Bernstein Polynomials of a Smooth Function Restricted to an Isolated Hypersurface Singularity

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Abstract

Let $f, g$ be two germs of holomorphic functions on $\mathbb{C}^n$ such that $f$ is smooth at the origin and $(f, g)$ defines an analytic complete intersection $(Z, 0)$ of codimension two. We study Bernstein polynomials of $f$ associated with sections of the local cohomology module with support in $X = g^{-1}(0)$, and in particular some sections of its minimal extension. When $(X, 0)$ and $(Z, 0)$ have an isolated singularity, this may be reduced to the study of a minimal polynomial of an endomorphism on a finite dimensional vector space. As an application, we give an effective algorithm to compute those Bernstein polynomials when $f$ is a coordinate and $g$ is non-degenerate with respect to its Newton boundary.

§1. Introduction

Let $n \geq 2$ be an integer. Let us denote $\mathcal{O} = \mathbb{C}\{x_1, \ldots, x_n\}$ the ring of germs at 0 of complex holomorphic functions, and $\mathcal{D} = \mathcal{O}(\partial/\partial x_1, \ldots, \partial/\partial x_n)$ the ring of linear differential operators with holomorphic coefficients.

Let $g \in \mathcal{O}$ be a nonzero germ such that $g(0) = 0$, and $\mathcal{R} = \mathcal{O}[1/g]/\mathcal{O}$ the local cohomology module with support in the hypersurface $(X, 0) \subset (\mathbb{C}^n, 0)$ defined by $g$. It is a regular holonomic $\mathcal{D}$-module such that its complex of holomorphic solutions is the perverse sheaf $\mathcal{C}_X[-1]$ (see [5], [6], [14]).

Given a germ of function $f \in \mathcal{O}$ nonzero on $X$, there are functional equations in $\mathcal{R}[1/f, s]f^s = \mathcal{R} \otimes_{\mathcal{O}} \mathcal{O}[1/f, s]f^s$ of the form:

$$b(s)f^s = P \cdot f^{s+1}$$
for every $\delta \in \mathbb{R}$, with $b(s) \in \mathbb{C}[s]$ nonzero and $P \in \mathcal{D}[s] = \mathcal{D} \otimes \mathbb{C}[s]$ (see [6]). We call Bernstein polynomial of $f$ associated with $\delta$, and we denote $b(\delta f^s, s)$, the unitary generator of the ideal of polynomials $b(s)$ verifying such an identity. When $f$ is not a unit, it is easy to check that $(s + r(\delta) + 1)$ is a factor of $b(\delta f^s, s)$, where $r(\delta) \in \mathbb{N}$ is such that $\delta \in f^{r(\delta)} \mathcal{R} - f^{r(\delta) + 1} \mathcal{R}$; let us denote $b(\delta f^s, s) \in \mathbb{C}[s]$ the quotient of $b(\delta f^s, s)$ by $(s + r(\delta) + 1)$.

Because of the algebraic theory of vanishing cycles, roots of these polynomials determine the eigenvalues of the monodromy of $f|_X : (X, 0) \to (\mathbb{C}, 0)$ (see [7], [12], and [20] for examples). In particular, the singular monodromy theorem implies that their roots are rational numbers ([8], [10]).

The effective determination of these polynomials is a difficult question. Following ideas of B. Malgrange ([11], [2] part A), we have investigated this problem in [21] when $X$ has an isolated singularity and $(f, g)$ defines a germ of complete intersection isolated singularity $(Z, 0)$. First, for $\delta \in \mathcal{R}$ of the form $\dot{a}/g^\ell$ with $a \in \mathcal{O}$ nonzero on the components of $Z$, the holonomic $\mathcal{D}$-module:

$$\mathcal{N}_\delta = (s + 1) \frac{\mathcal{D}[s] \delta f^s}{\mathcal{D}[s] \delta f^{s + 1}}$$

is supported by 0. Then the minimal polynomial of the action of $s$ on $\mathcal{N}_\delta$ - which is nothing else but $b(\delta f^s, s)$ - may be computed using its $n^{th}$-group of de Rham cohomology $H^n_{dR}(\mathcal{N}_\delta) = \mathcal{N}_\delta / \sum (\partial/\partial x_i) \mathcal{N}_\delta$. In order to do that, we need an explicit description of this group. So we imposed that the annihilator in $\mathcal{D}$ of $\delta$ is generated by operators of degree less or equal to one; but it is a very constraining condition, because this implies that $g$ is weighted-homogeneous and that $a \in \mathcal{O}$ is a unit (see [21], [23]).

In this paper, we study the particular case where $f$ is a germ of a smooth function. Let us recall that this contains the classical theory of the Bernstein polynomial of germs of holomorphic functions, because of the following relation:

$$b\left(\frac{1}{h} - z^s, s\right) = b(h^s, s)$$

for every $h \in \mathcal{O}$ nonzero, where $b(h^s, s)$ is the Bernstein polynomial of $h$ and $1/h - z \in \mathbb{C}[x, z][1/h - z]/\mathbb{C}[x, z]$ (see Proposition 2.8 for example).

Without further condition on $g$, we prove in Theorem 2.1 that for some $\delta \in \mathcal{R}$, the $\mathcal{D}[s]$-module $\mathcal{N}_\delta$ coincides with:

$$(1) \quad \mathcal{N}_\ell = \frac{\mathcal{D}[s](\text{jac}(g), g)\delta f^{s + 1}}{\mathcal{D}[s] J \delta f^{s + 1}}$$

for an integer $\ell \in \mathbb{N}^*$, where $\text{jac}(g) \subset \mathcal{O}$ is the jacobian ideal of $g$, $J \subset \mathcal{O}$ is the ideal generated by $g$ and by all the $2 \times 2$-minors of the jacobian matrix of
Bernstein Polynomials Of

\((f, g)\), and \(\delta \in \mathcal{R}\) is defined by \((-1)^{\ell+1}(\ell - 1)!/g^\ell \in \mathcal{O}[1/g]\). More precisely, \(\mathcal{N}_\delta\) is equal to \(\mathcal{N}_\ell\) (resp. \(\mathcal{N}_{\ell+1}\)) when \(\delta = v(g)\delta\ell\) (resp. \(\delta = \delta\ell\)) for every generic regular vector field \(v\) such that \(v(f) = 0\). This result enables us to treat in the same way the Bernstein polynomials of \(f\) associated with sections \(\delta\ell, \ell \in \mathbb{N}^*\), but also with certain generators of the minimal extension \(\mathcal{L} \subset \mathcal{R}\) of the local algebraic cohomology with support in \(X\) (since D. Barlet and M. Kashiwara prove in [1] that \(\mathcal{L}\) is generated by any nonzero section defined by \(v(g)/g\), where \(v \in \mathcal{D}\) is a vector field).

So we are interested in the determination of the minimal polynomial of the action of \(s\) on \(\mathcal{N}_\ell\), denoted by \(b_\ell(s)\), when \(f\) is smooth, \(X\) has an isolated singularity and \((f, g)\) defines a germ of complete intersection isolated singularity. In the third part, we express \(H^0_{\mathcal{D}\mathcal{R}}(\mathcal{N}_\ell)\) under these assumptions as a quotient of two finite dimensional vector spaces \(Z^*_\ell\) and \(Z_{\ell}\) defined in section 3.2. Therefore:

**Theorem 1.1.** For every \(\ell \in \mathbb{N}^*\), \(b_\ell(s)\) is the minimal polynomial of the action induced by \(s\) on \(Z^*_\ell/Z_{\ell}\).

This needs the knowledge of the annihilator in \(\mathcal{D}\) of \(\delta_k f^s\), \(\text{Ann}_D \delta_k f^s\), which authorizes the calculation of the \(n^{th}\)-group of the de Rham cohomology of the \(\mathcal{D}\)-module \(\sum_{k \geq 1} \mathcal{D}\delta_k f^{s+1}\) (into which \(\mathcal{D}[s](\text{jac}(g), g)\delta f^{s+1}\) injects). As an application, we develop in the last part an algorithm to compute \(b_\ell(s)\) when \(f = x_1\) and \(g\) is non-degenerate with respect to its Newton boundary in the sense of Kouchnirenko, which gives a generalization of [2]. Using the Newton function \(\rho\) on \(\mathcal{O}\), we define a weight function \(\rho^*\) by \(\rho^*(u\delta_k x_1^{s+1}) = \rho(ux_2 \cdots x_n) - k\). Then Kouchnirenko division theorem makes it possible to establish that the filtration induced by \(\rho^*\) is suited to our construction of \(H^0_{\mathcal{D}\mathcal{R}}(\sum_{k \geq 1} \mathcal{D}\delta_k f^{s+1})\). Moreover, the action of \(s\) respects the filtration induced by \(\rho^*\) on \(Z^*_\ell/Z_{\ell}\). Thus, if \(b_{\ell,q}(s)\) is the minimal polynomial of the action of \(s\) on \(\text{gr}_q Z^*_\ell/Z_{\ell}\), then the polynomial \(b_\ell(s)\) is the l.c.m. of \(b_{\ell,q}(s), q \in \mathbb{Q}\) (Theorem 4.9). The technics ‘rewriting by division’ and ‘increase in weight’ allow us to give an explicit computation of the spaces \(Z^*_\ell / Z_{\ell}\) and of the action of \(s\) on \(Z^*_\ell / Z_{\ell}\), and thus to determine \(b_\ell(s)\). In the particular case of semi-weighted-homogeneous germs, these computations are easier (Remark 4.12). On the way, we deduce from an algorithm for computing a multiple of the polynomials \(b_{\ell,q}(s)\) that the multiplicities of the roots of \(b_\ell(s)\) are strictly smaller than \(n\) (Theorem 4.10).

We end with the complete determination of the polynomials \(b_\ell(s)\) when \(g = x_1^4 + x_2^4 + x_3^4 + (x_1 x_2 x_3)^2, d \geq 9\).

Finally, we point out that the methods at the root of the algorithm may be adapted to compute Bernstein functional equations associated with an analytic
morphism - introduced by C. Sabbah ([15], [16]) - in the following case: 

\[(g, x_1, \ldots, x_p) : (\mathbb{C}^n, 0) \to (\mathbb{C}^{p+1}, 0), 1 \leq p \leq n - 1.\]

In particular, one can make explicit non trivial equations of the form:

\[
d_0(\underline{s})g^{s_0}x_1^{s_1} \cdots x_p^{s_p} \in \mathcal{D}[\underline{s}]g^{s_0+1}x_1^{s_1} \cdots x_p^{s_p}
\]

\[
d_j(\underline{s})g^{s_0}x_1^{s_1} \cdots x_p^{s_p} \in \mathcal{D}[\underline{s}]x_jg^{s_0}x_1^{s_1} \cdots x_p^{s_p}, 1 \leq j \leq p
\]

where \(d_0(\underline{s}), d_j(\underline{s}) \in \mathbb{C}[s_0, \ldots, s_p]\) and \(\mathcal{D}[\underline{s}] = \mathcal{D} \otimes \mathbb{C}[s_0, \ldots, s_p]\). This completes H. Maynadier-Gervais results about these functional equations ([13]).

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§2. Some Equivalences of Functional Equations

In this part, we denote \(f \in \mathcal{O}\) a germ of a smooth function and \(g \in \mathcal{O}\) a germ which is not a unit and does not belong to \(f\mathcal{O}\).

We first prove Theorem 2.1, where the \(\mathcal{D}\)-module \(\mathcal{N}_\delta\) is identified to \(\mathcal{N}_\ell\) for some \(\delta \in \mathbb{R}\). Then we give relations between some Bernstein polynomials of \(f\) associated with sections of \(\mathcal{R} = \mathcal{O}[1/g]/\mathcal{O}\).

§2.1. Some identifications of \(\mathcal{N}_\delta\) with \(\mathcal{N}_\ell\)

Let us state the result at the root of this study.

