Multiple Poles at Negative Integers for $\int_A f^2 \square$ in the Case of an Almost Isolated Singularity

By

Daniel Barlet*

Summary

We give a necessary and sufficient topological condition on $A \in H^0(\{ f \neq 0 \}, \mathbb{C})$, for a real analytic germ $f : (\mathbb{R}^{n+1}, 0) \rightarrow (\mathbb{R}, 0)$, whose complexification has an isolated singularity relatively to the eigenvalue 1 of the monodromy, in order that the meromorphic continuation of $\int_A f^2 \square$ has a multiple pole at sufficiently "large" negative integers. We show that if such a multiple pole exists, it occurs already at $\lambda = -(n+1)$ with its maximal order which is computed topologically.

Content

§ 1 Mellin Transform on $\mathbb{R}^r$
§ 2 Statement of the Result
§ 3 The Proof
References

Introduction

The aim of the present Note is to generalize the result of [5] and its converse proved in [6] to the case of the eigenvalue 1. So we shall give a

* Université Henri Poincaré Nancy 1 et Institut Universitaire de France, Institut Elie Cartan, UMR 7502 CNRS-INRIA-UHP, BP 239-F-54506 Vandœuvre-lès-Nancy Cedex, France
necessary and sufficient topological condition in order that the meromorphic
extension of the holomorphic current.
\[ \lambda \mapsto \int_A f^2 \square \]
defined in a neighbourhood of the origin in \( \mathbb{R}^{n+1} \) has a pole of order at least 2
at \( \lambda = -(n+1) \), in the case of a real analytic germ \( f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0) \)
satisfying the following condition: we assume that the complexification \( f_c \)
of \( f \) admits an isolated singularity at 0 for the eigenvalue 1 of the mono-
dromy. This notion, introduced in [2], means that for any \( x \neq 0 \) in \( f_c^{-1}(0) \) near
0, the monodromy of \( f_c \) acting on the reduced cohomology of the Milnor fiber
of \( f_c \) at \( x \) has no non zero invariant vector.
Of course this hypothesis is satisfied when \( f_c \) has an isolated singularity at
0, but it allows also much more complicated situations.

In our result the interplay between connected components of the semi-
analytic set \( \{ f \neq 0 \} \) is essential: we denote by \( A = \sum_{x=1}^a c_x A_x \) an element in
\( H^0(\{ f \neq 0 \}, \mathbb{C}) \) so \( A_x \) are connected components of \( \{ f \neq 0 \} \) and \( c_x \) are
complex numbers (we shall precise below the meaning of \( \int_{A_x} f^2 \square \) when \( A_x \subset
\{ f < 0 \} \)). Our topological necessary and sufficient condition is given on \( A \).

The main new point here, compare to [5] and [6] is the use of [3] which explains how to compute the variation map in this context of isolated singularity
for the eigenvalue 1, in term of differential forms.

I want to thank Prof. Guzein–Zade who point out to me that the
orientations are not enough precise in [5]; so I shall try to take them carefully in
account here. The reader will see that it is not so easy. I want also to thank
Prof. B. Malgrange who suggests several improvements to the first draw of this
article.

\section{Mellin Transform on \( \mathbb{R}^* \)}

Let \( \varphi \in C^\infty(\mathbb{R}^*) \) such that
\[
\begin{cases}
\text{(i)} & \text{supp } \varphi \subset [-A, A] \\
\text{(ii)} & \varphi \text{ is bounded}
\end{cases}
\]
We define for \( \text{Re } \lambda > 0 \)
\[
M\varphi(\lambda) := \frac{1}{i\pi} \left[ \int_0^{+\infty} x^\lambda \varphi(x) \frac{dx}{x} - e^{-i\pi \lambda} \int_0^{-\infty} x^\lambda \varphi(-x) \frac{dx}{x} \right].
\]

Examples. Let \( z \in \mathbb{C} \) with \( \text{Re}(z) \geq 0 \) and let \( \varphi_0(x) = |x|^z \text{ near } 0 \) and \( \varphi_1(x) = |x|^z \text{ sgn}(x) \text{ near } 0 \). Then we have
\[ M\varphi_0(\lambda) = \frac{1}{i\pi} \frac{1 - e^{-i\pi\lambda}}{\lambda + \alpha} \] is an entire function of \( \lambda \)

and

\[ M\varphi_1(\lambda) = \frac{1}{i\pi} \frac{1 + e^{-i\pi\lambda}}{\lambda + \alpha} \] is an entire function of \( \lambda \).

So for \( \alpha \notin \mathbb{N} \) we have a simple pole at \( \lambda = -\alpha \). For \( \alpha = 2k \) with \( k \in \mathbb{N} \), \( M\varphi_0 \) has no pole but \( M\varphi_1 \) has one at \( \lambda = -2k \).

For \( \alpha = 2k + 1 \) with \( k \in \mathbb{N} \), \( M\varphi_1 \) has no pole but \( M\varphi_0 \) has one at \( \lambda = -2k - 1 \). This is reasonable because \( |x|^{2k} \) is \( C^\infty \) at 0 and \( |x|^{2k+1} \text{sgn}(x) \) is also \( C^\infty \) at 0 for \( k \in \mathbb{N} \). So poles of \( M\varphi \) measure the singularity of \( \varphi \) at 0, as usual.

