Abelian state-closed subgroups of automorphisms of \( m \)-ary trees

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Abstract. The group \( A_m \) of automorphisms of a one-rooted \( m \)-ary tree admits a diagonal monomorphism which we denote by \( x \). Let \( A \) be an abelian state-closed (or self-similar) subgroup of \( A_m \). We prove that the combined diagonal and tree-topological closure \( A^* \) of \( A \) is additively a finitely presented \( \mathbb{Z}_m[[x]] \)-module, where \( \mathbb{Z}_m \) is the ring of \( m \)-adic integers. Moreover, if \( A^* \) is torsion-free then it is a finitely generated pro-\( m \) group. Furthermore, the group \( A \) splits over its torsion subgroup. We study in detail the case where \( A^* \) is additively a cyclic \( \mathbb{Z}_m[[x]] \)-module, and we show that when \( m \) is a prime number then \( A^* \) is conjugate by a tree automorphism to one of two specific types of groups.

Mathematics Subject Classification (2010). 20E08, 20F18.

Keywords. Automorphisms of trees, state-closed groups, self-similar groups, abelian groups, topological closure, \( p \)-adic integers, pro-\( p \) groups.

1. Introduction

Automorphisms of one-rooted regular trees \( T(Y) \) indexed by finite sequences from a finite set \( Y \) of size \( m \geq 2 \) have a natural interpretation as automata on the alphabet \( Y \), with states which are again automorphisms of the tree. A subgroup of the group of automorphisms \( A(Y) \) of the tree is said to be state-closed in the language of automata (or self-similar in the language of dynamics) of degree \( m \), provided that the states of its elements are themselves elements of the same group. If the group is not state-closed then we may consider its state-closure. The prime example of a state-closed group is the group generated by the binary adding machine \( \tau = (e, \tau_0) \sigma \), where \( \sigma \) is the transposition \((0, 1)\).

We study in this paper representations of general abelian groups as state-closed groups of degree \( m \). For this purpose we use topological and diagonal closure oper-
ations in the automorphism group of the tree. Representations of free abelian groups of finite rank as state-closed groups of degree 2 were characterized in [4].

An automorphism group $G$ of the tree group is said to be transitive, provided that the permutation group $P(G)$ induced by $G$ on the set $Y$ is transitive; actions of groups on sets will be applied on the right. It will be shown that the structure of state-closed groups can in a certain sense be reduced to those which are transitive.

The automorphism group $A(Y)$ of the tree is a topological group with respect to the topology inherited from the tree. This topology allows us to exponentiate elements of $A(Y)$ by $m$-ary integers from $\mathbb{Z}_m$. Given a subgroup $G$ of $A(Y)$, its topological closure $\bar{G}$ with respect to the tree topology belongs to the same variety as $G$. Also, if $G$ is state-closed then so is $\bar{G}$.

The diagonal map $\alpha \mapsto \alpha^{(1)} = (\alpha, \alpha, \ldots, \alpha)$ is a monomorphism of $\mathcal{A}_m$. Define inductively $\alpha^{(0)} = \alpha$, $\alpha^{(i+1)} = (\alpha^{(i)})^{(1)}$ for $i \geq 0$. It is convenient to introduce a symbol $x$ and write $\alpha^{(i)}$ as $\alpha^x$ for $i \geq 0$. This will permit more general exponentiation, by formal power series $p(x) \in \mathbb{Z}_m[[x]]$. Given a subgroup $G$ of $A(Y)$, its diagonal closure $\bar{G}$ is the group $\langle G^{(i)} | i \geq 0 \rangle$. Observe that the diagonal closure operation preserves the state-closed property.

We will show that given an abelian transitive state-closed group $A$, its diagonal closure $\bar{A}$ is again abelian. The composition of the diagonal and topological closures when applied to $A$ produces an abelian group denoted by $A^*$, which can be viewed additively as a finitely generated $\mathbb{Z}_m[[x]]$-module. This approach was first used in [2].

The prime decomposition $m = \prod_{1 \leq i \leq s} p_i^{k_i}$ provides us with the decomposition $\mathbb{Z}_m = \bigoplus_{1 \leq i \leq s} \mathbb{Z}_{p_i^{k_i}}$, where $\mathbb{Z}_{p_i^{k_i}}$ are orthogonal idempotents such that $1 = \sum_{1 \leq i \leq s} \varepsilon_i \mathbb{Z}_{p_i^{k_i}}[[x]] = \bigoplus_{1 \leq i \leq s} \varepsilon_i \mathbb{Z}_{p_i^{k_i}}[[x]]$.