**Theorem 2.1.** Let \(f \in \mathcal{O}\) be a germ of a smooth function at the origin, and \(g \in \mathcal{O}\) a germ which is not a unit nor a multiple of \(f\). Let us denote \((Z, 0) \subset (\mathbb{C}^n, 0)\), the complete intersection defined by \(f\) and \(g\).

i) For every non negative integer \(\ell \in \mathbb{N}^*\), the \(\mathcal{D}[\underline{s}]\)-module:

\[
(s + 1) \frac{\mathcal{D}[\underline{s}]\delta_\ell f^s}{\mathcal{D}[\underline{s}]\delta_\ell f^{s+1}}
\]

where \(\delta_\ell = (-1)^{\ell+1}(\ell - 1)!/(1/g^\ell) \in \mathcal{R}\), coincides with \(\mathcal{N}_{\ell+1}\).

ii) Let \(v \in \mathcal{D}\) be a regular vector field such that \(v(f) = 0\). Let us suppose that \(v\) is not tangent to \((Z, 0)\). Then, for every \(\ell \in \mathbb{N}^*\), the \(\mathcal{D}[\underline{s}]\)-module:

\[
(s + 1) \frac{\mathcal{D}[\underline{s}]v(g)\delta_\ell f^s}{\mathcal{D}[\underline{s}]v(g)\delta_\ell f^{s+1}}
\]

coincides with \(\mathcal{N}_\ell\). Moreover, when \((Z, 0)\) does not have any irreducible smooth component, the equality is verified if \(v\) is not tangent to \((\text{Sing}(Z), 0)\).
iii) Let us suppose that \( f = x_1 \). Let \( v \in D \) be a vector field of the form \( x_1(\partial/\partial x_1) + v \) where \( v \in C[x_2, \ldots, x_n] \langle \partial/\partial x_2, \ldots, \partial/\partial x_n \rangle \) is a regular vector field. Let us suppose that \( v \) is not tangent to \((Z, 0)\). Then, for every \( \ell \in N^* \), the \( D[s] \)-module:

\[
(s + 1) \frac{D[s]v(g)\delta f^s \delta f^{s+1}}{D[s]v(g)\delta f^{s+1}}
\]

coincides with \( N_\ell \). Moreover, if \((Z, 0)\) does not have any irreducible smooth component, the equality is verified if \( v \) is not tangent to \( (\text{Sing}(Z), 0) \).

Given \( \delta \in R \), the \( D[s] \)-module \( N_\delta \) coincides with \( N_\ell \), \( \ell \in N^* \), if and only if the following identities are verified:

\[(\dagger) \quad D[s]\delta f^{s+1} = D[s]J \delta f^{s+1} \]
\[(\ddagger) \quad D[s](s + 1)\delta f^s + D[s]\delta f^{s+1} = D[s](\text{jac}(g), g) \delta f^{s+1} \]

In order to prove the theorem, we will check that these identities are verified in any case.

Proof of Theorem 2.1, case i). The equality \((\dagger)\) results from the following identities:

\[(\ast) \quad g \delta_{\ell+1}f^{s+1} = -\ell \delta f^{s+1} \]
\[(\beta) \quad (f_{x_\ell}^r, g_{x_\ell}^r - f_{x_\ell}^r g_{x_\ell}^r) \delta_{\ell+1}f^{s+1} = \left( f_{x_\ell}^r \frac{\partial}{\partial x_i} - f_{x_\ell}^r \frac{\partial}{\partial x_j} \right) \delta f^{s+1} \]

So let \( r \) be an index such that \( f_{x_\ell}^r \) is a unit. From the identities:

\[
(s + 1)\delta f^s = (f_{x_\ell}^r)^{-1} \frac{\partial}{\partial x_r} \delta f^{s+1} - (f_{x_\ell}^r)^{-1} g_{x_\ell}^r \delta_{\ell+1}f^{s+1}
\]

and \( (\dagger) \), we deduce:

\[
D[s](s + 1)\delta f^s + D[s]\delta f^{s+1} = D[s](g_{x_\ell}^r, J) \delta_{\ell+1}f^{s+1}.
\]

Thus \( (\ddagger) \) is verified since the ideal \( (g_{x_\ell}^r, \{g_{x_\ell}^r, f_{x_\ell}^r - g_{x_\ell}^r f_{x_\ell}^r\}_{i \neq r})O \) coincides with \( \text{jac}(g) \).

Proof of Theorem 2.1, first part of ii). Let \( v \in D \) be a regular vector field such that \( v \) annihilates \( f \) and is not tangent to \((Z, 0)\). Up to a change of coordinates, we may assume that \( f = x_1 \) and \( v = \partial/\partial x_2 \) (in particular \( J = (g_{x_2}, \ldots, g_{x_n}, g)O \)). In algebraic terms, the geometrical assumption on \( v \) is: \( g \not\in (x_1, x_3, \ldots, x_n)O \). In other words, there exists \( N \in N^* \) such that \( v^N(g) \) is a unit.
First we prove that the inclusion $\mathcal{D}[s]v(g)\delta_x x_1^{s+1} \subset \mathcal{D}[s]\mathcal{J}\delta_x x_1^{s+1}$ is an equality. It is enough to see that the ideal $I = \mathcal{D}[s]v(g) + \text{Ann}_{\mathcal{D}[s]}\delta_x x_1^{s+1}$ contains $g'_x, \ldots, g_n'$ and $g$. Since the operators $(\partial/\partial x_i)v(g) - vg'_x$, $3 \leq i \leq n$, and $vg + (\ell - 1)v(g)$ annihilate $\delta_x x_1^{s+1}$, then $vg, vg'_x, \ldots, vg'_x \in I$. So we have $g, g'_x, \ldots, g'_x \in I$ by using the following lemma. Thus (i) is true.

**Lemma 2.2.** Let $\theta \in \mathcal{D}$ be a vector field and $h \in \mathcal{O}$ a nonzero germ such that $\partial^N(h)$ is a unit for a non negative integer $N \in \mathbb{N}^*$.

Then, for every $a, c \in \mathcal{O}[s]$, the ideal $\mathcal{D}[s](\theta + c)a + \mathcal{D}[s]ha$ contains $a$.

Proof. It is enough to prove that $\theta^k(h)a, k \in \mathbb{N}^*$, belong to the given ideal. This may be done by induction, using the identities: $\theta ah - h(\theta + c)a = \theta(h)a - cah$ and $v\theta^k(h)a - \theta^k(h)(\theta + c)a = \theta^{k+1}(h)a - \theta^k(h)ca, k \in \mathbb{N}^*$. □

Let us prove (i) for $\delta = v(g)\delta_x$. Since $\mathcal{D}[s]v(g)\delta_x x_1^{s+1}$ coincides with $\mathcal{D}[s]\mathcal{J}\delta_x x_1^{s+1}$, and using the equality:

$$(s + 1)v(g)\delta_x x_1^s = \left( v(g)\frac{\partial}{\partial x_1} - g'_x v \right)\delta_x x_1^{s+1} = \left( \frac{\partial}{\partial x_1}v(g) - vg'_x \right)\delta_x x_1^{s+1}$$

it is enough to remark that $g'_x$ belongs to $\mathcal{D}(v(g), vg'_x)$. But this is a consequence of Lemma 2.2. Then (i) is verified. □

**Proof of Theorem 2.1, first part of iii.** Let $\bar{v}$ be the vector field $x_1(\partial/\partial x_1) + v$ where $v \in \mathbb{C}\{x_2, \ldots, x_n\}(\partial/\partial x_2, \ldots, \partial/\partial x_n)$ is regular and such that $v^N(g)$ is a unit for a non negative integer $N \in \mathbb{N}^*$. From the case ii), the $\mathcal{D}$-module $\mathcal{D}[s]v(g)\delta_x x_1^{s+1}$ coincides with $\mathcal{D}[s]\mathcal{J}\delta_x x_1^{s+1}$. So, to prove (i), we just have to remark that $x_1g'_x\delta_x x_1^{s+1}$ belongs to $\mathcal{D}[s]\bar{v}(g)\delta_x x_1^{s+1}$ and to $\mathcal{D}[s]\mathcal{J}\delta_x x_1^{s+1}$. First, it is easy to check that if $v^N(g)$ is a unit, then $\bar{v}^N(h)$ is a unit too. Moreover, identity (4) implies that $(\bar{v} + (s + 1)x_1g'_x) \chi_1g'_x \in I$ (resp. $\chi_1g'_x$ belongs to $\bar{I} = \mathcal{D}[s]\bar{v}(g) + \text{Ann}_{\mathcal{D}[s]}\delta_x x_1^{s+1}$ (resp. $I = \mathcal{D}[s]v(g) + \text{Ann}_{\mathcal{D}[s]}\delta_x x_1^{s+1}$). Thus the germ $x_1g'_x\delta_x x_1^{s+1}$ belongs to $I$ and to $\bar{I}$ i.e. $x_1g'_x\delta_x x_1^{s+1} \in \mathcal{D}[s]\bar{v}(g)\delta_x x_1^{s+1}$ and $x_1g'_x\delta_x x_1^{s+1} \in \mathcal{D}[s]\mathcal{J}\delta_x x_1^{s+1}$.

The proof of (i) for $\delta = \bar{v}(g)\delta_x x_1^{s+1}$ is similar to the one of the previous case, using the identity:

$$(s + 1)\bar{v}(g)\delta_x x_1^s = \left( \frac{\partial}{\partial x_1}v(g) + (s + 1 - v)g'_x \right)\delta_x x_1^{s+1}$$

□

**Remark 2.3.** In the last case, we also prove that $\mathcal{D}[s](\text{jac}(g), g)\delta_x x_1^{s+1}$ is contained in $\mathcal{D}[s]\mathcal{J}\delta_x x_1^{s+1}$. 

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Proof of Theorem 2.1, second part of ii) and iii). We are going to prove that the equalities (1) and (4) are true for every regular vector field \( v \) or \( \tilde{v} = x_1(\partial/\partial x_1) + v \), where \( v \) is not tangent to the singular set of \((Z,0)\) and fulfills the conditions of the exposition. Let us take some coordinates such that \( f = x_1 \) and \( v = \partial/\partial x_2 \). Thus the geometrical assumption on \( v \) means that there is at least one monomial \( x_2^N \) or \( x_2^N x_i, i \geq 3 \), in the Taylor expansion of \( g |_{x_1=0} \in C\{x_2, \ldots, x_n\} \).

We start with the case \( \delta = v(g)\delta I \Phi^{\delta+1} \). Under our assumption, there exists an integer \( N \in \mathbb{N}^* \) such that \( v^N(g) = l + h \) where \( l \) is a linear form, non-zero and not proportional to \( x_1 \), and \( h \in (x_1, \ldots, x_n)^2 \mathcal{O} \). We let remark that if \( l \) depends on the variable \( x_2 \), \( v^N(g) \) is a unit and \( v \) is not tangent to \((Z,0)\). Without loss of generality, we can also suppose that \( n \geq 3 \), \( l = x_3 \) and that there is no monomial of the form \( x_2^N \) in the Taylor expansion of \( h \).

In order to get (1), we will prove that the ideal \( I = \mathcal{D}[s]v(g) + \text{Ann}_D[s] \) contains \( g'_{x_1}, \ldots, g'_{x_n} \) and \( g \) (following the proof of the case ‘\( v \) not tangent to \((Z,0)\)’). We start with the membership of \( I \) for \( g \). As above, we have \( v g, v g_{x_3}, \ldots, v g'_{x_n} \in I \); so \( v g g'_{x_1} - v(g)g'_{x_1} \in I \) and then \( v g'_{x_i}, 3 \leq i \leq n, \) belong to \( I \) too. Using that \( v g \in I \), we deduce: \( v(g'_{x_i}) g \in I \). Thus \( g \) belongs to the ideal \( I \) (Lemma 2.2).

It is more difficult to get the membership of \( I \) for \( g'_{x_1}, \ldots, g'_{x_n} \). Since \( v g'_{x_i}, v(g)g'_{x_1} \in I \), we remark - with the help of technics of Lemma 2.2 - that \( v g'_{x_i}, 3 \leq i \leq n, \) belong to \( I \). Multiplying the operators \( (\partial/\partial x_1)g'_{x_1} - (\partial/\partial x_1)g'_{x_2} \in \text{Ann}_D \delta x_1^{\delta+1} \) by \( v^N(g) = x_3 + h \), we deduce:

(5) \quad \text{for } i \neq 1, 3, (1 + h'_{x_i})g'_{x_i} - h'_{x_1}g'_{x_2} \text{ belongs to } I

Thus the operators \( (\partial/\partial x_3)h'_{x_1}(1 + h'_{x_1})^{-1} - \partial/\partial x_1)g'_{x_2} \) belong to the ideal \( I \). Dividing \( h'_{x_1}(1 + h'_{x_1})^{-1} \) by \( x_1 + h \), we get \( (\partial/\partial x_3)\tilde{h}_{1,3} - \partial/\partial x_1)g'_{x_2} \in I \) where \( \tilde{h}_i \in \mathcal{O} \) does not depend of \( x_3 \). Similarly, dividing \( g \) by \( x_3 + h \), we have \( g = q(x_3 + h) + \tilde{g} \), where \( \tilde{g} \in \mathcal{O} \) does not depend of \( x_3 \), and is not proportional to \( x_1 \) because \((Z,0)\) does not have any smooth irreducible component. Thus \( \tilde{g}g'_{x_2} \) belongs to \( I \). So the fact \( g'_{x_2} \) belongs to \( I \) comes from Lemma 2.2, taking \( a = g'_{x_2}, h = \tilde{g} \) and \( v = \sum_{i \neq 1,3} A_i((\partial/\partial x_3)\tilde{h}_{1,3} - \partial/\partial x_1), \lambda_i \in C \) generic. From (5), we have then \( g'_{x_i}, \ldots, g'_{x_n} \in I \).