Without the condition ii) the situation is slightly more complicated: we shall use the following elementary lemma.

**Lemma 1.** Let \( P \) and \( Q \) in \( C[x] \) of degrees at most \( k - 1 \) and let

\[ \varphi(x) = \begin{cases} P(\log x) & \text{for } x > 0 \\ Q(\log|x| - i\pi) & \text{for } x < 0 \end{cases} \]

near 0, and assume \( \varphi \) satisfies condition i) \( \varphi \in C^\infty(\mathbb{R}^+) \).

Then \( M\varphi \) has no pole at \( \lambda = 0 \) iff \( P = Q \). Moreover if \( P = Q \) \( M\varphi \) is entire.

**Proof.** For \( P = Q \) we have

\[ M\varphi(\lambda) = \frac{1}{i\pi} \int_{-1}^{+1} P(\log z)z^i \frac{dz}{z} \]

modulo an entire function of \( \lambda \), where \( \log z = \log|z| + i\text{Arg}z \) with \( -\pi < \text{Arg}z < \pi \). From Cauchy formula on the path

\[ -1 \rightarrow -\epsilon \rightarrow \epsilon \rightarrow 1 \]

this gives

\[ -\int_{-\pi}^{0} P(i\theta)e^{-i\lambda\theta} d\theta \]

which is an entire function of \( \lambda \).

If \( P \neq Q \), as we have already seen that

\[ \int_{-1}^{+1} Q(\log z)z^i \frac{dz}{z} \]

is entire in \( \lambda \), it is enough to show that

\[ \int_{0}^{1} (Q - P)(\log x)x^i \frac{dx}{x} \]

has a pole of order \( \geq 1 \) at \( \lambda = 0 \). But we have
\[
\int_0^1 \left(\log x\right)^l x^2 \frac{dx}{x} = \frac{d^l}{d\lambda^l} \left( \int_0^1 x^2 \frac{dx}{x} \right) = (-1)^{l-1} (l-1)! \frac{1}{\lambda^{l+1}}
\]

which gives the conclusion.

\section{Statement of the Result}

Let \( f : X_{\mathbb{R}} \rightarrow \] \(-\delta, \delta]\) a Milnor representative of the non zero real analytic germ \( f : (\mathbb{R}^{n+1}, 0) \rightarrow (\mathbb{R}, 0) \). This is, by definition, the restriction to \( \mathbb{R}^{n+1} \) of a Milnor representative of the complexification \( f_c : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0) \) of \( f \).

Let \((A_x)_{x \in [1,a]}\) be the connected components of the relatively compact semi-analytic open set \( \{f \neq 0\} \cap X_{\mathbb{R}} \) and denote by \( A = \sum_{x=1}^a c_x A_x \), where the \( c_x \) are complex numbers, a fixed element in \( H^0(\{f \neq 0\} \cap X_{\mathbb{R}}, \mathbb{C}) \).

**Definition.** For a compactly supported \( C^\infty \) \( n \)-form \( \varphi \) on \( X_{\mathbb{R}} \), and for \(-\delta < s < \delta\), set

\[
I_x(s) = \int_{\{f = s\} \cap A_x} \varphi
\]

where the orientation of \( \{f = s\} \cap A_x \) is chosen in such a way that we have

\[
in\pi M_{I_x}(\lambda) = \begin{cases} 
\int_{A_x} f^\lambda \varphi \wedge \frac{df}{f} & \text{if } A_x \subset \{f > 0\} \\
\int_{A_x} (-f)^\lambda \varphi \wedge \frac{df}{f} & \text{if } A_x \subset \{f < 0\}
\end{cases}
\]

(1)

where the open set \( A_x \) is oriented by the canonical orientation of \( \mathbb{R}^{n+1} \) (assumed to be fixed in the sequel).

For \( A = \sum_{x=1}^a c_x A_x \) we define

\[
I_A(s) := \sum_{x=1}^a c_x \int_{\{f = s\} \cap A_x} \varphi
\]

with the previous conventions. So we shall get, by definition,

\[
in\pi M_{I_A}(\lambda) = \int_A f^\lambda \varphi \wedge \frac{df}{f} \text{ where}
\]

\[
\int_A f^\lambda \varphi \wedge \frac{df}{f} := \sum_{A_x \subset \{f > 0\}} c_x \int_{A_x} f^\lambda \varphi \wedge \frac{df}{f} - e^{-\pi i \lambda} \sum_{A_x \subset \{f < 0\}} c_x \int_{A_x} (-f)^\lambda \varphi \wedge \frac{df}{f}
\]

with the natural orientations of the open sets \( A_x \).
Define now, for any \( a \in [1, a] \)

\[ F_{a} := f^{-1}(s_{0}) \cap A_{a} \quad \text{if } A_{a} \subset \{ f > 0 \} \]

and

\[ F_{a} := f^{-1}(-s_{0}) \cap A_{a} \quad \text{if } A_{a} \subset \{ f < 0 \} \]

where \( s_{0} \) is a base point in \( D_{\delta}^{*} \) chosen in \( D_{\delta}^{*} \cap \mathbb{R}^{+} \). Here we assume that we have a Milnor fibration for \( f_{\mathbb{C}} \):

\[ f_{\mathbb{C}} : X_{\mathbb{C}} - f_{\mathbb{C}}^{-1}(0) \to D_{\delta}^{*} \]

and we shall denote by \( F_{\mathbb{C}} \) the complex Milnor fiber (that is to say \( F_{\mathbb{C}} := f_{\mathbb{C}}^{-1}(s_{0}) \)). We define then \( F_{A} := \sum_{a=1}^{a} c_{a} F_{a} \), as a closed oriented \( n \)-cycle of \( X_{\mathbb{R}} \), the orientation of the \( F_{a} \), being given by (1).