When $m = p^k$ and $p$ a prime number, the rings $\mathbb{Z}_m[[x]]$ and $\mathbb{Z}_p[[x]]$ are isomorphic, yet when $k > 1$ they are different representations of the same object and for this reason we distinguish between them.

In Sections 3 and 4 we prove

**Theorem 1.** Let $A$ be an abelian transitive state-closed group of degree $m$. Then

1. the group $A^*$ is isomorphic to a finitely presented $\mathbb{Z}_m[[x]]$-module;
2. if $A^*$ is torsion-free then it is a finitely generated $\mathbb{Z}_m$-module which is also a pro-$m$ group.

Item (1) is part of Theorem 5 and item (2) is Corollary 1 of Theorem 6.

We consider in Section 5 torsion subgroups of state-closed abelian groups and use methods from virtual endomorphisms of groups (see [3], [1]; reviewed in Section 5.1) to prove the following structural result.

**Theorem 2.** Let $A$ be an abelian transitive state-closed group of degree $m$ and $\text{tor}(A)$ its torsion subgroup. Then
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(i) \( \text{tor}(A) \) is a direct summand of \( A \) and has exponent a divisor of the exponent of \( P(A) \);

(ii) the action of \( A \) on the \( m \)-ary tree induces transitive state-closed representations of \( \text{tor}(A) \) on the \( m_1 \)-tree and of \( \frac{A}{\text{tor}(A)} \) on the \( m_2 \)-tree, where \( m_1 = |P(\text{tor}(A))| \) and \( m_2 = |\frac{P(A)}{P(\text{tor}(A))}| \);

(iii) if \( A = \text{tor}(A) \) and \( P(A) \not\equiv L_1/Z_4 \cap L_2/Z_4 \cap Z_{m_1}/Z_{m_2} [x] \).

The above results are analogous to Theorem 4.3.4 of [5] on the structure of finitely generated pro-\( p \) groups. By item (i) of the theorem, an abelian torsion group \( G \) of infinite exponent cannot have a faithful representation as a transitive state-closed group for any finite degree. Put differently, the group \( G \) does not admit any simple virtual endomorphism. On the other hand, the group of automorphisms of the \( p \)-adic tree is replete with abelian \( p \)-subgroups of infinite exponent. Item (iii) follows from Theorem 7, which is a conjugacy result and therefore more general than isomorphism.

We focus our attention in Section 6 on transitive state-closed abelian groups \( A \), for which \( A^* \) is additively a cyclic \( \mathbb{Z}_m[[x]] \)-module. We show

**Theorem 3.** (1) Let \( q_1, \ldots, q_m \in \mathbb{Z}_m[[x]] \) and let \( \sigma \) be the cycle \((1, 2, \ldots, m)\). Then the expression
\[
\alpha = (\alpha^{q_1}, \ldots, \alpha^{q_m}) \sigma
\]
is a well-defined automorphism of the \( m \)-ary tree and the state-closure \( A \) of \( \langle \alpha \rangle \) is an abelian transitive group. The group \( A^* \) is additively isomorphic to the quotient ring \( \mathbb{Z}_m[[x]]/(r) \), where \( r = m - xq \) and \( q = q_1 + \cdots + q_m \).

(2) Let \( A \) be a transitive state-closed abelian group of degree \( m \) such that \( A^* \) is additively a cyclic \( \mathbb{Z}_m[[x]] \)-module. Then \( P(A) \) is cyclic, say generated by \( \sigma \), and \( A^* \) is the state-diagonal-topological closure of an element of the form \( \alpha = (\alpha^{q_1}, \ldots, \alpha^{q_m}) \sigma \) for some \( q_1, \ldots, q_m \in \mathbb{Z}_m[[x]] \).

Finally we provide a complete description of the group \( A^* \) for state-closed groups of prime degree. Let \( j \geq 1 \) and let \( D_m(j) \) be the group generated by the set of states of the generalized adding machine \( \alpha = (e, \ldots, e, \alpha^{x^{j-1}}) \sigma \) acting on the \( m \)-ary tree with \( \sigma = (1, 2, \ldots, m) \). The topological closure of \( D_m(j) \) seen as \( \mathbb{Z}_m \)-module is isomorphic to the ring \( \mathbb{Z}_m[[x]]/(r) \), \( r = m - xj \).

