic polynomial of $h^*: H_n(F; \mathbb{C}) \rightarrow H_n(F; \mathbb{C})$ by

$$A(t) = \text{determinant } (tI_n - h^*),$$

where $I$ is the identity map of $F$. This characteristic map $h_*$ and its characteristic polynomial $\Delta(t)$ are fundamental topological invariants. Brieskorn [2] computed $\Delta(t)$ for varieties defined by Brieskorn polynomials, and Milnor and Orlik [9] did it for weighted homogeneous polynomials.

The following theorem answers whether or not the $2n-1$ dimensional manifold $\Sigma_w = f^{-1}(0) \cap S^{2n+1}$ is a topological sphere.

**Theorem** ([8], Section 8). For $n \geq 3$, the followings are equivalent:

i) $\Sigma_w$ is a topological sphere.

ii) $H_{n-1}(\Sigma_w) = 0$.

iii) The intersection pairing $H_n(F; \mathbb{Z}) \otimes H_n(F; \mathbb{Z}) \rightarrow \mathbb{Z}$ has determinant $\pm 1$.

iv) $\Delta(1) = \pm 1$.

Furthermore, if $\Sigma_w$ is a topological sphere, the diffeomorphism class of $\Sigma_w$ is completely determined by the signature of the intersection pairing

$$H_n(F; \mathbb{Z}) \otimes H_n(F; \mathbb{Z}) \rightarrow \mathbb{Z}$$

if $n$ is even. If $n$ is odd, $\Sigma_w$ is

- the standard sphere if $\Delta(-1) = \pm 1 \pmod{8}$,
- the Kervaire sphere if $\Delta(-1) = \pm 3 \pmod{8}$.

Let $\Sigma_w = f^{-1}(0) \cap S^{2n+1}$ be a topological sphere, where $f$ is a weighted
homogeneous polynomial of weight $w = (w_1, w_2, \ldots, w_{n+1})$, say $w_i = u_i/v_i$ in irreducible form for $i=1,2,\ldots,n+1$, and let $p$ be an odd prime number relatively prime to each $u_i$. To define a free cyclic group $\mathbb{Z}_p$-action on $\Sigma_w$, choose natural numbers $b_i$ such that $b_i = h/w_i = h v_i u_i^{-1} \pmod{p}$ for some $h \neq 0 \pmod{p}$ and $(b_i, p) = 1$ for all $i$, where $(b_i, p)$ denotes the greatest common divisor of $b_i$ and $p$. Now, we define a map $T$ on $\Sigma_w$ by

$$T(z_1, z_2, \ldots, z_{n+1}) = (\zeta^{b_1} z_1, \zeta^{b_2} z_2, \ldots, \zeta^{b_{n+1}} z_{n+1}),$$

where $\zeta = e^{2\pi i/p}$. Then

$$f(T(z_1, z_2, \ldots, z_{n+1})) = \zeta^k f(z_1, z_2, \ldots, z_{n+1}).$$

This is a well-defined free action on $\Sigma_w$ generating the cyclic group $\mathbb{Z}_p$. Denote its orbit space by $L(p; w; b)$. Note that we may assume that $h = 1 \pmod{p}$, i.e., $w_i b_i = 1 \pmod{p}$ for all $i$ by taking a suitable generator $T$ of $\mathbb{Z}_p$.

§ 2. An Algebraic Characterization of Stable Parallelizability

Define a $\mathbb{Z}_p$-action on $\Sigma_w \times C$ by $T'(z, \eta) = (T(z), \zeta^k \eta)$, where $\zeta$ and $T(z)$ are the same as above, so that the natural projection from $\Sigma_w \times C$ to $\Sigma_w$ is equivariant, that is, it commutes with the $\mathbb{Z}_p$-actions. By taking quotients, one can get the canonical complex line bundle $\gamma$ over $L(p; w; b)$. Similarly, one can get $\gamma^b = \gamma \otimes \gamma \otimes \cdots \otimes \gamma$, $(b$ times) with a $\mathbb{Z}_p$-action on $\Sigma_w \times C$ given by $T'(z, \eta) = (T(z), \zeta^b \eta)$. It can be proved easily that

$$\gamma^{b_1} \otimes \gamma^{b_2} \cdots \otimes \gamma^{b_{n+1}} = \Sigma_w \times C^{n+1} / T \times T,$$

where

$$(T \times T) (z, (\eta_1, \eta_2, \ldots, \eta_{n+1})) = (T(z), (\zeta^{b_1} \eta_1, \zeta^{b_2} \eta_2, \ldots, \zeta^{b_{n+1}} \eta_{n+1})).$$

To reduce the question of stable parallelizability of the orbit space $L(p; w; b)$ to a purely algebraic one, we first describe the tangent bundle of $L(p; w; b)$.