We define \( \theta_{a} : F_{a} \to F_{\mathbb{C}} \) as the obvious inclusion if \( A_{a} \subset \{ f > 0 \} \); and for \( A_{a} \subset \{ f < 0 \} \) \( \theta_{a} \) is given by the closed embedding of \( F_{a} = f^{-1}(-s_{0}) \cap A_{a} \) in \( f_{\mathbb{C}}^{-1}(s_{0}) = F_{\mathbb{C}} \) given by a \( C^{\infty} \) trivialisation of \( F_{\mathbb{C}} \) along the half-circle \( |s| = s_{0} \) and \( \text{Arg}(s) \in [-\pi, 0] \).

For \( A = \sum_{a=1}^{a} c_{a} A_{a} \) define the closed oriented \( n \)-cycle \( G_{A} \) of \( F_{\mathbb{C}} \)

\[ G_{A} = G_{A^{+}} - G_{A^{-}} = \sum_{A_{a} \subset \{ f > 0 \}} (\theta_{a})_{*} F_{a} - \sum_{A_{a} \subset \{ f < 0 \}} (\theta_{a})_{*} F_{a} \]

The minus sign in this definition comes from the following facts:

In our definition of Mellin transform, \( \mathbb{R}^{*} \) is oriented by the natural orientation coming from \( \mathbb{R} \). Using the monodromy brings the orientation of \( \mathbb{R}^{*} \) we have chosen to the opposite orientation of \( \mathbb{R}^{+} \). If we want to keep the global orientation of \( \mathbb{R}^{n+1} \) in this transfert (we push the Milnor fiber \( F_{\mathbb{R}} := f^{-1}(-s_{0}) \) in \( F_{\mathbb{C}} \) we have to change the orientation in \( f^{-1}(-s_{0}) \). This explains our definition of the cycle \( G_{A} \) in \( F_{\mathbb{C}} \).

When \( \varphi \in C_{c}^{\infty}(F_{\mathbb{C}}) \) is a \( n \)-form, we have

\[ \int_{G_{A}} \varphi := \sum_{A_{a} \subset \{ f > 0 \}} c_{a} \int_{F_{a}} \theta_{a}^{*}(\varphi) - \sum_{A_{a} \subset \{ f < 0 \}} c_{a} \int_{F_{a}} \theta_{a}^{*}(\varphi) \]

where \( F_{a} \) is oriented as before.

This gives a linear form on \( H_{\mathbb{C}}^{n}(F_{\mathbb{C}}, \mathbb{C}) \) associated to the oriented \( n \)-cycle \( G_{A} \) in \( F_{\mathbb{C}} \):

\[ \varphi \to \int_{G_{A}} \varphi \]

where \( \varphi \in C_{c}^{\infty}(F_{\mathbb{C}}) \) is a \( d \)-closed \( n \)-form.
We shall denote by $\delta(A)$ the cohomology class in $H^n(F_\mathbb{C}, \mathbb{C})$ which gives this linear form on $H^n(F_\mathbb{C}, \mathbb{C})$ via the Poincare duality: $H^n(F_\mathbb{C}, \mathbb{C}) \times H^n(F_\mathbb{C}, \mathbb{C}) \to \mathbb{C}$. Our result is the following analogue of [5] and its converse [6].

Theorem. Let $f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$ a non zero real analytic germ. Assume that $0 \in \mathbb{C}^{n+1}$ is an isolated singularity relative to the eigenvalue 1 of the monodromy of $f_\mathbb{C}$ the complexification of $f$.

Let $A = \sum_{a=1}^a c_a A_a$ an element in $H^0(\{ f \neq 0 \}, \mathbb{C})$ and $\delta(A)$ the corresponding class in $H^n(F_\mathbb{C}, \mathbb{C})$ (see the definition above). Then we have an equivalence between:

(i) $\delta(A)$ has a non zero component on $H^n(F_\mathbb{C}, \mathbb{C})_{\lambda=1}$ in the spectral decomposition of the monodromy acting on $H^n(F_\mathbb{C}, \mathbb{C})$

(ii) the meromorphic extension to the complex plane of the distribution $\lambda \to \int f^\lambda \square$ (holomorphic in $\lambda$ for $\text{Re } \lambda > 0$), admits a pole of order $\geq 2$ at $\lambda = -(n+1)$. Moreover, the order of the pole $-v$ for $v \in \mathbb{N}$ and $v \geq n+1$ of the meromorphic continuation of $\frac{1}{\Gamma(\lambda)} \int A f^\lambda \square$ is exactly the nilpotency order of $T - 1$ acting on $\delta(A)_1$, the component of $\delta(A)$ on $H^n(F_\mathbb{C}, \mathbb{C})_{\lambda=1}$.