**Theorem 4.** Let \( A \) be an abelian transitive state-closed group of prime degree \( m \) and let \( \sigma \) be the \( m \)-cycle automorphism. If \( \text{tor}(A) \) is nontrivial then \( A^* \) is a torsion group conjugate to \( \langle \sigma \rangle^* (\equiv \mathbb{Z}_m[[x]]) \). If \( A \) is torsion-free then \( A^* \) is a torsion-free group conjugate to the topological closure of \( D_m(j) \) for some \( j \).

One of the questions that has remained unanswered is whether a free abelian group of infinite rank admits a faithful transitive state-closed representation, even of prime degree.
2. Preliminaries

We fix the notation $Y = \{1, 2, \ldots, m\}$, $\mathcal{T}_m = \mathcal{T}(Y)$, $\mathcal{A}_m = \mathcal{A}(Y)$ and we let $\text{Perm}(Y)$ be the group of permutations of $Y$. A permutation $\gamma \in \text{Perm}(Y)$ is extended to an automorphism of the tree by $\gamma : yu \to \gamma yu$, fixing the non-initial letters of every sequence. An automorphism $\alpha \in \mathcal{A}_m$ is represented as $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)\sigma(\alpha)$, where $\alpha_i \in \mathcal{A}_m$ and $\sigma(\alpha) \in \text{Perm}(Y)$. Successive developments of $\alpha_i$ produce for us $\alpha_u$ (a state of $\alpha$) for every finite string $u$ over $Y$.

The product of $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)\sigma(\alpha)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_m)\sigma(\beta)$ in $\mathcal{A}_m$ is

$$\alpha\beta = (\alpha_1\beta_1)\sigma(\alpha)\ldots(\alpha_m\beta_m)\sigma(\alpha)\sigma(\beta).$$

Let $G$ be a subgroup of $\mathcal{A}_m$. Denote the subgroup of $G$ which fixes the vertices of the $i$-th level of the tree by $\text{Stab}_G(i)$. Given $y \in Y$, denote by $\text{Fix}_G(y)$ the subgroup of $G$ consisting of the elements of $G$, which fix $y$. The group $G$ is said to be recurrent provided it is transitive and $\text{Fix}_G(1)$ projects in the 1st coordinate onto $G$.

The group $\mathcal{A}_m$ is the inverse limit of its quotients by the $i$-th level stabilizers $\text{Stab}_{\mathcal{A}_m}(i)$ of the tree and is as such a topological group where each $\text{Stab}_{\mathcal{A}_m}(i)$ is an open and closed subgroup. For a subgroup $G$ of automorphisms of the tree, its topological closure $\tilde{G}$ coincides with the set of all infinite products $\ldots g_1 \ldots g_1 g_0$ or alternately, $g_0 g_1 \ldots g_i \ldots$, where $g_i \in \text{Stab}_{\mathcal{A}_m}(i)$. The group $\tilde{G}$ satisfies the same group identities as $G$. We note that the property of being state-closed is also preserved by the topological closure operation.

Let $\alpha$ be an automorphism of the tree. Then $\langle \alpha \rangle = \{\alpha^p \mid p \in \mathbb{Z}_m\}$. More generally, for $q = \sum_{i \geq 0} q_i x^i \in \mathbb{Z}_m[[x]]$ with $q_i \in \mathbb{Z}_m$, we write the expression

$$\alpha^q = \alpha^{q_0} \alpha^{q_1 x} \ldots \alpha^{q_i x^i} \ldots,$$

which can be verified to be a well-defined automorphism of the tree.

We recall the reduction of group actions to transitive ones, with a view to a similar reduction for state-closed groups of automorphisms of trees. Let $G$ be a subgroup of $\text{Perm}(Y)$, let $\{Y_i \mid i = 1, \ldots, s\}$ be the set of orbits of $G$ on $Y$ and let $\{\rho_i : G \to \text{Perm}(Y_i) \mid i = 1, \ldots, s\}$ be the set of induced representations. Then, each $\rho_i$ is transitive and $\rho : G \to \prod_{1 \leq i \leq s} \text{Perm}(Y_i) \leq \text{Perm}(Y)$ defined by $g \to (g^{\rho_1}, \ldots, g^{\rho_s})$ is a monomorphism. The reduction for tree actions follows from

**Lemma 1.** Let $G$ be a state-closed group of automorphisms of the tree $\mathcal{T}(Y)$ and let $X$ be a $P(G)$-invariant subset of $Y$. Then $\mathcal{T}(X)$ is $G$-invariant and for the resulting representation $\mu : G \to \mathcal{A}(X)$ the group $G^\mu$ is state-closed. If $G$ is diagonally closed or is topologically closed then so is $G^\mu$.