Theorem 2.1. Over $L(p; w; b)$, $T \otimes e \otimes \text{re}(\gamma)$ is isomorphic to

$$\text{re}(\gamma^{b_1} \otimes \gamma^{b_2} \cdots \otimes \gamma^{b_{n+1}}),$$

where $T$ denotes the tangent bundle, $e$ the trivial 1-dimensional real bundle.
over $L(p;w;b)$, and $\text{re}$ the realification of a bundle.

Proof. Let $\tau(.)$ denote the tangent bundle and $\nu(.)$ the normal bundle of the space $(.)$ in $\mathbb{C}^{n+1}$, then the trivial bundle $\sum_w \times \mathbb{C}^{n+1}$ is isomorphic to
\[ \tau(\sum_w) \oplus \nu(\sum_w) = \tau(\sum_w) \oplus \nu(S^{2n+1}) \oplus \nu(f^{-1}(0)) \]
over $\sum_w$. But $\nu(S^{2n+1})$ is trivial and $\text{grad} f$ is a cross section of $\nu(f^{-1}(0))$, so that $\nu(f^{-1}(0)) = \mathbb{C} \cdot \text{grad} f$. Define
\[ \Phi: \tau(\sum_w) \oplus R \oplus C \longrightarrow \sum_w \times \mathbb{C}^{n+1} \]
by
\[ \Phi(v_z, r, \eta) = (z, v + rz + \eta \cdot \text{grad} f(z)), \]
where $v_z$ denotes a tangent vector at $z$ and $R, C$ represent the trivial bundles $R \times \sum_w, C \times \sum_w$ respectively. By using $\zeta \cdot \text{grad} f(Tz) = T(\text{grad} f(z))$, we can see that $\Phi$ is an equivariant isomorphism from $\tau(\sum_w) \oplus R \oplus C$ with $\mathbb{Z}_p$-action given by $dT \oplus I \oplus (-\zeta)$ to $\sum_w \times \mathbb{C}^{n+1}$ with $\mathbb{Z}_p$-action given by $T \times T$. Therefore, by taking quotients, it is proved.

Remark. In Theorem 2.1, if $L(p;w;b)$ is defined as an orbit space of a Brieskorn sphere, then we have
\[ \tau \oplus e \oplus \text{re}(\gamma) \simeq \text{re}(\gamma^1 \oplus \gamma^2 \oplus \ldots \oplus \gamma^{n+1}). \]
This is the correction of Orlik's theorem 3 ([12], p. 252).

Recall that the standard lens space $L^{2n-1}(p)$ is defined as the orbit space of $S^{2n-1}$ by the linear action. Since the principal $\mathbb{Z}_p$-bundles
\[ S^{2n-1} \longrightarrow L^{2n-1}(p) \quad \text{and} \quad \sum_w \longrightarrow L(p;w;b) \]
are $2n-1$ universal, there are maps
\[ f: L^{2n-1}(p) \longrightarrow L(p;w;b) \quad \text{and} \quad g: L(p;w;b) \longrightarrow L^{2n-1}(p) \]
such that the induced bundles $f^*\gamma = \gamma$ and $g^*\gamma = \gamma$, where $\gamma$ is the canonical bundle over the suitable orbit space. Hence, Theorem 2.1 implies the following:

Lemma 2.2. The space $L(p;w;b)$ is stably parallelizable if and only if $\text{re}(\gamma)$ is stably isomorphic to
\[ \text{re}(\gamma^1) \oplus \text{re}(\gamma^2) \oplus \ldots \oplus \text{re}(\gamma^{n+1}) \]
over the standard lens space \(L^{2n-1}(p)\), where \(\gamma\) represents the canonical bundle over \(L^{2n-1}(p)\).