Now we consider (4). Following the proof of the case ii) above, it is enough to remark that the ideal \( I' = \mathcal{D}[s](v g'_{x_1}, g'_{x_2}, \ldots, g'_{x_n}, g) + \text{Ann}_D[s] \delta x_1^{\delta+1} \) contains \( g'_{x_1} \). Multiplying \( v g'_{x_1} \) by \( g'_{x_2} \), we see that \( v(g'_{x_1})g'_{x_1} \) belongs to \( I' \). Then we conclude with Lemma 2.2 (with \( h = v(g'_{x_1}) \)).

In the case \( \delta = \tilde{v}(g)\delta I \Phi^{\delta} \), we can assume that \( f = x_1, \tilde{v} = x_1(\partial/\partial x_1) + v \) where \( v = \partial/\partial x_2 \) and \( \tilde{v}^N(g) = x_3 + h, h \in (x_1, \ldots, x_n)^2 \mathcal{O} \). Then the identities
(†) and (‡) may be got similarly, using that the operators $(\tilde{v} - (s + 1))g, (\tilde{v} - (s + 1))g'_{x_2}, \ldots, (\tilde{v} - (s + 1))g'_{x_n}$ belong to the ideal $I = \mathcal{D}[s] \tilde{v}(g) + \text{Ann}_{\mathcal{D}[s]} \delta_\ell x_1^{s+1}$.

This comes from the identities:

$$(s + 1)g\delta_\ell x_1^{s+1} = \left( (x_1 \frac{\partial}{\partial x_1} + \vartheta) g + (\ell - 1)(x_1 g'_{x_1} + \vartheta(g)) \right) \delta_\ell x_1^{s+1}$$

$$(s + 1)\vartheta(g)\delta_\ell x_1^{s+1} = \left( (x_1 \frac{\partial}{\partial x_1} + \vartheta) g + \vartheta(x_1 g'_{x_1} + \vartheta(g)) \right) \delta_\ell x_1^{s+1}$$

for every vector field $\vartheta \in \mathbb{C}[x_2, \ldots, x_n](\partial/\partial x_2, \ldots, \partial/\partial x_n)$.

**Remark 2.4.** From these identities, we deduce the following ones:

$$\mathcal{D}[s]_{\leq d} \delta_\ell f^{s+1} = \mathcal{D}[s]_{\leq d-1} f'_{x_2'}_\delta_\ell f^{s+1} + \mathcal{D} \delta_\ell f^{s+1}$$

$$\mathcal{D}[s]_{\leq d}(\text{jac}(g), g)\delta_\ell f^{s+1} = \mathcal{D}[s]_{\leq d} g'_{x_2} \delta_\ell f^{s+1} + \mathcal{D} \delta_\ell f^{s+1}$$

for every $d \in \mathbb{N}$, where $r$ is an index such that $f'_{x_r}$ is a unit and $\mathcal{D}[s]_{\leq d} \subset \mathcal{D}[s]$ is the subspace of the operators which the degree in $s$ is less or equal to $d$. This may be done by induction, and using that $f'_{x_2} \delta_\ell f^{s+1}$ belongs to $\mathcal{D}[s]_\ell f^{s+1} \mathcal{D}[s]_\ell f^{s+1}$ for every $\ell \in \mathbb{N}^*$ (Remark 2.3).

**Remark 2.5.** The identity (†) is not always true if $(Z, 0)$ has an irreducible smooth component. For example, if $f = x_1, g = x_1^2 + x_2 x_3, v = \partial/\partial x_2$ and $\ell = 1$, then $\mathcal{D}[s]v(g) + \text{Ann}_{\mathcal{D}[s]} \delta_\ell x_1^{s+1}$ is equal to $\mathcal{D}[s](x_1^2, x_3, (\partial/\partial x_2)x_2, s + 2 - (\partial/\partial x_1)x_1)$, and then it is different from the ideal $\mathcal{D}[s] \mathcal{J} + \text{Ann}_{\mathcal{D}[s]} \delta_\ell x_1^{s+1} = \mathcal{D}[s](x_1^2, x_2, x_3, s + 2 - (\partial/\partial x_1)x_1)$.

### §2.2. Some relations between Bernstein polynomials

We start with some relations between the Bernstein polynomials of $f$ associated with some elements of $\mathcal{R}$ and the polynomial $\tilde{b}_\ell(s)$, the minimal polynomial of the action of $s$ on $\mathcal{N}_\ell$.

**Corollary 2.6.** Let $f \in \mathcal{O}$ be a germ of a smooth function, and let $g \in \mathcal{O}$ be a germ which is neither a unit nor a multiple of $f$. Let us denote $(Z, 0) \subset (\mathbb{C}^n, 0)$, the complete intersection defined by $(f, g)$. Let $\ell \in \mathbb{N}^*$ be a non negative integer.

i) The polynomial $\tilde{b}(\delta_\ell f^*, s)$ coincides with $\tilde{b}_{\ell+1}(s)$.

ii) Let $v$ be a regular vector field such that $v(f) = 0$. If $v$ is not tangent to $(Z, 0)$, $\tilde{b}(v(g)\delta_\ell f^*, s)$ coincides with $\tilde{b}_\ell(s)$. Moreover, when $(Z, 0)$ does not have any irreducible smooth component, the equality is verified if $v$ is not tangent to $(\text{Sing}(Z), 0)$. 

iii) Assume that $f = x_1$. Let $v \in C\{x_2, \ldots, x_n\}(\partial/\partial x_2, \ldots, \partial/\partial x_n)$ be a regular vector field. If $v$ is not tangent to $(Z, 0)$, then $\tilde{b}(x_1 g'_{x_1} + v(g))\delta f^s$, $s$ coincides with $\tilde{b}_v(s)$. Moreover, when $(Z, 0)$ does not have any smooth component, this equality is true if $v$ is not tangent to $(\text{Sing}(Z), 0)$.

iv) Let $u \in \text{jac}(g) + g\mathcal{O}$ be a generator of the $\mathcal{O}$-module $(\text{jac}(g) + g\mathcal{O})/J$. Then the polynomial $b(u\delta f^s, s)$ is a multiple of $\tilde{b}_v(s - 1)$.

Proof. The first 3 points are easy consequences of Theorem 2.1 and of the fact that $v(g)$ is not divisible by $f$ for every $v$ verifying the requisite conditions. The last point is a consequence of the surjectivity of the following $\mathcal{D}[s]$-linear morphism:

$$\frac{\mathcal{D}[s]u\delta f^{s+1}}{\mathcal{D}[s]u\delta f^{s+2}} \rightarrow \frac{\mathcal{D}[s](\text{jac}(g), g)\delta f^{s+1}}{\mathcal{D}[s]f\delta f^{s+1}}$$

which is well defined from Remark 2.3.

Hence, for every generic vector field $v$ annihilating $f$, the polynomial $\tilde{b}(v(g)\delta, s)$ coincides with $\tilde{b}_v(s)$. However, because of iv), this is not true for every regular vector field $v$.

The following corollary gives a similar result for the classical Bernstein polynomial of a germ of function.

**Corollary 2.7.** Let $h \in \mathcal{O}$ be a germ neither zero nor a unit. Let us denote $(\mathcal{H}, 0) \subset (\mathbb{C}^n, 0)$ the hypersurface defined by $h$ and $\tilde{b}(s) \in \mathbb{C}[s]$ its reduced Bernstein polynomial.

Let $v \in \mathcal{D}$ be a regular vector field. If $v$ is not tangent to $(\mathcal{H}, 0)$, then the reduced Bernstein polynomial of $v(h)h^s$ is equal to $\tilde{b}(s + 1)$. Moreover, when $(\mathcal{H}, 0)$ does not have any smooth component, the equality is true if $v$ is not tangent to the singular set of $(\mathcal{H}, 0)$.

This shifting in the roots of $\tilde{b}(s)$ is very clear in terms of poles of analytic continuation of distributions $\int_{\mathbb{C}^n} |h|^{2\lambda} \varphi$, where $\varphi$ is a $(n, n)$-differential form with compact support around the origin, because:

$$\int_{\mathbb{C}^n} v(h)|h|^{2\lambda} \varphi = -\frac{1}{\lambda + 1} \int_{\mathbb{C}^n} h|h|^{2\lambda}(\nu, \varphi)$$

for every vector field $v$.

In order to prove this corollary, we will use the following result. This is the first explicit example of computation of the polynomials $\tilde{b}_v(s)$, $\ell \in \mathbb{N}^*$, and it generalizes a result of [19].
Proposition 2.8. Let \( h \in \mathcal{O} \) be a germ which is neither zero nor a unit. Let us denote \( \tilde{b}(s) \) its reduced Bernstein polynomial. Let \( N \in \mathbb{N}^* \) be a non negative integer and \( z \) a new variable.

Up to a multiplicative constant, the polynomial \( \tilde{b}(s) \), \( \ell \in \mathbb{N}^* \), associated with \( f = z \) and \( g = h - z^N \in \mathbb{C}[x,z] \) is equal to \( b(1 - \ell + (s + 1)/N) \).

Proof. Without loss of generality, we will prove the result for \( \tilde{h} = e^\tau h \), where \( \tau \) is a new variable. In fact, it does not change the value of the studied Bernstein polynomials.

To prove that \( \tilde{b}(s) \) is a multiple of \( \tilde{b}(1 - \ell + (s + 1)/N) \), we start with the ‘Bernstein identity’ of \( b(s) \), i.e.:

\[
\tilde{b}(s)z^{N-1} \in \mathcal{D}_{x,z}\{\tilde{h}, \tilde{h}_x, \ldots, \tilde{h}_{x^n}, \tilde{h} - z^N\} + \text{Ann}_\mathcal{D}_{x,z} \tilde{b}(s)z^{s+1}
\]

where \( \mathcal{D}_{x,z} \) is the ring of differential operators \( \mathbb{C}\{x,z,\tau\}\langle \partial/\partial x, \partial/\partial z, \partial/\partial \tau \rangle \). As the operator \( N(\partial/\partial \tau) + z(\partial/\partial z) - s - 1 + N\ell \) annihilates \( b(z^{s+1}) \), this equation may be rewritten:

\[
\tilde{b}(N \frac{\partial}{\partial \tau} + z \frac{\partial}{\partial z} - N + N\ell)z^{N-1} \in \mathcal{D}_{x,z}\{\tilde{h}, \tilde{h}_x, \ldots, \tilde{h}_{x^n}, \tilde{h} - z^N\} + \text{Ann}_\mathcal{D}_{x,z} \tilde{b}(s)z^{s+1}
\]

or:

\[
\tilde{b}(N \frac{\partial}{\partial \tau} - N - 1 + N\ell)z^{N-1} \in \mathcal{D}_{x,z}\{\tilde{h}, \tilde{h}_x, \ldots, \tilde{h}_{x^n}, \tilde{h} - z^N\} + \text{Ann}_\mathcal{D}_{x,z} \tilde{b}(s)z^{s+1}.
\]

Then we remark that \( \text{Ann}_\mathcal{D}_{x,z} \tilde{b}(s)z^{s+1} \) is generated by its operators which are not dependant of \( \partial/\partial z \). Indeed, if \( P = \sum_{i=0}^d (i^\ell/\partial z)^i P_i \) with \( P_i \in \mathbb{C}\{x,z,\tau\}\langle \partial/\partial x, \partial/\partial z, \partial/\partial \tau \rangle \) annihilates \( \tilde{b}(s)z^{s+1} \), so does \( [P, z] = \sum_{i=1}^d i(\partial/\partial z)^{i-1} P_i \).