**Proof.** Let $xu$ be a sequence from $X$ and let $\alpha \in G$. Then $(xu)^\alpha = x^{\sigma(\alpha)}u^{\alpha_x}$. As $x^{\sigma(\alpha)} \in X$ and $\alpha_x \in G$, it follows that $(xu)^\alpha$ is a sequence from $X$. Also, for any sequence $u$ from $X$, we have $(\alpha^\mu)_u = (\alpha_u)^\mu$. Thus, $G^\mu$ is state-closed. The last assertion is clear. \qed
We note the following important properties of transitive state-closed abelian groups $A$.

**Proposition 1.** Let $A$ be an abelian transitive state-closed group of degree $m$. Then $\text{Stab}_A(i) \leq A^{(i)}$ for all $i \geq 0$. The group $\tilde{A}$ is an abelian transitive state-closed group and is a minimal recurrent group containing $A$. Moreover, the topological closure and diagonal closure operations commute when applied to $A$. The diagonal-topological closure $A^*$ of $A$ is an abelian transitive state-closed group.

**Proof.** Let $\alpha = (\alpha_1, \ldots, \alpha_m)\sigma, \beta = (\beta_1, \ldots, \beta_m) \in A$. Then the conjugate of $\beta$ by $\alpha$ is $\beta^\alpha = (\beta_1^{\alpha_1}, \ldots, \beta_m^{\alpha_m})^\sigma$. As $\alpha_i, \beta_i \in A$ and $A$ is abelian, it follows that $\beta = (\beta_1, \ldots, \beta_m)^\sigma$. Furthermore, since $A$ is transitive, $\beta = (\beta_1, \ldots, \beta_1) = (\beta_1)^{(1)}$. Thus, $\text{Stab}_A(i) \leq A^{(i)}$ for all $i$. A similar verification shows that $Q_A = \langle \alpha \rangle$ is abelian.

Let $G$ be a recurrent group such that $A \leq G \leq \tilde{A}$. Given $\alpha \in G$, as $G$ is recurrent, there exists $\beta \in \text{Stab}_G(1)$ such that $\beta = (\beta_1, \ldots, \beta_m)$ with $\beta_1 = \alpha$. Since $G$ is transitive and abelian, we have $\beta_1 = \cdots = \beta_m = \alpha$; that is, $\beta = \alpha^{(1)}$. Hence, $A^{(i)} \leq G$ and $G = \tilde{A}$ follows.

The last two assertions of the proposition are clear. 

The following result indicates the smallness of recurrent transitive abelian groups from the point of view of centralizers.

**Proposition 2** (Theorem 7 [1]). (1) Let $A$ be a recurrent abelian group of degree $m$ and let $C_A(A)$ be the centralizer of $A$ in $A$. Then $C_A(A) = \tilde{A}$.

(2) Let $m$ be a prime number and $A$ be an infinite transitive state-closed abelian group. Then $C_A(A) = \tilde{A}$.

This result will be used in the proofs of Lemma 3 and step 4 of Theorem 9.

### 3. A presentation for $A^*$

Let $A$ be a transitive abelian state-closed group of degree $m$ and let $A^*$ be its diagonal-topological closure. Then $A^*$ is additively a $\mathbb{Z}_m[[x]]$-module having the following properties. Given $\alpha \in A^*$, then

(i) $x\alpha = 0$ implies $\alpha = 0$;

(ii) $m\alpha = x\gamma$ for some $\gamma \in A^*$.

Let $P(A)$ be given by its presentation

$$\langle \sigma_i \mid 1 \leq i \leq k \mid \sigma_i^{m_i} = e, \text{ abelian} \rangle.$$
Choose for each $\sigma_i$ an element $\beta_i$ in $A$, which induces $\sigma_i$ on $Y$; denote $\beta_i$ by $\beta(\sigma_i)$. Then, for any $n \geq 0$, the automorphism of the tree $\beta(\sigma_i)^{(n)}$ is an element of $A$ which induces $(\sigma_i)^{(n)}$ on the $(n + 1)$-th level of the tree. Although the notation $\beta_i$ has been used to indicate the $i$th entry in an automorphism $\beta$, we hope this new usage will not cause confusion.