Recall that the mod \(p\) cohomology ring of the standard lens space \(L^{2n-1}(p)\) is the tensor product

\[ H^*(L^{2n-1}(p) ; \mathbb{Z}_p) \cong \Lambda(u) \otimes \mathbb{Z}_p[v]/(v^n) \]

of the exterior algebra \(\Lambda(u)\) and the truncated polynomial ring generated by \(v\), where \(\deg u = 1\), \(\deg v = 2\), and \(\beta^*_p(u) = v\) for the Bockstein isomorphism

\[ \beta^*_p : H^1(L^{2n-1}(p) ; \mathbb{Z}_p) \longrightarrow H^2(L^{2n-1}(p) ; \mathbb{Z}_p). \]

**Lemma 2.3.** If the space \(L(p ; w ; b)\) is stably parallelizable, then

\[ 1 + v^2 = (1 + b_1^2 v^2)(1 + b_2^2 v^2) \cdots (1 + b_{n-1}^2 v^2) \]

in \(\mathbb{Z}_p[v]/(v^n)\).

**Proof.** By Lemma 2.2 and the hypothesis, the mod \(p\) reduction of the total Pontrjagin class of \(re(\gamma)\) is equal to that of

\[ re(\gamma^1) \oplus re(\gamma^2) \oplus \ldots \oplus re(\gamma^{n+1}), \]

where \(\gamma\) is the canonical line bundle over \(L^{2n-1}(p)\). Let \(\xi\) be the canonical line bundle over \(CP(n-1)\), then the induced bundle \(\pi^*(\xi)\) over \(L^{2n-1}(p)\) is clearly the line bundle \(\gamma\), where \(\pi : L^{2n-1}(p) \longrightarrow CP(n-1)\) is the natural projection. Note that \(H^*(CP(n-1) ; \mathbb{Z}_p) \cong \mathbb{Z}_p[w]/(w^n)\). The Gysin sequence of the principal bundle \(S^1 \longrightarrow L^{2n-1}(p) \longrightarrow CP(n-1)\) with \(\mathbb{Z}_p\) coefficients is

\[ \cdots \longrightarrow H^2(CP(n-1)) \xrightarrow{\pi^*} H^1(L^{2n-1}(p)) \longrightarrow H^0(CP(n-1)) \]

\[ \cdots \longrightarrow H^2(CP(n-1)) \xrightarrow{\pi^*} H^1(L^{2n-1}(p)) \longrightarrow H^0(CP(n-1)), \]

in which \(H^0(CP(n-1)) \cong H^2(L^{2n-1}(p))\) must be an isomorphism. By the naturality of Chern classes,

\[ c_1(\gamma) = c_1(\pi^*(\xi)) = \pi^*(c_1(\xi)) = \pi^*(w) = v. \]

The first Pontrjagin class \(P_1(\text{re}(\gamma))\) comes from the identity

\[ 1 - P_1(\text{re}(\gamma)) = (1 - c_1(\gamma))(1 + c_1(\gamma)) = 1 - v^2. \]

Hence, the total Pontrjagin class of \(\text{re}(\gamma)\) in mod \(p\) is \(P(\text{re}(\gamma)) = 1 + P_1(\text{re}(\gamma)) = 1 + v^2.\) Since \(c_1(\mu \otimes \nu) = c_1(\mu) + c_1(\nu)\) for any line bundles \(\mu, \nu,\)
Therefore, \( P(\text{re}(\gamma_j)) = 1 + b_j^2 \nu^2 \), and

\[
P(\text{re}(\gamma_j)) = 1 + b_j^2 \nu^2
\]

in \( \mathbb{Z}_p[\nu]/(\nu^n) \), by the product formula of the Pontrjagin class.

From theorem 2.1, one can also get the total Pontrjagin and Stiefel-Whitney classes of the space \( L(p; w; b) \).

**Corollary 2.4.**

\[
P(L(p; w; b)) = (1 + \nu^2)^{-1} \prod_{i=1}^{n+1} (1 + b_i \nu^2),
\]

\[
w(L(p; w; b)) = (1 + \nu)^{-1} \prod_{i=1}^{n+1} (1 + b_i \nu),
\]

where \( \nu \) is a preferred generator for \( H^2(L(p; w; b); \mathbb{Z}) \), so that the total Chern class of \( \gamma \) is \( 1 + \nu \), and \( u \) is its mod 2 reduction.