Remarks. 1) The notion of an isolated singularity relative to the eigenvalue 1 of the monodromy has been introduced in [2]. It means that for any $x \neq 0$ near 0 in $\mathbb{C}^{n+1}$ such that $f_\mathbb{C}(x) = 0$, the monodromy acting on the reduced cohomology of the Milnor fiber of $f_\mathbb{C}$ at $x$ has no non zero invariant vector.

2) In the case where $A$ is a connected component of the open set $\{ f \neq 0 \}$, (ii) is equivalent, in term of asymptotic expansion of integrals $s \to \int_\partial f^{-1}(s) \phi$ when $s \to 0$, with $\phi \in C^0_{\mathbb{R}}(x)$ is a $n$-form, to the non vanishing of the coefficient of $s^p(\log |s|)^j$ for some $p \in \mathbb{N}$ and some $j \in \mathbb{N}^*$ (for some choice of $\phi$).

3) The precise order of poles at large negative integers is describe in a purely topological way.

§ 3. The Proof

We shall use here the notations of [3]. For $A$ given, let $e$ be the component of $\delta(A)$ on $H^n(F_\mathbb{C}, \mathbb{C})_{\lambda=1}$, the spectral subspace of $H^n(F_\mathbb{C}, \mathbb{C})$ associated to eigenvalue 1 of the monodromy.

Assume $e \neq 0$ and let us prove that (i) $\Rightarrow$ (ii). As the canonical hermitian form $h$ is non degenerated on $H^n(F_\mathbb{C}, \mathbb{C})_{\lambda=1}$ (see [2]) there exists $e' \in H^n(F_\mathbb{C}, \mathbb{C})_{\lambda=1}$ such that $h(e', e) \neq 0$.

From [3] we know that $h$ is topological and can be computed by the following formula:

$$h(e', e) = I(\tilde{\text{var}}(e'), e)$$
where $I$ is the (hermitian) intersection form on $F_{\mathbb{C}}$ which gives the Poincare duality

$$ I : H^n_{c}(F_{\mathbb{C}}, \mathbb{C}) \times H^n(F_{\mathbb{C}}, \mathbb{C}) \rightarrow \mathbb{C} $$

which is invariant by the monodromy and where

$$ \var := H^n(F_{\mathbb{C}}, \mathbb{C})_{\lambda=1} \rightarrow H^n_{c}(F_{\mathbb{C}}, \mathbb{C})_{\lambda=1} $$

is the composition of the "ordinary variation map" (built in this context in [3]) and of the automorphism

$$ \Theta(x) := \frac{1}{x} \log(1 + x) \quad \text{with} \quad 1 + x := T_{|H^n(F_{\mathbb{C}}, \mathbb{C})_{\lambda=1}}, \quad \text{here } T \text{ is the monodromy.} $$

So, if $e'' := \Theta(e')$, we have

$$ I(\var(e''), \delta(A)) \neq 0, $$

using the fact that $I$ is monodromy invariant, which implies that the spectral decomposition of $H^n(F_{\mathbb{C}}, \mathbb{C})$ is $I$-orthogonal.

If now $\varphi \in C^\infty_c(F_{\mathbb{C}})$ if a closed $n$-form representing $\var(e'') = \var(e')$ in $H^n_c(F_{\mathbb{C}}, \mathbb{C})$ we shall have

$$ \int_{G_{\delta}} \varphi \neq 0 \quad (2) $$

But in [3] it is explained how to represent $\var(e') = \var(e'')$ by a de Rham representative (that is to say how to build such a $\varphi$) for a given class $e' \in H^n_{c}(F_{\mathbb{C}}, \mathbb{C})_{\lambda=1}$. Let us give a direct construction of the variation map in this context (as suggested by B. Malgrange) following [9].

Let $\Psi_1$ and $\Phi_1$ the spectral parts for eigenvalue 1 of the monodromy of respectively nearby and vanishing-cycle sheaves of $f$. The assumption says that $\Phi_1$ is concentrated at 0 and so we have an isomorphism

$$ R\Gamma_{\{0\}} \Phi_1 \sim \Phi_1. $$

Now the variation map $\var : \Phi_1 \rightarrow \Psi_1$ gives a map $R\Gamma_{\{0\}} \Phi_1 \rightarrow R\Gamma_{\{0\}} \Psi_1$. The composition

$$ \Psi_1 \xrightarrow{\text{can}} \Phi_1 \sim R\Gamma_{\{0\}} \Phi_1 \rightarrow R\Gamma_{\{0\}} \Psi_1 \rightarrow R\Gamma_c \Psi_1 $$

induces our variation map (see [3])

$$ H^n(F, \mathbb{C})_{\lambda=1} \rightarrow H^n_{c}(F, \mathbb{C})_{\lambda=1}. $$

Let $\delta$ the complex of semi-meromorphic forms with poles in $f = 0$ and
$\mathcal{E} \cdot [\log f]$ the complex given by polynomial in $\log f$ with coefficients in $\mathcal{E}$ and the differential

$$D(u.(\log f)^{(j)}) = du.(\log f^{(j)}) + \frac{df}{f} \wedge u.(\log f)^{(j-1)}$$

where $(\log f)^{(j)} := \frac{1}{j!}(\log f)^j$.

