The Ergodicity of the Convolution $\mu \ast \nu$ on a Vector Space

By

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Abstract

Let $G$ be a subgroup of a vector space $X$ and $\mu, \nu$ be two probability measures on $X$. If $\mu$ and $\nu$ are $G$-quasi-invariant and $G$-ergodic, then the convolution $\mu \ast \nu$ is also $G$-ergodic.

§ 1. Introduction

Let $X$ be a vector space, $\mathcal{B}$ be a subspace of $X^*$ (the algebraic dual of $X$) and $\mathcal{B}_{\mathcal{B}}$ be the smallest $\sigma$-algebra on $X$ which makes each $x' \in \mathcal{B}$ measurable. For probability measures $\mu$ and $\nu$ on $\mathcal{B}_{\mathcal{B}}$, $\mu$ is said to be absolutely continuous with respect to $\nu$ (denoted by $\mu < \nu$) if $\nu(A) = 0, A \in \mathcal{B}_{\mathcal{B}}$, implies that $\mu(A) = 0$. $\mu$ and $\nu$ are equivalent (denoted by $\mu \sim \nu$) if $\mu < \nu$ and $\nu < \mu$. Denote by $A \ominus B = (A \cap B') \cup (A^c \cap B)$ the symmetric difference.

Denote by $\tau_x(x \in X)$ the translation $\tau_x(z) = z + x$. $\tau_x: (X, \mathcal{B}_{\mathcal{B}}) \rightarrow (X, \mathcal{B}_{\mathcal{B}})$ is measurable and $\tau_x \mathcal{B}_{\mathcal{B}} = \mathcal{B}_{\mathcal{B}}$ for every $x \in X$. We put for $x \in X$,

$$\mu_x(A) = \tau_x(\mu)(A) = \mu(A - x), A \in \mathcal{B}_{\mathcal{B}}.$$

Let $A_\mu = \{x \in X; \mu_x \sim \mu\}$ be the set of all admissible translates of $\mu$. $A_\mu$ is an additive subgroup of $X$. For a subset $G \subseteq X$, $\mu$ is called $G$-quasi-invariant if $G \subseteq A_\mu$, and $\mu$ is called $G$-ergodic if $\mu(A \ominus (A - x)) = 0$ for every $x \in G$ implies that $\mu(A) = 0$ or 1.


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convolution $\mu \ast \nu$ of two probability measures $\mu$ and $\nu$ is defined as follows:

$$\mu \ast \nu(A) = \int_X \mu(A - x) d\nu(x).$$

$\mu \ast \nu$ coincides with the image measure $\phi(\mu \times \nu)$, where $\mu \times \nu$ is the product measure on $(E \times E, \mathcal{B} \otimes \mathcal{B})$.

It holds that $A_{\mu \ast \nu} \supset A_\mu + A_\nu$, see Yamasaki [2], p. 170, [3], Theorem 13.1. By this result, for a subgroup $G$ of $X$, it follows that if $\mu$ or $\nu$ is $G$-quasi-invariant, then $\mu \ast \nu$ is also $G$-quasi-invariant.

Concerning the ergodicity of $\mu \ast \nu$, Yamasaki [2], p. 170, raised the following problem.

**Problem.** Let $G$ be a subgroup of $X$ and let $\mu$, $\nu$ be probability measures on $(X, \mathcal{B})$ which are $G$-quasi-invariant and $G$-ergodic. Then is the convolution $\mu \ast \nu$ $G$-ergodic?

Yamasaki [2], Theorem 23.2, proved that if $G$ is either

(1) $G$ is a linear subspace of algebraically countable dimension, or

(2) $G$ is a complete separable metrizable topological vector subspace of $X$ such that the identity $G \to X_{\sigma(X, \mathcal{B})}$ is continuous, where $\sigma(X, \mathcal{B})$ is the weak topology determined by $\mathcal{B}$,

then the answer is affirmative.

In this paper, we shall prove that the answer to the above problem is affirmative without any assumption on $G$.