In [5], \( \widetilde{KO}(L^{2n-1}(p)) \) is computed. Setting \( \delta = \text{re}(\gamma) - 2 \), the \( p \)-torsion part of \( \widetilde{KO}(L^{2n-1}(p)) \) is a direct summand of cyclic groups generated by \( \delta^i \), \( 1 \leq i \leq (p-1)/2 \), where if \( n-1 = s(p-1) + r \), \( 0 \leq r < p-1 \), the order of \( \delta^i \) is \( p^{i+1} \) for \( i \leq [r/2] \) and \( p^i \) for \( i > [r/2] \).

**Lemma 2.5.** If \( L(p; w; b) \) is stably parallelizable, then \( n-1 \) is less than \( p \).

**Proof.** Let \( L(p; w; b) \) be stably parallelizable, then \( \text{re}(\gamma) \) is stably isomorphic to

\[
\text{re}(\gamma^1) \oplus \text{re}(\gamma^2) \oplus \ldots \oplus \text{re}(\gamma^{n+1})
\]

over the standard lens space \( L^{2n-1}(p) \), which gives

\[
\text{re}(\gamma) - 2 = \text{re}(\gamma^1) - 2 + \ldots + \text{re}(\gamma^{n+1}) - 2
\]

in \( \widetilde{KO}(L^{2n-1}(p)) \). Since \( \widetilde{KO}(L^{2n-1}(p)) \) is abelian, we can assume that \( b_1 \leq b_2 \leq \ldots \leq b_{n+1} \). By taking the diffeomorphic copy of \( L(p; w; b) \) under the complex conjugation of the \( i \)-th coordinate, if it is needed,
we may assume that \( b_1 \leq b_2 \leq \ldots \leq b_{n+1} \leq (p-1)/2 \). Let \( n-1 = s(p-1) + r, 0 \leq r < p-1 \), and set \( \bar{a} = \text{re}(\gamma) - 2 \), then \( \bar{a}^i, 1 \leq i \leq (p-1)/2 \), are generators of the cyclic subgroups of the \( p \)-torsion part of \( \tilde{K}(L^{2n-1}(p)) \), and their orders are \( p' \) or \( p' + 1 \). On the other hand,

\[
(\text{re}(\gamma^{b_1}) - 2) + (\text{re}(\gamma^{b_2}) - 2) + \ldots + (\text{re}(\gamma^{b_{n+1}}) - 2)
\]

can be written as a polynomial of \( \bar{a} \). So, we can set

\[
\bar{a} = \alpha_{s_{n+1}} + \alpha_{s_{n+1}-1} \bar{a} + \ldots + \alpha_0 \bar{a}^{s_{n+1}}
\]

with some coefficients \( \alpha_i \)'s, so that \( \alpha_{s_{n+1}-1} = 1 \pmod{p'} \), and all other coefficients are divided by \( p' \). And \( \alpha_0 \) is also the number of \( b_j \)'s such that \( b_j = b_{n+1} \) in \( b_1 \leq b_2 \leq \ldots \leq b_{n+1} \), because

\[
\text{re}(\gamma^{b_{n+1}}) - 2 = \bar{a}^{s_{n+1}} + \text{terms of lower degree of } \bar{a}.
\]

Similarly, for any \( b \) with \( 1 \leq b \leq (p-1)/2 \), the number of copies of \( \text{re}(\gamma^b) - 2 \) in

\[
(\text{re}(\gamma^{b_1}) - 2) + (\text{re}(\gamma^{b_2}) - 2) + \ldots + (\text{re}(\gamma^{b_{n+1}}) - 2)
\]

must be divided by \( p' \). Now, let \( \beta \) be the number of copies of \( \text{re}(\gamma) - 2 \) in

\[
(\text{re}(\gamma^{b_1}) - 2) + (\text{re}(\gamma^{b_2}) - 2) + \ldots + (\text{re}(\gamma^{b_{n+1}}) - 2),
\]

then \( \beta + \beta' = \alpha_{s_{n+1}-1} = 1 \pmod{p'} \), where \( \beta' \) is the coefficient of \( \bar{a} \) in the polynomial of \( \bar{a} \) for

\[
(\text{re}(\gamma^{b_1}) - 2) + \ldots + (\text{re}(\gamma^{b_{n+1}}) - 2) - \beta(\text{re}(\gamma) - 2).
\]

On the other hand, \( \beta' \) is divided by \( p' \), so \( \beta = 1 \pmod{p'} \). Since the total number of \( b_i \)'s is \( n+1 \), \( \beta + hp' = n+1 \) for some \( h \), so \( n = s(p-1) + r + 1 \pmod{p'} \). The only possibility is \( s = 0 \), or \( s = 1 \) and \( r = 0 \). In both cases, \( n-1 \) is less than \( p' \).