So we prove by induction that the operators \( P_0, \ldots, P_d \) annihilate \( \tilde{b}(s)z^{s+1} \). The identity becomes:

\[
(6) \quad \tilde{b}(N \frac{\partial}{\partial \tau} - N - 1 + N\ell)z^{N-1} \in \mathcal{D}_{x,z}\{\tilde{h}, \tilde{h}_x, \ldots, \tilde{h}_{x^n}, \tilde{h} - z^N\} + \text{Ann}_\mathcal{D}_{x,z} \tilde{b}(s)z^{s+1}.
\]

By division, an operator \( P \in \text{Ann}_\mathcal{D}_{x,z} \tilde{b}(s)z^{s+1} \) may be written:

\[
P = \tilde{Q}\left( \frac{\partial}{\partial \tau} (\tilde{h} - z^N) + (\ell - 1)\tilde{h} \right) + \sum_{i=1}^n Q_i\left( \frac{\partial}{\partial x_i} (\tilde{h} - z^N) + (\ell - 1)\tilde{h}_x \right) + q(\tilde{h} - z^N)^\ell + R' + \sum_{i=1}^{\ell} r_i(\tilde{h} - z^N)^{\ell-i}
\]

where \( \tilde{Q}, R' \), and \( r_i \) are polynomials in \( \tilde{h}, \tilde{h}_x, \ldots, \tilde{h}_{x^n} \).
where \( R' \in (\partial/\partial x, \partial/\partial \tau)\mathbb{C}\{x, \tau\}\{\partial/\partial x, \partial/\partial \tau\}[z] \) and \( r_1, \ldots, r_{\ell} \in \mathbb{C}\{x, \tau\}[z] \) have a degree in \( z \) strictly less than \( N \), and \( Q, Q_1, \in \mathcal{D}_{x, \tau}, q \in \mathbb{C}\{x, z, \tau\} \). So we have:

\[
R \frac{1}{(h - z^N)\ell} = \sum_{i=1}^{d} (-1)^i \frac{(\ell + i - 1)!}{(\ell - 1)!} \frac{r_i'}{(h - z^N)^{\ell+i}} + \sum_{i=1}^{\ell} \frac{r_i}{(h - z^N)^i}
\]

and

\[
R\hat{h}^s = \sum_{i=1}^{d} s(s - 1) \cdots (s - i + 1) \frac{r_i'}{h^i} + rh^s
\]

where \( d = \text{deg} R \) and \( r_i' \in \mathbb{C}\{x, \tau\}[z] \) has a degree in \( z \) strictly less than \( N \). As \( R \) annihilates \( \delta_r \), all the germs \( r_i \) and \( r_i' \) are necessarily equal to zero, and then \( R \) annihilates \( \hat{h}^s \). Hence (6) implies that:

\[
\bar{b}_\ell (\frac{\partial}{\partial \tau} - N - 1 + N\ell)z^{N-1} \in \mathcal{D}_{x, \tau}(\hat{h}, \hat{h}_{x_1}, \ldots, \hat{h}_{x_n}, z^N) + \mathcal{D}_{x, \tau}\text{Ann}_{\mathcal{D}_{x}} \hat{h}^s
\]

where \( \mathcal{D}_{x} = \mathbb{C}\{x, \tau\}\{\partial/\partial x, \partial/\partial \tau\} \). Consequently, \( \bar{b}_\ell (\frac{\partial}{\partial \tau} - N - 1 + N\ell) \) belongs to the ideal \( \mathcal{D}_{x}(\hat{h}, \hat{h}_{x_1}, \ldots, \hat{h}_{x_n}) + \text{Ann}_{\mathcal{D}_{x}} \hat{h}^s \) i.e. \( \bar{b}_\ell (Ns - N - 1 + N\ell) \) is definitely a multiple of \( \hat{b}(s) \).

The proof of the converse relation is similar (see [19]).

**Proof of Corollary 2.7.** By similar computations, we prove easily that the polynomial \( \hat{b}(\delta/\partial (h - z)z^s, s) \) coincides with the Bernstein polynomial of \( ah^s \).

So the assertion is a direct consequence of Corollary 2.6 and Proposition 2.8.

We end with a relation between the Bernstein polynomial of \( f \) associated with some particular element of \( \mathcal{O}[1/g] \) and of \( R = \mathcal{O}[1/g]/\mathcal{O} \). From the point of view of the monodromy, it is very clear (because \( \Phi_f(\mathcal{O}) \) is zero when \( f \) is smooth).

**Proposition 2.9.** Let \( f \in \mathcal{O} \) be a germ of a smooth function, and \( g \in \mathcal{O} \) a germ which is neither a unit nor a multiple of \( f \).

For every \( \ell \in \mathbb{N}^* \), the Bernstein polynomial of \( (1/g^\ell)f^s \) coincides with \( b(\delta_\ell f^s, s) \).

**Proof.** We just prove that the Bernstein polynomial of \( (1/g^\ell)f^s \in \mathcal{O}[1/fg, s]f^s \), denoted by \( b((1/g^\ell)f^s, s) \), is a factor of \( b(\delta_\ell f^s, s) \) (the converse relation is evident). Let \( R \in \mathcal{D}[s] \) be an operator realizing the functional equation of \( \delta_\ell f^s \): \( b(\delta_\ell f^s, s)\delta_\ell f^s = R\delta_\ell f^{s+1} \). So there are an integer \( d \in \mathbb{Z} \) and
where $r$ is an index such that $f_{x_r}'$ is a unit. So the equation (7) implies that $b(\delta f, s)(1/g')f^s \in D[s](1/g')f^{s+1}$, and our assertion is proved. \hfill \Box

§3. The Case of Isolated Singularities

In this part, the germ $g \in O$ defines an isolated singularity, and $f \in O$ is a germ of smooth function such that $f(0) = 0$ and $(f, g)$ defines a complete intersection isolated singularity.

Following [2], [21], we give an explicit description of $H_D^0(N)$ in order to study the polynomials $\delta_k(s)$ (Theorem 1.1). So we introduce the $D$-module $\sum_{k \geq 1} D\delta_kf^{s+1}$.

§3.1. A suitable $D$-module

First, we remark that for every $\ell \in \mathbb{N}^*$, the $D[s]$-module $D[s]\delta_\ell f^{s+1}$ is a submodule of $\sum_{k \geq 1} D\delta_kf^{s+1}$. This comes from the identities:

\begin{equation}
(s + 2)\delta_kf^{s+1} = (f_{x_r}')^{-1}\frac{\partial}{\partial x_r} f\delta_kf^{s+1} - (f_{x_r}')^{-1}g_{x_r}' f\delta_{k+1}f^{s+1}, \quad k \in \mathbb{N}^*
\end{equation}

where $r$ is an index such that the germ $f_{x_r}'$ is a unit. Indeed, the $D$-module $\sum_{k \geq 1} D\delta_kf^{s+1}$ coincides with $\sum_{k \geq 1} \sum_{i \geq 0} D\delta_k\xi_i \subset \mathcal{R}[1/f, s]f^{s+1}$, where $\delta_k\xi_i$ is the element $(s + 2)i + 1 = 2i + 1 \delta_k f^{s+1}$, because:

$$
\delta_k\xi_i = (f_{x_r}')^{-1}\frac{\partial}{\partial x_r} \delta_k\xi_{i-1} - (f_{x_r}')^{-1}g_{x_r}' \delta_{k+1}\xi_{i-1}, \quad k \in \mathbb{N}^*
$$

for $i \in \mathbb{N}$.

We give now some results about the $D$-module $\sum_{k \geq 1} D\delta_kf^{s+1}$. 

\[ a \in O[s], \ a \not\in fO[s] - \{0\}, \text{ such that:} \]

\[(7) \quad b(\delta f^s, s)\frac{1}{g} f^s = R\frac{1}{g} f^{s+1} + a f^{s+d} \]

in $O[1/f, s]f^s$. If $a$ is zero, $b((1/g')f^s, s)$ divides definitely $b(\delta f^s, s)$. Otherwise, let us prove that $af^{s+d}$ belongs to $D[s]f^{s+1}$. If $d \geq 1$, it is trivial. So we suppose that $d \leq 0$. By specializations of $s$ in $-1, 0, \ldots, -d - 1$, we remark that $(s + 1)s \cdots (s + d + 1)$ is a factor of $a$. Hence $af^{s+d}$ belongs to $D[s]f^{s+1}$, because:

$$
\left[(f_{x_r}')^{-1}\left(\frac{\partial}{\partial x_r}\right)\right]^{-d+1} f^{s+1} = (s + 1)\cdots (s + d + 1) f^{s+d}
$$

This comes from the identities:
Lemma 3.1. For every non negative integer \( \ell \in \mathbb{N}^* \), the \( \mathcal{D} \)-module:

\[
\sum_{k \geq 1} \mathcal{D} \delta_k f^{s+1} \bigg/ \mathcal{D} \mathcal{J} \delta_\ell f^{s+1}
\]

is supported by the origin.

Proof. Under our assumptions, the ideal \( \mathcal{J} \) defines zero (see its definition page 798). So we have to prove that for every \( P \in \mathcal{D} \) and every non negative integer \( k \geq \ell \), there is an integer \( m \in \mathbb{N}^* \) such that \( hP \delta_k f^{s+1} \) belongs to \( \mathcal{D} \mathcal{J} \delta_\ell f^{s+1} \) for every \( h \in \mathcal{J}^m \). This may be done by induction on \( k - \ell \) and on the degree \( d \) of the operator \( P \), using that \( hP \in \mathcal{D} \mathcal{J} \) for \( h \in \mathcal{J}^{d+1} \) and that \( u \delta_k f^{s+1} \in \mathcal{D} \delta_k f^{s+1} \) for \( u \in \mathcal{J} \) (with the help of identities (2) & (3), page 801).

Let \( E \) be a \( \mathbb{C} \)-vector subspace of \( \mathcal{O} \) isomorphic to \( \mathcal{O} / \mathcal{J} \) by projection, \( \mathcal{D} \subset \mathcal{D} \) the ring of differential operators with constant coefficients, \( \mathcal{D} E \subset \mathcal{D} \) the subspace generated by \( \partial^\beta e \), \( e \in E \), and \( \mathcal{D} \mathcal{J} \subset \mathcal{D} \) the left ideal generated by \( \mathcal{J} \).

Proposition 3.2. For every \( \ell \in \mathbb{N}^* \), there is a decomposition:

\[
\sum_{k \geq 1} \mathcal{D} \delta_k f^{s+1} = \mathcal{D} \mathcal{J} \delta_\ell f^{s+1} \oplus \bigoplus_{k \geq \ell} \mathcal{D} E \delta_k f^{s+1}
\]

Proof. First remark that the \( \mathcal{D} \)-modules \( \mathcal{D} \delta_k f^{s+1} \), \( 1 \leq k \leq \ell - 1 \), are contained in \( \mathcal{D} \mathcal{J} \delta_\ell f^{s+1} \) (since \( g \in \mathcal{J} \)). So, to get the existence of the decomposition, it is enough to prove it only for the elements \( u \delta_k f^{s+1} \), \( u \in \mathcal{O} \), \( k \geq \ell \). By division by \( \mathcal{J} \), there exists a uniquely defined element \( e \in E \), and \( h, \lambda_{i,j} \in \mathcal{O} \), \( 1 \leq i < j \leq \ell \) such that \( u = e + hg + \sum_{i<j} \lambda_{i,j} (f'_{x_j} g'_{x_i} - f'_{x_i} g'_{x_j}) \). Hence we have:

\[
u \delta_k f^{s+1} = e \delta_k f^{s+1} - (k - 1)h \delta_{k-1} f^{s+1}
\]

\[
+ \left[ \sum_{i<j} \left( \frac{\partial}{\partial x_i} f'_{x_j} - \frac{\partial}{\partial x_j} f'_{x_i} \right) \lambda_{i,j} - \left( f'_{x_j} \frac{\partial \lambda_{i,j}}{\partial x_i} - f'_{x_i} \frac{\partial \lambda_{i,j}}{\partial x_j} \right) \right] \delta_{k-1} f^{s+1}
\]

for \( k \geq \ell + 1 \). So, by induction on \( k \), every element of \( \sum_{k \geq 1} \mathcal{D} \delta_k f^{s+1} \) may be decomposed in \( \mathcal{D} \mathcal{J} \delta_\ell f^{s+1} \oplus \bigoplus_{k \geq \ell} \mathcal{D} E \delta_k f^{s+1} \).