**Theorem 5.** Let $A$ be a transitive abelian state-closed group of degree $m$. Then $A^*$ is additively a $\mathbb{Z}_m[[x]]$-module generated by

$$\{\beta_i \mid 1 \leq i \leq k\}$$

subject to the set of defining relations

$$\{r_i = \sum_{1 \leq j \leq k} m_i \beta_i - p_{ij} \beta_j x = 0 \mid 1 \leq i \leq k\}$$

for some $p_{ij} \in \mathbb{Z}_m[[x]]$.

Moreover, there exist $r, q \in \mathbb{Z}_m[[x]]$ such that $r = m - xq$ and $rA^* = (0)$. The elements of $A^*$ can be represented additively as $\sum_{1 \leq i \leq k} p_i \beta_i$, where $p_i = \sum_{j \geq 0} p_{ij} x^j$ and each $p_{ij} \in \mathbb{Z}$ with $0 \leq p_{ij} < m$.

**Proof.** Let $\alpha \in A^*$ and $\sigma(\alpha) = \prod_{1 \leq i \leq k} \sigma_i^{r_{i1}}$, $0 \leq r_{i1} < m_i$. Then either $\alpha(\prod_{1 \leq i \leq k} \beta_i^{r_{i1}})^{-1}$ is the identity element or there exists $l_2 \geq 1$ such that

$$\alpha(\prod_{1 \leq i \leq k} \beta_i^{r_{i1}})^{-1} \in \text{Stab}(l_2) \setminus \text{Stab}(l_2 + 1)$$

and so, $\alpha(\prod_{1 \leq i \leq k} \beta_i^{r_{i1}})^{-1} = (\gamma)^{(l_2)}$ for some $\gamma \in A^*$. We treat $\gamma$ in the same manner as $\alpha$. In the limit, we obtain

$$\alpha = \prod_{1 \leq i \leq k} (\beta_i^{r_{i1}}(\beta_i^{r_{i2}}(\beta_i^{r_{i3}}(\ldots \beta_i^{r_{ij}}(\ldots))) = \prod_{1 \leq i \leq k} \beta_i^{q_i},$$

where $0 \leq r_{ij} < m_i$, $1 \leq l_2 < l_3 < \cdots < l_j < \cdots$, and $q_i = r_{i1} + \sum_{j \geq 2} r_{ij} x^{l_j}$ are formal power series in $x$. Additively we then have

$$\alpha = \sum_{1 \leq i \leq k} q_i \beta_i \in \sum_{1 \leq i \leq k} \mathbb{Z}_m[[x]] \beta_i.$$
Let $F = \bigoplus_{1 \leq i \leq k} \mathbb{Z}_m[[x]] \hat{\beta}_i$ be a free $\mathbb{Z}_m[[x]]$-module of rank $k$. Define the $\mathbb{Z}_m[[x]]$-homomorphism

$$
\phi: \sum_{1 \leq i \leq k} \mathbb{Z}_m[[x]] \hat{\beta}_i \rightarrow A^*, \quad \sum_{1 \leq i \leq k} p_i \hat{\beta}_i \mapsto \prod_{1 \leq i \leq k} \hat{\beta}_i^{p_i},
$$

and let $R$ be the kernel $\phi$. Define $J$ to be the $\mathbb{Z}_m[[x]]$-submodule of $R$ generated by

$$
\hat{r}_i = m_i \hat{\beta}_i - x(\sum_{1 \leq j \leq k} p_{ij} \hat{\beta}_j) \quad (1 \leq i \leq k).
$$

We will show that $J = R$. So let $v \in R$ and write $v = \sum_{1 \leq i \leq k} v_i \hat{\beta}_i$, where

$$
v_i = \sum_{j \geq 0} v_{ij} x^j, \quad v_{ij} = v_{ij,0} + mw_{ij} \in \mathbb{Z}_m.
$$