Then the exact sequence of complexes

$$0 \to C^\infty \to \mathcal{E} \cdot [\log f] \to \mathcal{E} \cdot [\log f]/C^\infty \to 0$$
corresponds to the distinguished triangle

$$\begin{array}{c}
\mathbb{C} \\
\downarrow +1 \\
\Phi_1 \\
\downarrow \\
\Psi_1 \\
\end{array}$$

and $\mathcal{E} \cdot [\log f]$ is a complex of fine sheaves representing $\Psi_1$.

Let consider now a $n$-cycle $x$ in $\mathcal{E}^n[\log f]$

$$x = x_k + x_{k-1} \cdot (\log f) + \cdots + x_1 \cdot (\log f)^{(k-1)}$$

(this strange way of indexation will be compatible with notations in [3]!).

Then $Dx = 0$ gives $dx_k + \frac{df}{f} \wedge x_{k-1} = 0, \ldots, dx_2 + \frac{df}{f} \wedge x_1 = 0$ and $dx_1 = 0$.

To compute $\var$ on $[x]$ we have to write

$$x = y + z + Dt.$$ 

Where $t \in \mathcal{E}^{n-1}[\log f], z$ is $C^\infty$ of degree $n$ and $y$ is in $\mathcal{E}^n[\log f]$ with compact support. So that $D(y + z) = 0$ and $\var[x]$ (see above $\var = \Theta \circ \var$) is given by

$$N(y + z) = N(y) = y_{l-1} + y_{l-2} \cdot \log f + \cdots + y_1 \cdot (\log f)^{(l-2)}$$

if

$$y = y_l + y_{l-1} \cdot \log f + \cdots + y_1 \cdot (\log f)^{(l-1)}.$$ 

Now in [3] this is performed in an “explicit” way for a given $w (= x)$

$$w = w_k + w_{k-1} \cdot \log f + \cdots + w_1 \cdot (\log f)^{(k-1)}$$

such that $w_k|_{F_e} = e'$ in $H^n(F, \mathbb{C})_{x=1}$.

In a first step $w$ is replaced by a cocycle $\tilde{w}$ in $\mathcal{E}^n[\log f]$ with degree $k$ in $\log f$ such that $N\tilde{w}$ has compact support in the Milnor ball $X$ and such that
\[ \hat{w}_k = w_k + \frac{df}{f} \wedge \xi_k \] still induces \( e' \) in \( H^n(F, \mathbb{C})_{\lambda=1} \). For a \( C^\infty \) function \( \sigma \) on \( X \) which is equal to 1 on a large enough compact set, we will have

\[ D(\sigma \hat{w}) = W = d\sigma \wedge \left( w_k + \frac{df}{f} \wedge \xi_k \right) \]

which is in \( \mathcal{E}^{n+1} \) and has compact support near \( \partial X \) and is \( d \)-closed. Using a Leray residu on \( \{ f = 0 \} \) near \( \partial X \) (where 1 is not an eigenvalue of the monodromy of \( f \) in positive degrees) we write

\[ W = \omega + D(\alpha + \eta \log f) \]

where \( \eta \) is \( C^\infty \) \( d \)-closed of degree \( n \) with compact support near \( \partial X \), \( \omega \) is \( C^\infty \) of degree \( n + 1 \) with compact support near \( \partial X \) and also \( d \)-closed, and where \( \alpha \in \mathcal{E}^n \) has compact support near \( \partial X \).

Then \( \var\bar{\var}(e') \) is given by \( \hat{w}_k \) where

\[ \hat{w} = \sigma \hat{w} - \alpha + \eta \log f \]

induces a \( n \)-cocycle with compact support in \( \mathcal{E}^n[\log f]/C^{\infty} \); so that \( \hat{w} \in \mathcal{E}^n[\log f] \) has degree \( k \geq 1 \) in \( \log f \) and coincide with \( \hat{w} \) on a large compact set (\( N\hat{w} = N\hat{w} \) and \( \hat{w}_k = \sigma \hat{w}_k + \eta = v_k + \eta \) with the notation in [3] p. 20).

Now (2) gives

\[ \int_{G_A} v_k + \eta \neq 0 \]  

This will show that the meromorphic extension of \( \int_A f^{\lambda}(v_k + \eta) \frac{df}{f} \) will have at \( \lambda = 0 \) a pole of order \( \geq 1 \) (see lemma 2 below). Consider now the meromorphic function

\[ \int_A f^{\lambda} \hat{w}_{k+1} + \frac{df}{f} = \int_A f^{\lambda} \sigma w_k + \frac{df}{f} \]

as

\[ \frac{1}{\lambda} d(f^{\lambda} \hat{w}_{k+1}) = f^{\lambda} \frac{df}{f} \wedge \hat{w}_{k+1} + \frac{1}{\lambda} f^{\lambda} d\hat{w}_{k+1} \quad \text{for } \Re(\lambda) \gg 0, \]