The proof of the uniqueness uses that the ideals \( \text{Ann}_{\mathcal{D}} \delta_k f^{s+1} \), \( k \in \mathbb{N}^* \), are contained in \( \mathcal{D} \mathcal{J} \) (see [19], [21]). Suppose that \( V \delta_\ell f^{s+1} + \sum_{k=\ell}^\ell U_k \delta_k f^{s+1} = 0 \)
with $V \in \mathcal{D}J$ and $U_k \in DE$. This may be written:

$$\left[ (-1)^{L+\ell} \frac{(\ell - 1)!}{(L - 1)!} V g^{L-\ell} + U_L + \sum_{k=\ell}^{L-1} (-1)^{L+k} \frac{(k - 1)!}{(L - 1)!} U_k g^{L-k} \right] \delta_L f^{s+1} = 0$$

As $\text{Ann}_D \delta_L f^{s+1} \subseteq \mathcal{D}J$, the operator $U_L$ belongs to $DE$ and to $\mathcal{D}J$ in the same time, and so it is zero. By induction, we prove that $U_k$, $\ell \leq k \leq L - 1$, are zero too, and then $V \delta_\ell f^{s+1} = 0$. Consequently, we get the assertion. \qed

Let $D' \subset D$ be the ideal of operators without nonzero constant term. Given $\kappa \in \mathbb{N}^*$, we consider the linear morphism:

$$c_\kappa : \bigoplus_{k \geq \kappa} D\delta_k f^{s+1} = \mathcal{D}J \delta_\kappa f^{s+1} \oplus \bigoplus_{k \geq \kappa} DE \delta_k f^{s+1} \longrightarrow \bigoplus_{k \geq \kappa} E \delta_k f^{s+1}$$

defined by $c_\kappa(D\delta_k f^{s+1}) = 0$ and if $Q = Q' + e$ with $Q' \in D'E, e \in E$, then $c_\kappa(\delta_k f^{s+1}) = e \delta_k f^{s+1}$ for every $k \geq \kappa$. Its kernel is $\mathcal{D}J \delta_\kappa f^{s+1} \oplus \bigoplus_{k \geq \kappa} D'E \delta_k f^{s+1}$. So we have the inclusion: $\bigoplus_{k \geq 1} D'O \delta_k f^{s+1} \subseteq \ker c_\kappa$. Hence $c_\kappa$ induces an isomorphism:

$$\hat{c}_\kappa : H^n_{DR} \left( \bigoplus_{k \geq \kappa} \mathcal{D}J \delta_k f^{s+1} \right) \longrightarrow \bigoplus_{k \geq \kappa} E \delta_k f^{s+1}. \quad (9)$$

§3.2. The spaces $\mathcal{Z}_\ell$, $\mathcal{Z}_\ell'$ and the polynomial $\tilde{b}_\ell(s)$

Given $\ell \in \mathbb{N}^*$, let us denote $\mathcal{Z}_\ell' = c_\ell(D[s] \{ \text{jac}(g), g \} \delta_\ell f^{s+1})$ and $\mathcal{Z}_\ell = c_\ell(D[s]J \delta_\ell f^{s+1}) \subseteq \mathcal{Z}_\ell'$. Now we give some general results on these $\mathbb{C}$-vector spaces.

**Lemma 3.3.** For every $\ell \in \mathbb{N}^*$, there are the following identifications:

$$\mathcal{Z}_\ell' = c_\ell(D[s]g'_s \delta_\ell f^{s+1}), \quad \mathcal{Z}_\ell = c_\ell(D[s]f'_s \delta_\ell f^{s+1})$$

where $r$ is an index such that $f'_r$ is a unit.

It is a consequence of Remark 2.4.

**Proposition 3.4.** For every $\ell \in \mathbb{N}^*$, the dimensions of the spaces $\mathcal{Z}_\ell$ and $\mathcal{Z}_\ell'$ are finite.

**Proof.** From regularity of the holonomic $\mathcal{D}$-module $\mathcal{R}$, there exist good operators in $s$ in the annihilator of $\delta f^s$, $\delta \in \mathcal{R}$, i.e. of the form $s^N + P_1 s^{N-1}$
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+ \cdots + P_N \in \mathcal{D}[s] where the degree of \( P_i \in \mathcal{D} \) is less or equal to \( i \) (see [4], [18]).

If \( N \) is the degree of such an operator annihilating \( \delta f^{s+1} \), then:

\[
\mathcal{D}[s] \delta f^{s+1} = \sum_{i=0}^{N-1} s^i \mathcal{D} \delta f^{s+1} \subset \sum_{k=1}^{N+\ell-1} \mathcal{D} \delta_k f^{s+1}
\]

(see identity (8)). In particular, the dimension of \( c_\ell(D[s] \delta f^{s+1}) \) is finite, and the one of \( Z_\ell, Z'_\ell \) are finite too.

Remark that the dimension of \( Z_\ell, Z'_\ell \) and \( Z'_\ell/Z_\ell \) depends on the integer \( \ell \) (see the example studied in the last part).

Given \( \ell \in \mathbb{N}^* \), we define the action of \( s \) on \( \bigoplus_{k \geq \ell} E \delta_k f^{s+1} \) by \( sU = c_\ell(sU) \).

Remark that \( c_\ell(sU) \in \mathcal{Z}_\ell \) when \( U \in \ker c_\ell \). Indeed, \( s \bigoplus_{k \geq \ell} D' E \delta_k f^{s+1} \) is contained in the kernel of \( c_\ell \). Hence, the action of \( s \) on \( \bigoplus_{k \geq \ell} E \delta_k f^{s+1} \) is well defined on \( \mathcal{Z}_\ell, \mathcal{Z}'_\ell \), and then on \( \mathcal{Z}'_\ell/\mathcal{Z}_\ell \).

The proof of Theorem 1.1 is the very same as the one of [21], Theorem 1.1. It uses Lemma 3.1, the identification (9) and the fact that the functor \( H^n_{DR} \), from the category of \( \mathcal{D} \)-modules supported by zero to the category of \( \mathbb{C} \)-vector spaces, is an exact and faithful functor ([11]).

§4. The Computational Algorithm for Non Degenerate Hypersurfaces

Here we adapt to the case of polynomials \( \tilde{b}_\ell(s) \) the algorithm of computation of Bernstein polynomial of a non-degenerate convenient germ with respect to its Newton boundary in the sense of Kouchnirenko (see [2]). We invite the reader to see [2] for the proof of some results which may be easily extended.

§4.1. Division by \( J \) and increase in weight

Let \( g \in \mathcal{O} \) be a nonzero germ of an holomorphic function with \( g(0) = 0 \). Its Taylor expansion is written \( \sum_{A \in \mathbb{N}^n} g_A x^A \) where \( g_A \in \mathbb{C} \) and \( x^A = x_1^{a_1} \cdots x_n^{a_n} \) for \( A = (a_1, \ldots, a_n) \in \mathbb{N}^n \).

Let \( N(g) = \{ A \in \mathbb{N}^n \mid g_A \neq 0 \} \) be the Newton cloud of \( g \) and \( \Gamma(g) \subset (\mathbb{R}^+)^n \) its Newton boundary, the union of compact faces of the convex hull of \( N(g) + \mathbb{N}^n \). For every face \( \Delta \subset \Gamma(g) \) and every \( u = \sum_{A \in \mathbb{N}^n} u_A x^A \in \mathcal{O} \), we denote \( u|_{\Delta} = \sum_{A \in \Delta} u_A x^A \) the restriction of \( u \) to \( \Delta \).

We make the following assumptions on \( g \):
- $g$ is convenient: each coordinate line has a point contained in $\Gamma(g)$.

- $g$ is non-degenerate with respect to its Newton boundary: for every face $\Delta \subset \Gamma(g)$, the system:

$$\left( x_1 \frac{\partial g}{\partial x_1} \right)_{\Delta} = \cdots = \left( x_n \frac{\partial g}{\partial x_n} \right)_{\Delta} = 0$$

does not have any solution in $(\mathbb{C}^*)^n$.

Under these conditions, $g$ defines an isolated singularity. We will suppose that $f = x_1$. In particular, the ideal $J$ is $(g, g_{x_2}, \ldots, g_{x_n})\mathcal{O}$. Moreover the morphism $(x_1, g)$ defines a isolated singularity too, because the restriction of $g$ to $x_1 = 0$ is also convenient and non-degenerate.

Remark that the system of equations in the definition of the non-degeneracy condition is equivalent to the following one:

$$g|_{\Delta} = \left( x_2 \frac{\partial g}{\partial x_2} \right)_{\Delta} = \cdots = \left( x_n \frac{\partial g}{\partial x_n} \right)_{\Delta} = 0$$

because $g|_{\Delta}$ is a weighted-homogeneous polynomial in restriction to every face $\Delta \subset \Gamma(g)$. Let us recall that a nonzero polynomial is weighted-homogeneous of weight $d \in \mathbb{Q}^+$ for a system $\alpha \in (\mathbb{Q}^+)^n$ if it is a $\mathbb{C}$-linear combination of monomials $x^\alpha$ with $(\alpha, A) = d$.

Now we introduce some notations before giving the division theorem by the ideal $J$ which is adapted to our situation.

**Notation 4.1.** Let $\mathcal{F}$ be the set of $n - 1$ dimensional faces of $\Gamma(g)$. Given $F \in \mathcal{F}$, we consider the vector $\alpha_F = (\alpha_{F,1}, \ldots, \alpha_{F,n}) \in (\mathbb{Q}^+)^n$ such that $\langle \alpha_F, A \rangle = 1$ for every $A \in F$. The weight $\rho_F(u)$ in relation to the face $F \in \mathcal{F}$ of a nonzero germ $u = \sum_{A \in \mathbb{N}^n} u_A x^A \in \mathcal{O}$ is also defined by $\rho_F(u) = \inf \{ \langle \alpha_F, A \rangle | u_A \neq 0 \} \in \mathbb{Q}^+$. By agreement, we fix $\rho_F(0) = +\infty$. Then we define the weight of a germ $u \in \mathcal{O}$ in relation to $\Gamma(g)$ by $\rho(u) = \inf_{F \in \mathcal{F}} \rho_F(u)$.

For every rational $q \in \mathbb{Q}$, let us denote $\mathcal{O}_{\geq q} = \{ u \in \mathcal{O} | \rho(u) \geq q \}$ and $\text{gr} \mathcal{O} = \bigoplus_{q \in \mathbb{Q}^+} \mathcal{O}_{\geq q}$. We define another weight function, $\rho^* : \mathcal{O} \to \mathbb{Q}^+ \cup \{ +\infty \}$, by $\rho^*(u) = \inf_{F \in \mathcal{F}} \rho^*_F(u)$ where $\rho^*_F(u) = \rho_F(ux_2 \cdots x_n)$ for every $u \in \mathcal{O}$. As above, we have the spaces $\mathcal{O}_{\geq q}^*, \mathcal{O}_{\geq q}^*$, $q \in \mathbb{Q}$. If $\mathcal{O}_{\geq q}$ is the set of germs $u \in \mathcal{O}$ such that $ux_2 \cdots x_n$ is a polynomial supported by $q\Gamma(g)$, then $\text{gr}^* \mathcal{O} = \bigoplus_{q \in \mathbb{Q}} \mathcal{O}_{\geq q}^*/\mathcal{O}_{q}^*$ may be identified to $\bigoplus_{q \in \mathbb{Q}} \mathcal{O}_{q}^*$.

For every $u \in \mathcal{O}$ nonzero, let $\text{in}^*(u)$ be the coset of $u$ in $\mathcal{O}_{\geq \rho^*(u)}^*/\mathcal{O}_{\rho^*(u)}^*$ identified to $\mathcal{O}_{\rho^*(u)}^*$. For every $q \in \mathbb{Q}^+$, let $E_q^* \subset \mathcal{O}_q^*$ be a supplementary of
$O_q^* \cap \text{in}^*(\mathcal{J})$ in $O_q^*$, where $\text{in}^*(\mathcal{J}) \subset \mathbb{C}[x]$ is the ideal generated by the initial parts of the elements of $\mathcal{J}$. Finally, let $E_{2q}^* \subset E$ be the space $\bigoplus_{q \geq q} E_q^*$.