The next lemma will be useful to prove the main theorem.

**Lemma 2.6([3]).** Let \( \xi, \eta \) be oriented vector bundles over a finite CW complex \( X \), and suppose that

i) \( \dim(X) < 2p + 2 \), \( p \) an odd prime, and

ii) \( H^{*}(X; \mathbb{Z}) \) has no \( q \)-torsion for any \( q < p \).

If their Pontrjagin classes \( P(\xi), P(\eta) \) are equal, then \( (\xi - \eta) - (\dim \xi - \dim \eta) \in \tilde{K}(X) \) is a 2-torsion element.
Theorem 2.7. The space $L(p; w; b)$ is stably parallelizable if and only if

i) $n - 1$ is less than $p$, and

ii) $(1 + b_2 v^2)(1 + b_3 v^2) \cdots (1 + b_{n+1} v^2) = 1 + v^2$ in $\mathbb{Z}_p[v]/(v^2)$, or equivalently

$$b_1^{2j} + b_2^{2j} + \ldots + b_{n+1}^{2j} = 1 \pmod{p} \text{ for } j = 1, 2, \ldots, [\frac{n-1}{2}].$$

Proof. The “only if” part comes from Lemmas 2.3–2.5. Let us assume i) and ii). Then, the mod $p$ Pontrjagin class of $re(\gamma)$ is equal to that of $re(\gamma^1) \oplus re(\gamma^2) \oplus \ldots \oplus re(\gamma^{n+1})$. By Lemma 2.6,

$$re(-\gamma \oplus \gamma^1 \oplus \gamma^2 \oplus \ldots \oplus \gamma^{n+1}) - 2n$$

is a 2–torsion element in $\widetilde{KO}(L^{n-1}(p))$. But it is clearly in the image of

$$re: \widetilde{K}(L^{n-1}(p)) \longrightarrow \widetilde{KO}(L^{n-1}(p)),$$

which does not contain any 2–torsion element. So it must be a zero element. Therefore, $re(\gamma)$ is stably isomorphic to $re(\gamma^1) \oplus re(\gamma^2) \oplus \ldots \oplus re(\gamma^{n+1})$ over $L^{n-1}(p)$, and $L(p; w; b)$ is stably parallelizable.

§3. Some Examples

Milnor and Orlik [9] gave the computation of $A(1)$ as follows: Let $C^* = C - \{0\}$ denote the group with the multiplication. To each monic polynomial

$$(t - \alpha_1) (t - \alpha_2) \cdots (t - \alpha_k), \quad \alpha_i \in C^*,$$

assign the divisor

$$\text{divisor } ((t - \alpha_1) (t - \alpha_2) \cdots (t - \alpha_k))$$

$$= \langle \alpha_1 \rangle + \langle \alpha_2 \rangle + \ldots + \langle \alpha_k \rangle$$

as an element of the rational group ring $QC^*$. Denote

$$A_k = \text{divisor } (t^k - 1)$$

$$= \langle 1 \rangle + \langle \xi \rangle + \ldots + \langle \xi^{k-1} \rangle,$$

where $\xi = e^{2\pi i/k}$. Note that $A_k A_{k+1} = (a, b) A_{[a, b]_C}$, where $[a, b]$ denotes their least common multiple and $(a, b)$ the greatest common divisor. Then, for a weighted homogeneous polynomial $f(z_1, z_2, \ldots, z_{n+1})$ of type $w = (w_1, w_2, \ldots, w_{n+1})$, the characteristic polynomial $A(t) = \text{determinant}$
(tI_n - h_n) of the linear transformation $h_n: H_n(F; C) \rightarrow H_n(F; C)$ is determined by

\[ \Delta = (v_1^{-1} A_{u_1} - 1) (v_2^{-1} A_{u_2} - 1) \cdots (v_{n+1}^{-1} A_{u_{n+1}} - 1), \]

where $w_i = u_i/v_i$, $i = 1, 2, \ldots, n+1$, is the expression in irreducible form.