Then $m_i | v_{i0,0}, v_{i0,0} = m_i v'_{i0,0}$; factor $m = m_i m'_i$. Therefore,

$$
v_i = v_{i0} + (\sum_{j \geq 1} v_{ij} x^{j-1})x,
$$

$$
v_{i0} = m_i v'_{i0,0} + mw_{i0} = (v'_{i0,0} + m_i w_{i0})m_i,
$$

$$
v_i \hat{\beta}_i = (v'_{i0,0} + m_i w_{i0})(m_i \hat{\beta}_i) + (\sum_{j \geq 1} v_{ij} x^{j-1})x \hat{\beta}_i,
$$

$$
\equiv (v'_{i0,0} + m_i w_{i0})(x \sum_{1 \leq j \leq k} p_{ij} \hat{\beta}_j) + (\sum_{j \geq 1} v_{ij} x^{j-1})x \hat{\beta}_i \mod J.
$$

Hence

$$
v = \sum_{1 \leq i \leq k} v_i \hat{\beta}_i + x \mu + J, \quad \mu = \sum_{1 \leq i \leq k} \mu_i \hat{\beta}_i \in R.
$$

Hence, by repeating the argument, we obtain

$$
v \in \left( \bigcap_{i \geq 1} x^i R \right) + J = J, \quad J = R.
$$

On re-writing the relations $m_i \hat{\beta}_i = \sum_{1 \leq j \leq k} p_{ij} x \hat{\beta}_j$ in the form

$$
p_{i1} x \hat{\beta}_1 + \cdots + (p_{ii} x - m_i) \hat{\beta}_i + \cdots + p_{kk} x \hat{\beta}_k = 0
$$

we see that the $k \times k$ matrix of coefficients of these equations has determinant $r = m - qx$ for some $q \in \mathbb{Z}_m[[x]]$ and thus $r$ annihilates $A^*$.

The last assertion of the theorem follows by using $r = m - qx \in R$ to reduce the coefficients modulo $m$.

4. The $m$-congruence property

A group $G$ of automorphisms of the $m$-ary tree is said to satisfy the $m$-congruence property, provided that given $m^i$ there exists $l(i) \geq 1$ such that $\text{Stab}_G(l(i)) \leq G^{m^i}$ for all $i$, in which case the topology on $G$ inherited from $\mathcal{A}(Y)$ is equal to the pro-$m$ topology. Since when $A^*$ is written additively, we have $\text{Stab}_G(l(i)) = x^{l(i)} A^*$, the $m$-congruence property reads $x^{l(i)} A^* \leq m^i A^*$. 
Theorem 6. Let \( r = m - qx^j \in \mathbb{Z}_m[[x]] \) with \( q \in \mathbb{Z}_m[[x]] \) and \( j \geq 1 \). Let \( S \) be quotient ring \( \mathbb{Z}_m[[x]]_{(r)} \). Suppose that \( S \) is torsion-free. Then \( S \) is a finitely generated pro-\( m \) group.

Proof. From the decomposition \( \mathbb{Z}_m[[x]] = \bigoplus_{1 \leq i \leq s} \mathbb{Z}_{p_i^{k_i}}[[x]] \) corresponding to the prime decomposition \( m = \prod_{1 \leq i \leq s} p_i^{k_i} \), we obtain

\[
\begin{align*}
  r &= \sum_{1 \leq i \leq s} r_i, \\
  r_i &= \varepsilon_i r = p_i^{k_i} - q_i(x)x^j, \\
  S &= \sum_{1 \leq i \leq s} S_i, \quad S_i = \frac{\mathbb{Z}_{p_i^{k_i}}[[x]]}{(r_i)},
\end{align*}
\]

where each \( S_i \) is torsion-free. Thus, it is sufficient to address the case where \( m \) is a prime power \( p^k \).

(1) First, we show that \( S \) is a pro-\( m \) group.

So let \( r = p^k - qx^j \) and decompose \( q = q(x) = s(x) + p \cdot t(x) \), where each non-zero coefficient of \( s(x) \) is an integer relatively prime to \( p \). If \( s(x) = 0 \) then \( q(x) = p \cdot t(x) \) and

\[
r = p^k - q(x)x^j = p^k - p \cdot t(x)x^j = p(p^{k-1} - t(x)x^j) \in (r);
\]

but as by hypothesis \( S \) is torsion free, we have \( p^{k-1} - t(x)x^j \in (r) \), which is not possible.