Stokes formula and the analytic continuation give

\[ \int_A f^{\lambda} \sigma w_k + \frac{df}{f} = \frac{(-1)^{n-1}}{\lambda} \int_A f^{\lambda} (v_k + \eta) + \frac{df}{f} - \frac{1}{\lambda} \int_A f^{\lambda} \omega \]  

(5)
using $d\tilde{w}_{k+1} = d\sigma \wedge \left( w_k + \frac{df}{f} \wedge \tilde{\xi}_k \right) + \sigma \frac{df}{f} \wedge v_k$, (3) and the fact that $\sigma v_k = v_k$ ($\sigma \equiv 1$ on the support of $v_k$). As $\omega$ is $C^\infty$, the meromorphic function $\int_A f^\lambda \omega$ has no pole at $\lambda = 0$, and so $\frac{1}{\lambda} \int f^\lambda \omega$ has at most a simple pole at 0. We conclude from (3) and (4) that $\int_A f^\lambda \sigma w_k \wedge \frac{df}{f}$ has at least a pole of order 2 at $\lambda = 0$ from the following lemma:

**Lemma 2.** Let $\tilde{v} \in H^0_c(f(X_\mathfrak{f}, \mathcal{E}^n(k)))$ such that $\delta \tilde{v} = 0$ and $\int_{G^*} \tilde{v}_k \neq 0$. Then the meromorphic extension of $\int_A f^\lambda \tilde{v}_k \wedge \frac{df}{f}$ has a pole of order $\geq 1$ at $\lambda = 0$.

**Proof.** For $x \in \mathbb{R}$ near 0 define $\varphi(x) = \int_{(f=x) \cap A} \tilde{v}_k$. Then we shall have $\left(x \frac{d}{dx} \right)^k \varphi \equiv 0$ on $\mathbb{R}^*$ near 0 because of the assumption $\delta \tilde{v} = 0$. So we can apply lemma 1 to $\varphi$. The main point is now to show that if $P, \overline{Q} \in \mathbb{C}[x]$ of degree $\leq k - 1$ are such that

$$\varphi(x) = P(\log|x|) \quad \text{for } 0 < x \ll 1$$
$$\varphi(x) = Q(\log|x|) - \overline{Q}(\log|x|) \quad \text{for } -1 \ll x < 0$$

we have $P \neq Q$!

The hypothesis $\int_{G^*} \tilde{v}_k \neq 0$ can be written $\int_{G^*} \tilde{v}_k - \int_{G^-} \tilde{v}_k \neq 0$ if $A = A^+ + A^-$ with $A^+ = \sum_{A_\mathfrak{s} \subset \{f > 0\}} c_\mathfrak{s} A_\mathfrak{s}$ and $A^- = \sum_{A_\mathfrak{s} \subset \{f < 0\}} c_\mathfrak{s} A_\mathfrak{s}$. We have $\int_{G^*} \tilde{v}_k = \varphi(s_0)$ by definition. To compute $\int_{G^-} \tilde{v}_k$ we have to follow, along the half-circle $s_0 e^{i\theta}, \theta \in [-\pi, 0]$, the holomorphic multivalued function given by $\int_{(f = s) \cap A^-} \tilde{v}_k$ where $(f = s) \cap A^-$ is a notation for the horizontal family of oriented, closed $n$-cycles in the fibers of $f_X$ with value

$$(f = -s_0) \cap A^- \quad \text{at } s = -s_0 = s_0 e^{-i\pi}.$$ 

From the fact that $\varphi(x) = Q(\log|x| - i\pi)$ for $-1 \ll x < 0$, this multivalued function is $Q(\log s)$ for the choice $-\pi \leq \text{Arg}s \leq 0$. So we get $\int_{G^-} \tilde{v}_k = Q(\log s_0)$ and then $\int_{G^*} \tilde{v}_k = (P - \overline{Q})(\log s_0) \neq 0$.

So we have $P \neq Q$ and by lemma 1 we get the desired pole of order $\geq 1$ at $\lambda = 0$. 

So (i) $\Rightarrow$ (ii) is proved if we can choose $\tilde{v}$ in order to have

$$f^{n+1} \tilde{v}_k \wedge \frac{df}{f} \in C^\infty(X_\mathfrak{f}).$$

In fact $\tilde{v}_k = v_k + \eta$ where $\eta$ is $C^\infty$ so we only need to satisfy $f^{n+1} v_k \wedge \frac{df}{f} \in C^\infty(X_\mathfrak{f})$. 

So (i) $\Rightarrow$ (ii) is proved if we can choose $\tilde{v}$ in order to have

$$f^{n+1} \tilde{v}_k \wedge \frac{df}{f} \in C^\infty(X_\mathfrak{f}).$$
MULTIPLE POLES AT NEGATIVE INTEGERS

But from [3] p. 20 we have

\[ v_k = w_{k-1} - d\xi_k + \frac{df}{f} \wedge \xi_{k-1} \]

and so \( \frac{df}{f} \wedge v_k = \frac{df}{f} \wedge w_{k-1} - \frac{df}{f} \wedge d\xi_k \). Now

\[ \int_A f^\lambda \frac{df}{f} \wedge d\xi_k = \int_A d \left( \frac{f^{\lambda+1}}{\lambda+1} d\xi_k \right) \equiv 0 \]

by Stokes formula (for Re \( \lambda \gg 0 \) so everywhere) and it is enough to choose \( w \) such that \( f^w \) is holomorphic.