§ 2. Main Result

**Theorem.** Let $\mu$, $\nu$ be probability measures on $(X, \mathcal{B})$ and $G_\mu$, $G_\nu$ be two subgroups of $X$. Suppose that $\mu$, $\nu$ be $G_\mu$, $G_\nu$-quasi-invariant and $G_\mu$, $G_\nu$-ergodic, respectively. Suppose also that $G_\nu \subseteq A_\mu$, that is, $\mu$ is $G_\nu$-quasi-invariant. Then $\mu \ast \nu$ is $G_\mu$-ergodic.

**Corollary.** Let $\mu$, $\nu$ be probability measures on $(X, \mathcal{B})$ and $G$ be a
subgroup of \( X \). If \( \mu \) and \( \nu \) are \( G \)-quasi-invariant and \( G \)-ergodic, then \( \mu \ast \nu \) is \( G \)-ergodic.

To prove the theorem, we use the following lemma due to Yamasaki [2], Theorem 25. 6, p. 182. The proof given here is a modification of Shimomura [1], p. 706–707.

**Lemma.** Let \( \{x_n\} \subset A_\mu \) be a net satisfying that

\[
\int |(d\mu_{x_n}/d\mu)(z)| - 1 |d\mu(z)| \to 0.
\]

Then it follows that \( \mu(B \ominus (B - x_n)) \to 0 \) for every \( B \in \mathcal{B}_G \).

**Proof.** First we show that \( x_n \to 0 \) in \( \sigma(X, \mathcal{B}) \). For every \( x' \in \mathcal{R} \), take \( \delta \) so that \( |t| < \delta \) implies that

\[
|\sum \exp(it\langle z', x'\rangle) d\mu(z)| > 1/2. \]

By

\[
|1 - \exp(it\langle x_n, x'\rangle)| \left| \int \exp(it\langle z, x'\rangle)d\mu(z) \right| = \left| \int \exp(it\langle z, x'\rangle)(d\mu_{x_n}/d\mu)(z) - 1 |d\mu(z)|, \right.
\]

we have

\[
|1 - \exp(it\langle x_n, x'\rangle)| < 2 \int |(d\mu_{x_n}/d\mu)(z)| - 1 |d\mu(z) |
\]

for every \( t \) with \( |t| < \delta \). Thus \( \langle x_n, x'\rangle \to 0 \).

Next we claim that

\[
\int |(d\mu_{x_n}/d\mu)(z)f(z - x_n) - f(z) |d\mu(z)| \to 0 \quad \text{for every } f \in L^1(X, \mathcal{B}_G).
\]

In fact, for each function of the form

\[
f(z) = \sum \gamma_j \exp(it\langle z, x_j'\rangle), \quad \gamma_j \text{ are real numbers and } x_j \in \mathcal{R},
\]

the assertion holds since \( \langle x_n, x_j'\rangle \to 0 \) for every \( j \). Since these functions are dense in \( L^1(X, \mathcal{B}_G) \), we get the claim.

For every \( B \in \mathcal{B}_G \), we have

\[
\mu(B \ominus (B - x_n)) = \int |\chi_{B-x_n}(z) - \chi_B(z) |d\mu(z) \leq \int |(d\mu_{x_n}/d\mu)(z) - 1 |d\mu(z) + \int |d\mu(z) + \int |(d\mu_{x_n}/d\mu)(z)\chi_B(z + x_n) - \chi_B(z) |d\mu(z)| \to 0
\]

remarking that

\[
\int |(d\mu_{x_n}/d\mu)(z) - 1 |d\mu(z) = \int |(d\mu_{x_n}/d\mu)(z) - 1 |d\mu(z)|,
\]

where \( \chi_B \) is the characteristic function of \( B \). This proves the Lemma.