**Theorem 4.2.** ([2], [9]) For every $u \in \mathcal{O}$, there exists a unique element $v \in E = \bigoplus_q E_q^*$ and $\lambda_1, \ldots, \lambda_n \in \mathcal{O}$ such that:

$$u = v + \lambda_1 g + \sum_{i=2}^n \lambda_i g_{x_i}^s,$$

where $\rho^*(v) \geq \rho^*(u)$, $\rho^*(\lambda_1) \geq \rho^*(u) - 1$, and for $2 \leq i \leq n$: $\rho^*(\lambda_i g_{x_i}^s) \geq \rho^*(u)$, $\rho^*(\lambda_i) \geq \rho^*(u) - 1 + \rho(x_i)$, $\rho^*(\partial \lambda_i / \partial x_i) \geq \rho^*(u) - 1$.

The proof is a direct adaptation of the one of Proposition B.1.2.2, B.1.2.3, B.1.2.6 of [2], which need Theorems 2.8 and 4.1 of [9]. In particular, the multiplication by $x_2 \cdots x_n$ induces a strict isomorphism $\lambda$ from $(\mathcal{O}/\mathcal{J}, \rho^*)$ to $(\mathcal{O}x_2 \cdots x_n/\mathcal{O}x_2 \cdots x_n \cap I(g), \rho)$ where $I(g) = (g, x_2 g_{x_2}, \ldots, x_n g_{x_n})\mathcal{O}$.

Indeed, these Kouchnirenko results are true for every non-degenerate family $h_1, \ldots, h_n \in \mathcal{O}$, i.e. satisfying the non-degeneracy condition and such that $\rho(h_i) = 1$ for $1 \leq i \leq n$. In particular, the family $\{g, x_2 g_{x_2}, \ldots, x_n g_{x_n}\}$ is non-degenerate.

Let us denote $\Pi^* = \{q \in \mathbb{Q}^+ \mid E_q^* \neq 0\}$ and $\sigma^* = \sup \{q \mid E_q^* \neq 0\}$. Rewriting [2, p. 566], we get:

$$n - \sup_{F \in \mathcal{F}} \rho_F(x_1 \cdots x_n) \leq \sigma^* < n$$

The estimation is obtained by using the Rees function $\mathcal{P}_{I(g)}$, which coincides with the weight function $\rho$ under our assumptions ([3], [17]).

We end by giving the technical lemmas at the root of the algorithm. First we give a filtered version of Proposition 3.2.

**Lemma 4.3.** Given $N, \ell \in \mathbb{N}^*$, $q \in \mathbb{Q}$, there is the following identity in $\sum_{k \geq 1} D \delta k x_1^{s+1}$:

$$\sum_{k=1}^N D O_q^* \supseteq q+k \delta k x_1^{s+1} = D J_{q+\ell} \delta \ell x_1^{s+1} \oplus D E_q^* \supseteq q+k \delta k x_1^{s+1}$$

where $J_{q+\ell} = J \cap O_{2q+\ell}^*$.

For every face $F \in \mathcal{F}$, let us denote $|\alpha_F| \in \mathbb{Q}^+$ the sum $\sum_{i=1}^n \alpha_{F,i}$, $\chi_F = \sum_{i=1}^n \alpha_{F,i} x_i (\partial / \partial x_i)$ the Euler vector field associated with $F$, $\overline{\chi}_F = \sum_{i=1}^n \alpha_{F,i} (\partial / \partial x_i) x_i = \chi_F + |\alpha_F|$ and $h_F = \chi_F(g) - g \in \mathcal{O}$. 

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Lemma 4.4. Given \( w \in C, F \in F, u \in O \) and \( k \in \mathbb{N}^* \), there is an identity:

\[
(\alpha_{F,1}(s+1) + |\alpha_F| + w)u \delta_k x_1^{s+1} = [\chi_F u + [(w+k)u - \chi_F(u)]] \cdot \delta_k x_1^{s+1}
\]

and the following identities, for every \( F' \in F' \):

\[
\rho_{F'}(x_j u) > \rho^{*}(u), \quad \rho_{F'}((w+k)u - \chi_F(u)) \geq \rho^{*}(u), \quad \rho_{F'}(uh_F) \geq \rho^{*}(u) + 1
\]

If \( F' = F \), then \( \rho_{F'}(uh_F) > \rho^{*}(u) + 1 \). Moreover, if \( \rho_{F'}(u) > \rho^{*}(u) \) or \( \rho_{F'}(u) = \rho^{*}(u) = w + k + |\alpha_F| - \alpha_{F,1} \), then \( \rho_{F'}((w+k)u - \chi_F(u)) > \rho^{*}(u) \).

For every monomial \( u \), let \( F^*(u) \subset F \) be the set of the faces \( F \) with \( \rho_{F}(u) = \rho^{*}(u) \); if \( u \in O \) is nonzero, then \( F^*(u) \subset F \) is the set of \( F \in F \) such that there exists a monomial \( v \) in in \( F \) with \( \rho_{F}(v) = \rho^{*}(u) \). Using Lemma 4.4, we get the following formula:

Lemma 4.5. For every \( u \in O \) nonzero and \( k \in \mathbb{N}^* \):

\[
\prod_{F \in F^*(u)} (\alpha_{F,1}(s+2) + \rho^{*}(u) - k) \left[ u \delta_k x_1^{s+1} \right] \in \sum_{i=0}^{\#F^*(u)} \mathcal{DO}_{\rho^*(u)+i}^* \delta_{k+i} x_1^{s+1}
\]

Remark that the multiplicity of a factor \( (\alpha_{F,1}(s+2) + \rho^{*}(u) - k) \) in the given polynomial may be arbitrarily high. The next result states the existence of a polynomial such that the multiplicities are strictly smaller than \( n \).

Proposition 4.6. Let \( u \in O \) nonzero and \( k \in \mathbb{N}^* \). Let \( A^*(u) \subset \mathbb{Q}^{s*} \) be the set of \( \alpha_{F,1} \) with \( F \in F^*(u) \). Then:

\[
\prod_{a \in A^*(u)} (a(s+2) + \rho^{*}(u) - k) \left[ u \delta_k x_1^{s+1} \right] \in \sum_{i=0}^{(n-1) \times \#A^*(u)} \mathcal{DO}_{\rho^*(u)+i}^* \delta_{k+i} x_1^{s+1}
\]

We prove this result in the next paragraph.

§4.2. Proof of Proposition 4.6

We need some additional notations.
Let us attach to any face $F \in \mathcal{F}$ the closed cone $C(F) \subset (\mathbb{R}^+)^n$, the union of linear half-lines going through $F$. In particular, $A \in (\mathbb{R}^+)^n$ belongs to $C(F)$ if and only if $\inf_{F' \in \mathcal{F}} \langle \alpha_{F'}, A \rangle = \langle \alpha_F, A \rangle$. Let us denote $\mathcal{C}$ the fan with support in $(\mathbb{R}^+)^n$ associated with the Newton boundary $\Gamma(g)$. We recall that it is the smallest family of convex polyhedral rational convex cones of $(\mathbb{R}^+)^n$ which contains the cones $C(F), F \in \mathcal{F}$, and verifies the conditions:

- if $C$ is a facet of a cone of $\mathcal{C}$ then $C \in \mathcal{C}$;
- if $C_1, C_2 \in \mathcal{C}$, then $C_1 \cap C_2$ is a facet of $C_1$ and $C_2$.

For every $A \in (\mathbb{R}^+)^n$ nonzero, we note $C(A) \in \mathcal{C}$ the cone of smallest dimension which contains $A$, and $d(A) \in \mathbb{N}$ its dimension. In particular, we have $1 \leq d(A) \leq n$ and $d(A) = n$ if and only if $A$ belongs to the interior of a cone $C(F)$.

The proof of the proposition uses the following elementary results.

**Lemma 4.7.** Let $F \in \mathcal{F}$ and let $A, A' \in C(F)$ be two nonzero vectors such that $A' \not\in C(A)$. Then $A, A' \in C(A + A')$ and so $d(A + A') \geq d(A) + 1$.

**Lemma 4.8.** Let $F_1, \ldots, F_m \in \mathcal{F}$ be faces such that $\alpha_{F_1,1}, \ldots, \alpha_{F_m,1}$ are equal. Let $A \in (\mathbb{R}^+)^n$ be a vector belonging to the cone $C(F_1, \ldots, F_m) = C(F_1) \cap \cdots \cap C(F_m)$ and such that $\inf_{F \in C(F)} \langle \alpha_{F}, A \rangle = \alpha_{F_1,1}$. Then, for every $\epsilon \in \mathbb{R}^+$ small enough, the vector $A + \epsilon(1, 0, \ldots, 0)$ belongs to $C(F_1, \ldots, F_m)$.

**Proof of Proposition 4.6.** Without loss of generality, we assume that $u$ is a monomial; we denote $A \in \mathbb{N} \times (\mathbb{N}^*)^{n-1}$ the $n$-uptlet such that $ux_2 \cdots x_n$ is $\mathbb{C}$-proportional to $x^A$.

Let $F_1 \in \mathcal{F}^+(u)$. Using Lemma 4.4, we have:

$$
\langle \alpha_{F_1,1}(s+2) + \rho^*(u) - k \rangle u \delta_k x_1 \delta_{k+1} x_1^{s+1} = \bar{x}_{F_1} \cdot u \delta_k x_1 \delta_{k+1} x_1^{s+1} - u h_{F_1} \delta_{k+1} x_1^{s+1}
$$

where $\bar{x}_{F_1} \cdot u \delta_k x_1 \delta_{k+1} x_1^{s+1} \in DO_{\rho^*(u)} \delta_{k+1} x_1^{s+1}$. If $w_1 = x^{A_1}$ is a monomial of the Taylor expansion of $h_{F_1}$, then two cases are possible:

- First case: $\rho^*(uw_1) > \rho^*(u) + 1$. Then $uw_1 \delta_{k+1} x_1^{s+1} \not\in O_{\rho^*(u)+1} \delta_{k+1} x_1^{s+1}$.

- Second case: $\rho^*(uw_1) = \rho^*(u) + 1$. As $\rho_F(h_{F_1}) \geq 1$ with an equality if and only if $F \neq F_1$, we have also $\mathcal{F}^+(uw_1) = \{ F \in \mathcal{F}^+(u) | A_1' \in F \}$ and this set does not contain $F_1$. From Lemma 4.7 applied with $A \in C(F_1) \cap C(F_2)$, $A' = A_1' \in C(F_2) - C(F_1)$ for $F_2 \in \mathcal{F}^+(uw_1)$, we get $d(A + A_1') \geq d(A) + 1$. 


Hence, up to an element of the \( D \)-module \( \sum_{i=0}^n \mathcal{DO}_{>\rho}(u) \cdot \delta_k + x_1^{k+1} \), the element \((\alpha_{F,1}(s+2)+\rho(u)-k)\delta_k \cdot x_1^{k+1}\) is equal to a \( C \)-linear finite combination of terms \( uw_1 \delta_k + x_1^{k+1}\) with weight \( \rho(u) - k \) such that \( \mathcal{F}^*(uw_1) \subset \mathcal{F}^*(u) - \{ F_1 \} \) and \( d(A + A'_1) \geq 2 \) if \( w_1 a x_2 \cdots x_n = x^{A+1} A'_1 \).

Remark that if \( d(A + A') = n \) then \( \mathcal{F}^*(uw) \) has necessarily one element. So, when a polynomial \( c(s) \in C[s] \) allows to use \( n \) times this process, we prove that \( c(s) \delta_k \cdot x_1^{k+1} \) belongs to \( \mathcal{D}[s]_{\leq \deg c(s)-n} \sum_{i=0}^n \mathcal{DO}_{>\rho}(u) \cdot \delta_k + x_1^{k+1} \) then to \( \sum_{i=0}^{\deg c(s)} \mathcal{DO}_{>\rho}(u) \cdot \delta_k + x_1^{k+1} \) (Lemma 4.4). In particular, the polynomial \( \prod_{a \in \mathcal{A}^*(u)} (a(s+2) + \rho(u) - k) \) is suitable. We will prove that the power \( n - 1 \) is sufficient.