This is possible from [3] (see the begining of the proof of theorem 2) and this complete the proof of (i) \( \Rightarrow \) (ii).

We shall prove now that \( S(A) = 0 \) implies in fact that \( f \) has no pole at negative integers.

**Proposition.** Let \( f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0) \) a non zero real analytic germ such that 1 is not an eigenvalue of the monodromy of \( f_\mathbb{C} \) acting on the reduced cohomology of the Milnor fiber of \( f_\mathbb{C} \) at any \( x \in f_\mathbb{C}^{-1}(0) \) close enough to the origine.

Let \( A_0 \) be a connected component of the open set \( \{ f \neq 0 \} \) in \( X_\mathbb{R} \).

Then, the meromorphic extension of \( \frac{1}{\Gamma(\lambda)} \int_A |f|^{\lambda} \Box \) has no pole at a negative integers.

**Proof.** The point is to explain that the Bernstein–Sato polynomial \( b \) of \( f_\mathbb{C} \) at 0 has only one simple root in \( \mathbb{Z} \) which is \(-1\). For that propose, remark that our hypothesis implies that the vanishing cycles sheaf \( \Phi \) of \( f \) satisfies \( \Phi_0 = 0 \), and so \( \Psi \), the nearby-cycles sheaf of \( f \) satisfies \( \Psi_1 \sim (\mathbb{C}, T = 1) \).

From [8] or [7] we conclude that all integral roots of \( b \) are simple (using that 0 is a simple root of \( b' \) and the final inequalities of [8]). If \( b \) has two different integral roots, then using the De Rham functor, we obtain a non trivial decomposition of \( (\mathbb{C}, T = 1) \sim \Psi_1 \). Of course this allows us to conclude that \( b \) has exactly one integral (simple) root. But of course \(-1\) is a root of \( b \). So we obtain that \( b(s) = (s+1)b_1(s) \) where \( b_1 \) has no integral root. Using now a Bernstein identity to perform the analytic continuation of \( \int_{A_0} |f|^{\lambda} \Box \) leads to, at most, simple poles at negative integers (because \( b(\lambda) \ldots b(\lambda + k) \) has, at most, a simple root at \( -\delta \) for \( \delta \in \mathbb{N}^* \)).

\[ \square \]
Corollary. If $0$ is an isolated singularity for the eigenvalue $1$ of $f_{\mathcal{E}}$, for any $A \in H^0(\{f \neq 0\}, \mathbb{C})$ the Laurent coefficients of the poles of $\frac{1}{\Gamma(\lambda)} \int_A f^\lambda \mathbb{D}$ at negative integers have there supports in $\{0\}$.

Proof. This is an obvious consequence of the proposition. 

Assume now that we have a pole of order $j \geq 2$ at $\lambda = -k$ ($k \in \mathbb{N}^*$) for $\int_A f^\lambda \mathbb{D}$. Let $\mathcal{I}$ be the coefficient of $\frac{1}{(\lambda + k)^j}$ in the Laurent expansion at $\lambda = -k$ of $\int_A f^\lambda \mathbb{D}$. Then $\mathcal{I} \neq 0$ by assumption.

Let $N = \text{order}(\mathcal{I})$ (recall that supp $\mathcal{I} \subset \{0\}$ by the corollary) ant let $\varphi \in C_c^\infty(X_{\mathcal{E}})$ such that $\langle \mathcal{I}, \varphi \rangle \neq 0$.

Using a Taylor expansion at order $N$ at $0$ for $\varphi$, we get a $\omega \in \Omega_{X_{\mathcal{E}}}^{n+1}$ such that $\langle \mathcal{I}, \omega |_{X_{\mathcal{E}}} \rangle = \langle \mathcal{I}, \varphi \rangle \neq 0$. Let $\rho \in C_c^\infty(X_{\mathcal{E}})$ with $\rho \equiv 1$ near $0$. So the meromorphic extension of $\int_A f^\lambda \rho \omega$ has a pole of order $j \geq 2$ at $\lambda = -k$. Now using the fact that $f^l \Omega_{X_{\mathcal{E}}}^{n+1} = \frac{df}{f} \wedge \Omega_{X_{\mathcal{E}}}^n$ near $0$ in $\mathbb{C}_{\mathcal{E}}^{n+1}$ for some $l \in \mathbb{N}$, we can assume that there exist $\alpha \in \Omega_{X_{\mathcal{E}}}^n$ such that $\int_A f^\lambda \frac{df}{f} \wedge \rho \alpha$ has a pole of order $j \geq 2$ at $\lambda = -k - l$.

Let $\omega_1 \ldots \omega_\mu$ be a meromorphic Jordan basis form the Gauss-Manin system in degree $n$ near $0$ for $f_{\mathcal{E}}$. We can write

$$\alpha = \sum_{p=1}^\mu a_p \omega_p + df \wedge \xi + d\eta,$$

where $a_p \in \mathbb{C}\{f\}[[f^{-1}]]$ and where $\xi$ and $\eta$ are meromorphic $(n-1)$-forms with poles in $\{f_{\mathcal{E}} = 0\}$.