To make the computation of $\Delta(1)$ easy, we cite two Milnor–Orlik’s theorems.

**Theorem 3.1** ([9]). By using $\Delta_a \Delta_b = (a, b) \Delta_{a+b}$, divisor $\Delta$ can be expressed as a linear combination of the divisors $\Delta_a$. Let

\[ \text{divisor } \Delta = a_1 \Delta_1 + a_2 \Delta_2 + \cdots + a_s \Delta_s, \]

and define two numbers $k(\Delta)$ and $\rho(\Delta)$ by the formula

\[ k(\Delta) = a_1 + a_2 + \cdots + a_s, \quad \text{and } \rho(\Delta) = a_1 a_2 \cdots a_s. \]

Then, $k(\Delta)$ and $\rho(\Delta)$ are non-negative integers, and

\[ \Delta(1) = \rho(\Delta) \quad \text{if } k(\Delta) = 0, \]
\[ \Delta(1) = 0 \quad \text{if } k(\Delta) \neq 0. \]

**Theorem 3.2** ([9]). Let

\[ f(z_1, \ldots, z_{n+1}) = f_1(z_1, \ldots, z_k) + f_2(z_{k+1}, \ldots, z_{n+1}) \]

where $f_1$ and $f_2$ are weighted homogeneous polynomials, and let $\Delta_1$ and $\Delta_2$ be the characteristic polynomials associated to $f_1$ and $f_2$. For the weight $w = (w_1, w_2, \ldots, w_{n+1})$, express $w_i = u_i/v_i$, $i = 1, 2, \ldots, n+1$, in an irreducible form. Suppose that each of the numbers $u_1, \ldots, u_k$ is relatively prime to each of $u_{k+1}, \ldots, u_{n+1}$. Then the numbers $k(\Delta)$, $\rho(\Delta)$ correspond to the polynomial $f = f_1 + f_2$ are determined by the integers $k_j = k(\Delta_j)$ and $\rho_j = \rho(\Delta_j)$ correspond to $f_j$, $j = 1, 2$ according to the formulas

\[ k(\Delta) = k_1 k_2 \quad \text{and } \rho(\Delta) = \rho_1 \rho_2. \]

The next theorems show how one can construct topological spheres using the weighted homogeneous polynomial.

**Theorem 3.3.** Let $g(z_1, z_2, \ldots, z_m)$ be a weighted homogeneous polynomial with weight $w = (w_1, w_2, \ldots, w_m)$, $w_i = u_i/v_i$ as before, $i = 1, 2, \ldots, m$. Choose any two positive integers $w_{m+1}$ and $w_{m+2}$ such that $(w_{m+j}, u_i) = 1$ for all $i = 1, 2, \ldots, m; j = 1, 2$. Then a polynomial $f$ defined by
f(z_1, \ldots, z_m, z_{m+1}, z_{m+2}) = g(z_1, \ldots, z_m) + z_{m+1}^{w_{m+1}} + z_{m+2}^{w_{m+2}}

is also a weighted homogeneous polynomial of weight \((w_i)\), and \(\Sigma_w = f^{-1}(0) \cap S^{2m+3}\) is a topological sphere.

Proof. Let \(k, k(g), k_1, k_2\) and \(\rho, \rho(g), \rho_1, \rho_2\) be numbers defined in Theorem 3.1 associated to \(f, g, z_{m+1}^{w_{m+1}}, z_{m+2}^{w_{m+2}}\) respectively. Clearly, divisor \(D_i = A_{w_i} - 1\), for \(i = 1, 2\), so that \(k_1 = 1 - 1 = 0\). Hence \(k = k_1k_2k(g) = 0\), and then \(A(1) = \rho = (\rho_g^{-1} \rho_k k(g)) b_2 \rho_2 k(g) = 1\). Therefore, \(\Sigma_w\) is a topological sphere.