It is easy to see that it is true if \( d(A) \geq 2 \). Remark that it is again true when there exists \( a \in \mathcal{A}^*(u) \) such that \( c_{F,1} = a \) for at most \( n-1 \) faces \( F \in \mathcal{F}^*(u) \) (this is true if \( n = 2 \)). Indeed, by taking such a face \( F_1 \in \mathcal{F}^*(u) \), the polynomials of degree less or equal to \( n \) so used to get terms \( uw_1 \cdot w_i \delta_k \cdot x_1^{k+1}, i \leq n \), with a weight strictly greater than \( \rho(u) - k \), are multiples of \((a(s+2) + \rho(u) - k)\), but they cannot be equal to \((a(s+2) + \rho(u) - k)\). A similar argument allows us to conclude that there exists \( F_1 \in \mathcal{A}^*(u) \) such that, for every monomial \( u_1 \) of the Taylor expansion of \( h_{F,1} \) with \( \rho(u_1) = \rho(u) - 1 \), the set \( \mathcal{A}^*(u_1) \) is not reduced to \( \{ \alpha_{F,1} \} \).

So we have just to consider the following case: \( n \geq 3, d(A) = 1 \), and, for every \( F \in \mathcal{F}^*(u) \), there exists at least one monomial \( w = x^{A'} \) in the Taylor expansion of \( h_{F,1} \) such that \( \rho(uw) = \rho(u) + 1, d(A + A') = 2, \mathcal{A}^*(uw) \subset \{ \alpha_{F,1} \} \) and the set \( \mathcal{F}^*(uw) \) has at least \( n - 1 \) elements. We will prove that after at least \( n - 1 \) iterations of the general process given above, we get a sum of terms \( uw_1 \cdots w_i \delta_k \cdot x_1^{k+1}, i \leq n - 1 \) with a weight strictly greater than \( \rho(u) - k \).

Let \( F_1 \in \mathcal{F}^*(u) \) such that \( \alpha_{F,1} \) is the smallest element of \( \mathcal{A}^*(u) \). Let \( w_1 = x^{A'_1} \) be a monomial in the Taylor expansion of \( h_{F,1} \) which verifies the requisite conditions, and let \( \mathcal{F}^*(uw_1) = \{ F_2, \ldots, F_m \} \). Let us prove that \( A + A'_1 \) is necessarily in the interior of the cone \( \{ 0 \} \times (\mathbb{R}^+)^{n-1} \). Otherwise the vector \( A + A'_1 \in (\mathbb{N}^+)^n \) is in the interior of the cone \( C(F_2, \ldots, F_m) = C(F_2) \cap \cdots \cap C(F_m) \) and \( A'_1 \notin C(F_1) \). As \( A \in C(A + A'_1) \cap C(F_1) \) and \( A'_1 \neq C(F_1) \), the cone \( C(F_1, F_2, \ldots, F_m) \) is contained in a facet of \( C(A + A'_1) \). Then for a dimensional argument, it coincides with \( C(A) \). But, from Lemma 4.8, this is not possible because \( d(A) = 1 \) and \( A \in \mathbb{N} \times (\mathbb{N}^+)^{n-1} \). So the assertion is proved.

Now we apply this process for the face \( F_2 \). If \( d(A + A'_1 + A'_2) \geq 4 \), at least \( n - 3 \) additional iterations are enough for ending. So we can assume that \( d(A + A'_1 + A'_2) = 3 \). But \( d(A + A'_1) = 2 \) and \( C(A + A'_1) \subset \{ 0 \} \times (\mathbb{R}^+)^{n-1} \).
So, using again the above argument, we obtain also that $A'_2 \in \{0\} \times (\mathbb{R}^+)^{n-1}$ necessarily, and then $C(A + A'_1 + A'_2) \subset \{0\} \times (\mathbb{R}^+)^{n-1}$. Iterating again at least $n - 4$ times this process and the argument, if it is not finished, then $C(A + A'_1 + \cdots + A'_{n-2})$ is a cone in $\{0\} \times (\mathbb{R}^+)^{n-1}$ of dimension $n-1$. But also $\mathcal{F}^*(uw_1 \cdots w_{n-2})$ is reduced to $\{F\}$ and after a last iteration, $\rho^*(uw_1 \cdots w_{n-2})$ is strictly greater than $\rho^*(u) - k$. This ends the proof.

§4.3. Filtrations and roots of $\tilde{b}_t(s)$

For every $\ell \in \mathbb{N}^*$, the weight function $\rho^*$ may be extend to $\bigoplus_{k \geq \ell} E\delta_k x_1^{s+1}$ by $\rho^*(\sum_k u_k \delta_k x_1^{s+1}) = \min_{k \leq \ell} \{\rho^*(u_k) - k\}$. It induces the decreasing filtration $(\bigoplus_{k \geq \ell} E\delta_k x_1^{s+1})_{\geq q} = \bigoplus_{k \geq \ell} E_{s+1,q+k}^* \delta_k x_1^{s+1}$, $q \in \mathbb{Q}$. Then the spaces $Z_{\ell}$, $Z_{\ell}'$ and $Z_{\ell}/Z_{\ell}$ get the induced filtrations and we have:

$$
\text{gr}^* Z_{\ell} \hookrightarrow \text{gr}^* Z_{\ell}' \hookrightarrow \text{gr}^* \left( \bigoplus_{k \geq \ell} E\delta_k x_1^{s+1} \right) \cong \bigoplus_{q \geq \ell} \left( \bigoplus_{k \geq \ell} E_{s+1,q+k}^* \delta_k x_1^{s+1} \right)
$$

For every $U = \sum_k u_k \delta_k x_1^{s+1} \in \bigoplus_{k \geq \ell} E\delta_k x_1^{s+1}$ nonzero, the initial part of $U$ is the element in $\text{in}^*(U) \in \bigoplus_{k \geq \ell} E_{s+1}^* (U) + \delta_k x_1^{s+1}$ defined by:

$$
\text{in}^*(U) = \sum_{\rho^*(u_k) - k = \rho^*(U)} \text{in}^*(u_k) \delta_k x_1^{s+1}
$$

If $G \subset \bigoplus_{k \geq \ell} E\delta_k x_1^{s+1}$ is a nonzero subspace, we will denote $\text{in}^*(G)$ the subspace of $\bigoplus_q (\bigoplus_{k \geq \ell} E_{s+1,q+k}^* \delta_k x_1^{s+1})$ generated by the initial parts of the nonzero vectors of $G$. For $q \in \mathbb{Q}$, let us denote $Z_{\ell,q} = \text{in}^*(Z_{\ell}) \cap \bigoplus_{k \geq \ell} E_{s+1,q+k}^* \delta_k x_1^{s+1}$, and $Z_{\ell,q}' = \text{in}^*(Z_{\ell}') \cap \bigoplus_{k \geq \ell} E_{s+1,q+k}^* \delta_k x_1^{s+1}$. In particular, the rational numbers $q$ with $Z_{\ell,q}' \neq 0$ are contained in $\{q \in \mathbb{Q} \mid \exists k \in \mathbb{N}^*, q + k \in \Pi^*\}$.

Using (8) and Lemma 4.3, we prove that the action of $s$ on $Z_{\ell}'/Z_{\ell}$ respects the filtration by $\rho^*$ and induces an action of degree zero on $\text{gr}^* (Z_{\ell}'/Z_{\ell})$. For every $q \in \mathbb{Q}$, let us denote $\tilde{b}_{\ell,q}(s)$ the minimal polynomial of $s$ on $\text{gr}^* (Z_{\ell}'/Z_{\ell})$. So, from Theorem 1.1, we have:

**Theorem 4.9.** *The polynomial $\tilde{b}_t(s)$ is the l.c.m. of the polynomials $\tilde{b}_{\ell,q}(s)$:*

$$
\tilde{b}_t(s) = \text{l.c.m.} Z_{\ell,q} \subseteq Z_{\ell}' \tilde{b}_{\ell,q}(s)
$$

Remark that, contrary to the classical case, the polynomials $\tilde{b}_{\ell,q}(s)$ are not a power of an affine form (see Lemma 4.5). In Proposition 4.6, we have proved that the multiplicities of their roots are strictly smaller than $n$. Thus:
Theorem 4.10. The multiplicity of a root of $\tilde{b}_t(s)$ is at most $n - 1$.

Remark 4.11. Up to a change of notations, the first part of the proof of Proposition 4.6 allows to prove in the case of a non-degenerate convenient germ that the multiplicities of its reduced Bernstein polynomial are raised by $n$.

§4.4. The effective computation

Thus the determination of $\tilde{b}_t(s)$ needs the one of spaces $Z^*_t$, and $Z^*_{t,q}$, $q \in \mathbb{Q}$. Here we adapt the method given in [2], and we apply it on an example.

Using the following formula:

$$
(\alpha F_1(s + 1) + w - (\alpha F_1) - \chi F_1) \partial x^p a_1 x^{p+1} = \partial x^p [(w + k - |\alpha F_1|)u - \chi F_1(u)] \partial x^{p+1} - \partial x^p u \partial x_{k+1} x^{p+1}
$$

for $u \in \mathcal{O}$, $k \in \mathbb{N}^*$, $w \in \mathbb{C}$, $\beta \in \mathbb{N}$, and Lemma 4.3, we construct a sequence $(S_{t,m})_{1 \leq m \leq M_t}$ of good operators $S_{t,m}$ in degree $m$, a creasing sequence of rational numbers $(q_{t,m})_{1 \leq m \leq M_t - 1}$ with $q_{t,1} \geq \rho(s) x_{1} g_{x_1}'$, and a sequence $(H_{t,m})_{1 \leq m \leq M_t - 1}$ of elements of $D_{k \geq t} \partial \delta x_{1}^{p+1}$ such that:

- $S_{t,m} x_{1} g_{x_1}' \delta x_{1}^{p+1} - H_{t,m} \in D_{\partial} \delta x_{1}^{p+1}$ for $1 \leq m \leq M_t - 1$;
- $S_{t,M_t} x_{1} g_{x_1}' \delta x_{1}^{p+1} \in D_{\partial} \delta x_{1}^{p+1}$;
- $H_{t,m} = \sum_{\ell \leq k \leq m - 1} H_{t,m,k} \delta x_{1}^{p+1}$ with $H_{t,m,k} \in D_{\delta, \partial}^{\ast}$ of degree at least $m + \ell - k - 1$.

Then this sequence $(H_{t,m})$ determines $Z_t$:

$$Z_t = \left\{ \sum_{m=1}^{M_t-1} c_m(a_m H_{t,m}) + c_\ell(a_0 x_{1} g_{x_1}' \delta x_{1}^{p+1}) \mid a_m \in \mathcal{O} \right\}
$$

because $Z_t$ coincides with $c_\ell(D[s] x_{1} g_{x_1}' \delta x_{1}^{p+1})$ (Lemma 3.3) and, for every $P(s) \in D[s]$:

$$P(s) x_{1} g_{x_1}' \delta x_{1}^{p+1} = \sum_{m=1}^{M_t-1} D S_{t,m} x_{1} g_{x_1}' \delta x_{1}^{p+1} + D x_{1} g_{x_1}' \delta x_{1}^{p+1} + D_{\partial} \delta x_{1}^{p+1}
$$

Indeed, by division we have: $P(s) = P_{Mt} S_{t,M_t} + \sum_{m=1}^{M_t-1} P_m S_{t,m} + P_0$ where $P_m \in D$, $0 \leq m \leq M_t - 1$, and $P_{Mt} \in D[s]_{\leq d-M_t}$ if $d \in \mathbb{N}$ is the degree in $s$ of $P(s)$. An induction on $d$ allows us to conclude, using Remark 2.4 and that $S_{t,M_t} x_{1} g_{x_1}' \delta x_{1}^{p+1} \in D_{\partial} \delta x_{1}^{p+1}$.