**Proof of the Theorem.** Suppose that for \( A \in \mathcal{B}_G \), \( \mu \ast \nu \) is \( \sigma(A \ominus (A - x)) \)
= 0 for every \( x \in G_n \). By the definition of the \( \sigma \)-algebra \( \mathcal{B}_\mathcal{G} \), there exists a countable subset \( \Gamma = \{ x'_i \}_{i=1}^{\infty} \subset \mathcal{G} \) such that \( A \in \mathcal{B}_\Gamma \); \( \mathcal{B}_\Gamma \) is the minimal \( \sigma \)-algebra on \( X \) which makes each \( x'_i \,(i=1,2,\ldots) \) measurable. The measures \( \mu, \nu \) are also \( G_n, G_{\nu} \)-quasi-invariant and \( G_n, G_{\nu} \)-ergodic on the sub-\( \sigma \)-algebra \( \mathcal{B}_\Gamma \subset \mathcal{B}_\mathcal{G} \). Consequently, in order to show \( \mu \ast \nu(A) = 0 \) or 1, we can suppose in advance that the \( \sigma \)-algebra \( \mathcal{B}_\mathcal{G} \) is countably generated. In particular, \( L^1(X, \mathcal{B}_\mathcal{G}) \) is separable.

Take a countable dense subset \( \{ d\mu_x/d\mu \}_{x=1}^{\infty} \) of \( \{ d\mu_x/d\mu \; ; \; x \in G_n \} \) in \( L^1(X, \mathcal{B}_\mathcal{G}) \). We claim that for each \( A \in \mathcal{B}_\mathcal{G} \), if \( \mu(A \ominus (A-x_n)) = 0 \) for every \( n \), then \( \mu(A \ominus (A-x)) = 0 \) for every \( x \in G_n \). Let \( x \in G_n \) be arbitrary. By \( \mu(A \ominus (A-x)) = 0 \) for every \( n \), it follows that \( \mu(A \ominus (A-x)) = \mu((A-x_n) \ominus (A-x)) \) for every \( n \). By the preceding Lemma (putting \( B = A - x \)), for every \( \varepsilon > 0 \), there exists \( \delta = \delta(A, x, \varepsilon) \) such that \( \int |(d\mu_x/d\mu)(z) - 1| d\mu(z) < \delta \) implies \( \mu((A-x) \ominus (A-x-y)) < \varepsilon \). By the definition of the sequence \( \{ x_n \} \), there exists \( n = n(\delta) \) such that \( \int |(d\mu_x/d\mu)(z) - (d\mu_x/d\mu)(z)| d\mu(z) = \int |(d\mu_x/d\mu)(z)| d\mu(z) < \delta \). Thus we have \( \mu((A-x) \ominus (A-x_n)) = \mu((A-x) \ominus (A-x - (x + x_n))) < \varepsilon \), that is, \( \mu(A \ominus (A-x)) < \varepsilon \) for every \( \varepsilon > 0 \) which proves \( \mu(A \ominus (A-x)) = 0 \).

By \( \mu \ast \nu(A \ominus (A-x_n)) = \int_x \mu((A-z) \ominus (A-z-x_n)) d\nu(z) = 0 \) for every \( n \), there exists a subset \( \Omega \in \mathcal{B}_\mathcal{G} \) satisfying that \( \nu(\Omega) = 1 \) and \( \mu((A-z) \ominus (A-z-x_n)) = 0 \) for every \( n \) and for every \( z \in \Omega \). By the second step of this proof and by the ergodicity of \( \mu \), it follows that \( \mu(A-z) = 0 \) or 1 for every \( z \in \Omega \). We set \( B = \{ z \in X ; \mu(A-z) = 1 \} \). Since \( G_\nu \subset A_\nu \), \( B \) is a \( G_\nu \)-invariant subset. By the \( G_\nu \)-ergodicity of \( \nu \), it follows that \( \nu(B) = 0 \) or 1. If \( \nu(B) = 1 \), then we have \( \mu \ast \nu(A) = 1 \) and if \( \nu(B) = 0 \) then \( \mu \ast \nu(A) = 0 \). Thus \( \mu \ast \nu(A) = 0 \) or 1. This completes the proof.

References