Now

$$\int_A f^\lambda \frac{df}{f} \wedge \rho(df \wedge \xi + d\eta) = \pm \int_A f^\lambda \frac{df}{f} \wedge d\rho \wedge \eta$$

will have, at most, simple poles at negative integers because $d\rho \equiv 0$ near $0$ (and the corollary). As is it enough to consider the case $a_p = f^m$ where $m \in \mathbb{Z}$ and this only shift $\lambda$ by an integer, we are left only with integrals like $\int_A f^\lambda \frac{df}{f} \wedge \rho \omega$

where $\omega$ is an element of the Jordan basis (*) for the monodromy acting on $H^n(F_{\mathcal{E}}, \mathbb{C})$ where $F_{\mathcal{E}}$ is the Milnor fiber of $f_{\mathcal{E}}$ at $0$. If $\omega$ belongs to an eigenvalue $\neq 1$ we can assume $\omega = w_k$ with

(*) see the computations with the sheaves $\Omega(k)$ in [1]
\[ dw_k = \frac{df}{f} \wedge w_k + \frac{df}{f} \wedge w_{k-1} \]
\[ dw_{k-1} = \frac{df}{f} \wedge w_{k-1} + \frac{df}{f} \wedge w_{k-2}, \text{ etc.} \]
and \[ w_0 = 0, \quad 0 < u < 1. \]

But
\[ u \int_A f^\lambda \frac{df}{f} \wedge pw_1 = \int_A f^\lambda pdw_1 = -\lambda \int_A f^\lambda \frac{df}{f} \wedge pw_1 - \int_A f^\lambda dp \wedge w_1 \]
gives
\[ (\lambda + u) \int_A f^\lambda \frac{df}{f} \wedge pw_1 = -\int_A f^\lambda dp \wedge w_1 \]
and \( dp \equiv 0 \) near 0 with \( u \in ]0, 1[ \) gives that \( \int_A f^\lambda \frac{df}{f} \wedge pw_1 \) has at most simple poles at negative integers \( \left( \text{as } \frac{1}{\lambda + u} \text{ is holomorphic near } \mathbb{Z} \right) \). An easy induction leads to the same result for \( \int_A f^\lambda \frac{df}{f} \wedge pw_k \).

So we are left with the eigenvalue 1 Jordan blocs, that is to say the \( u = 0 \) case; but then, we are back to the computation made in the direct part of the theorem. The point is now that \( \int_A f^\lambda dp \wedge w_k \) will not have (simple) pole at \( \lambda = 0 \) because \( \delta(A)_1 = 0 \) will give \( I(\text{var}(e'), \delta(A)) = 0 \). So these Jordan blocks for the eigenvalue 1 does not give pole, for \( \frac{1}{f(\lambda)} \int_A f^\lambda \Box \) at negative integers from our assumption \( \delta(A)_1 = 0 \) and the equivalence of i) and ii) is proved because we have contradicted our assumption \( \Xi \neq 0 \). Let us prove now the last statement of the theorem:

Let \( e = \delta(A)_1 \) and let \( h \in \mathbb{N}^* \) be the nilpotency order of \( T - 1 \) acting on \( \delta(A)_1 \). So we have \( N^{h-1}(e) \neq 0 \) and \( N^h(e) = 0 \) \( (N = T - 1) \).

Then we choose \( e' \) such that \( h(e', N^{h-1}(e)) \neq 0 \) and so \( I(\text{var}(e''), N^{h-1}(e)) \neq 0 \).

Then, as \( \text{var} \) commutes with \( N \), we have \[ I(\text{var}[N^{h-1}(e'')], \delta(A)) \neq 0. \]
So we get now for \( h \geq 2 \)
\[ \int_{G_A} v_{k-h+1} \neq 0 \quad \text{(notations as above)} \]

and then a pole of order \( \geq 2 \) at \( \lambda = 0 \) for \( \int_A f^\lambda \tilde{w}_{k-h+1} \wedge \frac{df}{f} \).

Now, using \( \delta \tilde{w} = \begin{pmatrix} \omega \\ 0 \\ \vdots \\ 0 \end{pmatrix} \), we conclude that \( \int_A f^\lambda \sigma_{w_k} \wedge \frac{df}{f} \) has a pole of order \( \geq 2 + h - 1 = h + 1 \) at \( \lambda = 0 \). So we obtain that the order of poles of \( \frac{1}{\Gamma(\lambda)} \int_A f^\lambda \square \) at (big) negative integers is at least the nilpotency order of \( T - 1 \) acting on \( \delta(A) \). The fact that this happens for \( v = -(n+1) \) is obtained as in the case \( h = 1 \). Conversely, if we have a nilpotency order equal to \( h \geq 1 \), arguing in the same way that in the proof of ii) \( \Rightarrow \) i), we conclude that the poles of \( \frac{1}{\Gamma(\lambda)} \int_A f^\lambda \square \) are of order at most \( h \).

References