The determination of $Z^*_t$ is similar, using sequences $(S^*_{t,m})_{1 \leq m \leq M_t}$, $(q^*_{t,m})_{1 \leq m \leq M_t - 1}$ with $q^*_{t,1} \geq \rho(s) x_{1} g_{x_1}'$, and $(H^*_{t,m})_{1 \leq m \leq M_t - 1}$.
Remark 4.12. If the Newton polyhedron of $g$ has only one $(n-1)$-dimensional face $F$ - with normal vector $\alpha \in (\mathbb{Q}^+)^n$ - the algorithm is very simple, exactly as in [2], part 2. In fact, it is enough to suppose that $g|_F$ and $(g|_F,x_1)$ define some isolated singularities, i.e. $g$, $(g,x_1)$ are semi-weighted-homogeneous morphism. Then the division theorem used in [2], p. 593, is sufficient, and so the weight function $\rho = \rho_F$ is enough. Moreover, II is also the set of the weights of a weighted-homogeneous co-basis of the ideal $\text{in}(\mathcal{J}) = (\text{in}(g), \text{in}(g_{x_2}), \ldots, \text{in}(g_{x_n}))\mathbb{C}[x]$, with $\sigma = n - 2|\alpha| + 1$, and the formula given in Lemma 4.4 ends in one time:

$$(\alpha_1(s+1) + |\alpha| + \rho(u) - k)u^{\delta_k} x_1^{s+1}$$

$$\in \mathcal{D}O_{>\rho(u)}^{\delta_k} x_1^{s+1} + \mathcal{D}O_{\geq \rho(u) + \rho(h)}^{\delta_k+1} x_1^{s+1}$$

where $h = \chi(g) - g$. Hence $\langle \alpha_1(s+1) + |\alpha| + q \rangle$ annihilates $gr_q \mathcal{Z}_i/\mathcal{Z}_t$, and the polynomial $\tilde{b}_t(s)$ is given by:

$$\tilde{b}_t(s) = \prod_{z_i \in \mathcal{Z}_i} \left(s + 1 + \frac{|\alpha| + q}{\alpha_1}\right)$$

When $g$ is in fact a weighted-homogeneous polynomial, we easily get:

$$\tilde{b}_t(s) = \prod_{p \in \Pi'} \left(s + \frac{|\alpha| + 1 + p - \ell}{\alpha_1}\right)$$

where $\Pi' \subset \mathbb{Q}^+$ is the set of the weights of a weighted homogeneous co-basis of $(x_1, g_{x_2}, \ldots, g_{x_n})\mathcal{O}$ (see [22]).

Example. Let $g$ be the germ $x_1^d + x_2^d + x_3^d + x_1^2 x_2^2 x_3^2$ with $d \geq 9$, and $f = x_1$. The computation of the Bernstein polynomial of $g$ is done in [2]. Here we determinate the polynomials $b_t(s)$, $\ell \in \mathbb{N}^*$.

The Newton polyhedron of $g$ has exactly three 2-dimensional faces $F_1$, $F_2$, $F_3$, with normal vectors associated:

$$\alpha_{F_1} = \left(\frac{1}{2} - \frac{2}{d}, \frac{1}{d}, \frac{1}{d}\right), \quad \alpha_{F_2} = \left(\frac{1}{d}, \frac{1}{d}, \frac{1}{2} - \frac{1}{d}\right), \quad \alpha_{F_3} = \left(\frac{1}{d}, \frac{1}{2}, \frac{1}{d} - \frac{2}{d}\right)$$

So $|\alpha_{F_1}| = 1/2$ and $h_{F_1} = (d/2 - 3)x_i^d$, $1 \leq i \leq 3$.

The ideal $\mathcal{J}$ is generated by $g$, $g'_{x_2} = dx_2^{d-1} + x_1^2 x_2 x_3^2$ and $g'_{x_3} = dx_3^{d-1} + 2x_1^2 x_2 x_3$. By taking away the non multiple of $x_2 x_3$ monomials from the monomial basis of $\mathcal{I}(g) = (g, x_2 g'_{x_2}, x_3 g'_{x_3})\mathcal{O}$ given in [2], B.4.2.2.3, we obtain (using the isomorphism $\lambda$) the following monomials:
\[\begin{array}{|c|c|c|}
\hline
u & \rho^*(u) & \\
\hline
(x_1 x_2 x_3)^{\varepsilon} x_1 & (\varepsilon + 1)/2 & 0 \leq \varepsilon \leq 4 \\
(x_1 x_2 x_3)^{\varepsilon} x_1 x_1' & (\varepsilon + 1)/2 + i/d & 0 \leq \varepsilon \leq 2, 1 \leq i \leq d - 1, 1 \leq \theta \leq 3 \\
(x_1 x_2 x_3)^{\varepsilon} x_2' x_3' & \varepsilon/2 + (i + j + 2)/d & 0 \leq \varepsilon \leq 1, 0 \leq i, j \leq d - 2 \\
x_1'^{j} x_0' & 1/2 + (i + j)/d & 1 \leq i, j \leq d - 1, \theta = 2, 3 \\
\hline
\end{array}\]

So this gives a basis of a supplementary \( E \subset \mathcal{O} \) of the ideal \( \mathcal{J} \). Thus \( \sigma^* = 5/2 \), and \( \Pi^* = \{1/2 + k/d \mid 0 \leq k \leq 2d\} \cup \{k/d \mid 2 \leq k \leq 2d\} \).

Now we determinate the space \( Z_\ell = c_\ell(D[s]_x g_{x_1}^d \delta_{\ell x_1^{\varepsilon+1}}) \). First we remark that the division of \( x_1 g_{x_1}^d \) by \( \mathcal{J} \) is given by:

\[x_1 g_{x_1}^d = dx_1 + \frac{2}{d-4} (dg - x_2 g_{x_2}^d - x_3 g_{x_3}^d)\]

Without loss of generality, it is also enough to find the sequence \((H_{\ell,m})\) associated with \( x_1^d \delta_{x_1^{\varepsilon+1}} \). We have the identities:

\[
\left(\frac{1}{d}(s + 1) + \frac{3}{2} - \ell - \mathcal{F}_s\right) x_1^d \delta_{x_1^{\varepsilon+1}} = \left(\frac{6 - d}{2}\right) x_1^d x_2^d \delta_{x_1^{\varepsilon+1}}
\]

\[
\left(\frac{1}{d}(s + 1) + \frac{3}{2} - \ell - \mathcal{F}_s\right) x_1^d x_2^d \delta_{x_1^{\varepsilon+1}} = \left(\frac{6 - d}{2}\right) x_1^d x_2^d \delta_{x_1^{\varepsilon+1}}
\]

where \( \rho^*((x_1 x_2 x_3)^d) = d/2 + 2/d > \sigma^* + 2 \) because \( d \geq 9 \). Hence the term \( (x_1 x_2 x_3)^d \delta_{x_1^{\varepsilon+1}} \) belongs to \( D\mathcal{J} \delta_{x_1^{\varepsilon+1}} \) and so \( M_\ell = 2 \). We get \( H_{\ell,1} \) by rewriting \((d(6 - d)/2)x_1^d x_2^d \delta_{x_1^{\varepsilon+1}}\). As \( dx_1^d x_2^d = x_1^d x_2 g_{x_2}^d - 2(x_1 x_2 x_3)^2 x_1^d \), we obtain:

\[H_{\ell,1} = (d - 6)(x_1 x_2 x_3)^2 x_1^d \delta_{x_1^{\varepsilon+1}} + d\left(\frac{d - 6}{2}\right) \left[x_1^d - \frac{\partial}{\partial x_2} x_1^d x_2^d\right] \delta_{x_1^{\varepsilon+1}}\]

Consequently, \( Z_\ell \) is equal to \( c_\ell(O x_1^d x_2^d x_3^d + O(x_1 x_2 x_3)^2 x_1^d \delta_{x_1^{\varepsilon+1}}) \). So we find:

\[Z_\ell = G \delta_{x_1^{\varepsilon+1}} \oplus C(x_1 x_2 x_3)^2 x_1^d \delta_{x_1^{\varepsilon+1}} \oplus C(x_1 x_2 x_3)^4 x_1^d \delta_{x_1^{\varepsilon+1}}\]

where \( G \subset E \) is the subspace generated by the monomials:

\[
\begin{aligned}
(x_1 x_2 x_3)^2 x_1 & \quad 2 \leq \varepsilon \leq 4 \\
(x_1 x_2 x_3)^2 x_1' & \quad \varepsilon = 0, \ i = d, \text{ or } \varepsilon = 1, \ i = d - 1, d, \text{ or } \varepsilon = 2, \ 2 \leq i \leq d \\
(x_1 x_2 x_3)^2 x_1 x_1' & \quad \varepsilon = 1, \ i = d - 1 \text{ or } \varepsilon = 2, \ 1 \leq i \leq d - 1 (\theta = 2, 3) \\
(x_1 x_2 x_3)^2 x_2' x_3' & \quad \varepsilon = 0, \ i = j = d - 2 \text{ or } \varepsilon = 1, \ d - 3 \leq i, j \leq d - 2 \\
x_1'^i x_0' & \quad i = d, \ 1 \leq j \leq d - 1 \text{ or } d - 2 \leq i, j \leq d - 1 (\theta = 2, 3)
\end{aligned}
\]
The determination of the sequence \( (H_{\ell,m}) \) associated with \( g_{x_2} \delta x_1^{x_1+1} \) is similar (for more details, see [22]). So we obtain that the quotient space \( \mathcal{Z}_\ell^*/\mathcal{Z}_\ell \) may be identified to:

\[
G' \delta \ell x_1^{x_1+1} \oplus \mathbb{C}(x_1 x_2 x_3)^2 x_1^{d-1} \delta \ell x_1^{x_1+1}
\]

where \( G' \subset E \) is the \( \mathbb{C} \)-vector space generated by the \( d(d-2) \) monomials:

\[
\begin{align*}
(x_1 x_2 x_3)^{x_1} & \quad \varepsilon = 0, \ i = d - 1, \ \text{or} \ \varepsilon = 1, \ i = d - 2 \\
(x_1 x_2 x_3)^{x_2 x_3} & \quad 1 \leq i, j \leq d - 2 \ \text{except} \ d - 3 \leq i, j \leq d - 2 \\
x_1^{x_1} x_3^j & \quad i = d - 1, \ 1 \leq j \leq d - 3, \ \text{or} \ i = d - 3, \ d - 1 \leq j \leq d - 2
\end{align*}
\]

for every \( \ell \in \mathbb{N}^* \), expect if \( d \) is even and \( \ell = 2 \). In this case, the four monomials \( x_1^{d-1} x_3^{d/2+1} \), \( x_3^{d/2+1} x_2 x_3(x_1 x_2 x_3) \), \( \theta = 2, 3 \), do not belong to \( G' \), and \( G' \) have the following two vectors in addition \( x_3^{d/2+1} g_{x_2} = d x_1^{d-1} x_3^{d/2+1} + 2 x_3^{d/2+1} x_2 x_3(x_1 x_2 x_3) \), \( \theta = 2, 3 \).

In order to study the action of \( s \) on nonzero spaces \( \mathcal{Z}_\ell^*/\mathcal{Z}_\ell \), we use the relation:

\[
(\alpha_{F_i}(s + 2) + \rho^*(u) - k) u \delta_{k} x_1^{x_1+1} = \frac{6 - d}{2} u x_1^{x_1+1}
\]

where \( u \) is a monomial and \( F_i \in \mathcal{F} \) such that \( \rho^*(u) = \rho_{F_i}(u) \), and we compute the image by \( c_{\ell} \) after rewriting by division. For every \( u \delta_{k} x_1^{x_1+1}, \ u \in G' \), the computation gives zero - in \( \text{gr} \rho^*(u) - \mathcal{Z}_\ell^*/\mathcal{Z}_\ell \) - with one exception if \( u = x_1^{d-1} \):

\[
\left( \frac{1}{d} (s + 2) + \frac{3}{2} - \frac{2}{d} - \ell \right) x_1^{d-1} \delta \ell x_1^{x_1+1} = \frac{d - 6}{2d} (x_1^{d-1} \delta \ell x_1^{x_1+1} + 2 x_1 x_2 x_3)^{d-1} \delta \ell x_1^{x_1+1}
\]

and \((1/d)(s + 2) + 3/2 - 2/d - \ell)^2 \delta \ell x_1^{x_1+1} = 0. Consequently, \( b_i(s) \) is the l.c.m. of \((1/d)(s + 2) + 3/2 - 2/d - \ell)^2\) and of \((\alpha_{F_i}(s + 2) + \rho^*(u) - \ell)\) with \( F \in \mathcal{F}^*(u) \), \( u \neq x_1^{d-1} \) in the given basis of \( G' \). Then in the general case, we have:

\[
b_i(s) = \text{l.c.m.} \left\{ \begin{array}{l} s + d(2 - \ell) - 1, \left( s + d\left(\frac{3}{2} - \ell\right) \right)^{2d-3} \prod_{i=1}^{2d-8} \left( s + d\left(\frac{3}{2} - \ell\right) + i \right) \end{array} \right\}
\]

where the last polynomial is the one of the monomials \( u \) with \( \mathcal{F}^*(u) = \{ F_i \} \).
References