Theorem 3.4 ([11]). Let \(g(z)\) be a weighted homogeneous polynomial in \(\mathbb{C}^n\) with an isolated critical point at the origin, and let \(f(z, w)\) be a weighted homogeneous polynomial in \(\mathbb{C}^n \times \mathbb{C}^2\) defined by \(f(z, w) = g(z) + w_1w_2\). Then \(g^{-1}(0) \cap S^{2n-1}\) is a topological sphere if and only if \(f^{-1}(0) \cap S^{2n+3}\) is a topological sphere. (Here, \(n > 3\)).

We conclude with an example. Let

\[ f(z_1, z_2, \ldots, z_7) = f_1(z_1, \ldots, z_5) + f_2(z_6, z_7), \]

where

\[ f_1(z_1, z_2, \ldots, z_5) = z_1^3 + z_2^{k-1} + z_3^2 + z_4^2 + z_5^2, \]
\[ f_2(z_6, z_7) = z_6z_7. \]

Then, \(f\) is a weighted homogeneous polynomial with weight \((w_i) = (3, 6k - 1, 2, 2, 2, 1/2, 1/2)\). By Theorem 3.4, \(\Sigma_w = f^{-1}(0) \cap S^3\) is an 11-dimensional topological sphere. First, we are interested in the diffeomorphic type of this sphere \(\Sigma_w\). Let \(F, F_1\), and \(F_2\) be the fibre in the Milnor’s fibering corresponding to the polynomials \(f, f_1,\) and \(f_2\) respectively. Then \(F, F_1\), and \(F_2\) are diffeomorphic to \(f^{-1}(1), f_1^{-1}(1),\) and \(f_2^{-1}(1)\) respectively (cf. [8], Lemma 9.4.), and \(f^{-1}(1)\) is homotopy equivalent to the join \(f_1^{-1}(1) \ast f_2^{-1}(1)\) (cf. [11]). Note that \(f_2^{-1}(1)\) has the same homotopy type as \(S^3\). Hence,

\[ H_5(F; \mathbb{Z}) = H_5(F_1 \ast F_2; \mathbb{Z}) = \sum_{i+j=5} \tilde{H}_i(F_1; \mathbb{Z}) \otimes \tilde{H}_j(F_2; \mathbb{Z}) = \sum_{p+q=4} \tilde{H}_p(F_1; \mathbb{Z}) \otimes \tilde{H}_q(F_2; \mathbb{Z}) = H_4(F_1; \mathbb{Z}) \otimes H_1(F_2; \mathbb{Z}) = H_4(F_1; \mathbb{Z}). \]

(See [7] for the 2nd isomorphism). Hence, the signature of the intersection pairing of \(F\) is equal to that of \(F_1\). Also it is well-known
that \( f^{-1}(0) \cap S^8 = k \cdot g_2 \) and the signature of \( F_1 \) is equal to \( 8k \), where \( g_2 \) is a generator of the cyclic group of all 28 7-dimensional homotopy spheres. Therefore, we get \( \sum w = f^{-1}(0) \cap S^9 = k \cdot g_3 \) for a generator of the cyclic group of all 992 11-dimensional homotopy spheres.

To get a cyclic group action on these spheres which induces stably parallelizable orbit spaces, it is required to choose a prime \( p \) and numbers \( b_i \)'s such that

\[
\begin{align*}
    w_1b_1 &= w_2b_2 = \ldots = w_7b_7 \pmod{p}, \\
    b_1^2 + b_2^2 + \ldots + b_7^2 &= 1 \pmod{p}, \\
    b_1^4 + b_2^4 + \ldots + b_7^4 &= 1 \pmod{p}.
\end{align*}
\]

Hence,

\[
\begin{align*}
    (1/3)^2 + b_2^2 + 3(1/2)^2 &= -7 \pmod{p}, \\
    (1/3)^4 + b_2^4 + 3(1/2)^4 &= -31 \pmod{p}
\end{align*}
\]

must be satisfied. Accordingly, \( 120524 = 0 \pmod{p} \), so \( p = 29 \) or \( p = 1039 \).

For example, if \( p = 29 \), then we can take

\( (b_1, b_2, \ldots, b_7) = (10, 1, 15, 15, 2, 2) \),

and then, for \( k = 10 + 29q, q = 1, 2, \ldots, 992 \), \( \sum w \) represent all 992 11-dimensional homotopy spheres. Furthermore, on these 992 homotopy spheres, the cyclic group action defined by the given \( b_i \)'s is well defined, and all their orbit spaces are stably parallelizable.

**Bibliography**