Write \( s(x) = x^lu(x) \), where \( l \geq 0 \) and \( u(x) \) is invertible in \( \mathbb{Z}_m[[x]] \) with inverse \( u'(x) \). Then \( q(x) = x^lu(x) + p \cdot t(x) \) and

\[
r = p^k - (x^lu(x)x^j + p \cdot t(x)x^j) = p(p^{k-1} - t(x)x^j) - x^{j+l}u(x).
\]

Therefore, on multiplying by \( u'(x) \), the inverse of \( u(x) \), we obtain

\[
p(p^{k-1} - t(x)x^j)u'(x) \equiv x^{j+l} \mod r.
\]

It follows that

\[
x^{j+l}S \leq pS, \quad x^{n(j+l)}S \leq p^nS.
\]

(2) Now we show that \( S \) is finitely generated as a \( \mathbb{Z}_m \)-module.

By the previous step there exist \( l \geq 1 \) and \( v(x) \in \mathbb{Z}[[x]] \) such that

\[
x^l \equiv mv(x) \mod r.
\]

Decompose \( v(x) = v_1(x) + v_2(x)x^l \) where the degree of \( v_1(x) \) is less than \( l \). Then
we deduce modulo $r$:

\[\begin{align*}
v(x) & \equiv v_1(x) + v_2(x)mv(x), \\
v_2(x)v(x) & \equiv w(x) \in \mathbb{Z}[x], \\
w(x) & = w_1(x) + w_2(x)x^l, \\
v(x) & \equiv v_1(x) + mw(x) \\
& \equiv v_1(x) + mw_1(x) + mw_2(x)x^l \\
& \vdots \\
v(x) & \equiv a_0 + a_1x + \cdots + a_{l-1}x^{l-1}, \quad a_i \in \mathbb{Z}_m.
\end{align*}\]

We have shown that $S$ is generated by $1, x, \ldots, x^{l-1}$ as a pro-$m$ group. \qed

**Corollary 1.** Let $A$ be an abelian transitive state-closed group of degree $m$. Suppose that the group $A^*/\text{ETX}$ is torsion-free. Then $A^*/\text{ETX}$ is a finitely generated pro-$m$ group.

**Proof.** With previous notation, the group $A^*$ is a $\mathbb{Z}_m[[x]]$-module generated by

\[\{\beta_i = \beta(\sigma_i) \mid 1 \leq i \leq k\}\]

and is annihilated by $r = m - qx^j \in \mathbb{Z}_m[[x]]$ for some $q \in \mathbb{Z}_m[[x]]$ and $j \geq 1$.

It follows that $A^*$ is an $S$-module, where $S = \mathbb{Z}_m[[x]](r)$. Since $S$ satisfies the $m$-congruence property, it follows that $A^*$ is a pro-$m$ group.

That $A^*$ is a finitely generated $\mathbb{Z}_m$-module, is a consequence of $S$ being a finitely generated $\mathbb{Z}_m$-module. \qed

5. **Torsion in state-closed abelian groups**

5.1. **Preliminaries on virtual endomorphisms of groups.** Let $G$ be a transitive state-closed subgroup of $\mathcal{A}(Y)$, where $Y = \{1, 2, \ldots, m\}$. Then $[G : \text{Fix}_G(1)] = m$ and the projection on the 1st coordinate of $\text{Fix}_G(1)$ produces a subgroup of $G$; that is, $\pi_1 : \text{Fix}_G(1) \to G$ is a virtual endomorphism of $G$. This notion has proven to be effective in studying state-closed groups. We give a quick review below.

Let $G$ be a group with a subgroup $H$ of finite index $m$ and a homomorphism $f : H \to G$. A subgroup $U$ of $G$ is **semi-invariant** under the action of $f$, provided that $(U \cap H)^f \leq U$. If $U \leq H$ and $U^f \leq U$ then $U$ is **$f$-invariant**.

The largest subgroup $K$ of $H$ which is normal in $G$ and $f$-invariant is called the **$f$-core($H$)**. If the $f$-core($H$) is trivial then $f$ and the triple $(G, H, f)$ are said to be a **simple**.

Given a triple $(G, H, f)$ and a right transversal $L = \{x_1, x_2, \ldots, x_m\}$ of $H$ in $G$, the permutational representation $\pi : G \to \text{Perm}(1, 2, \ldots, m)$ is $g^\pi : i \to j$, which
is induced from the right multiplication $H x_i g = H x_j$. We produce recursively a representation $\varphi: G \to \mathcal{A}(m)$ as follows:

$$g^\varphi = ((x_i g \cdot (x(i)g)^{-1})^f \varphi)^{1 \leq i \leq m g^\pi}.$$  

One further expansion of $g^\varphi$ is

$$g^\varphi = (((x_j g_i \cdot x_i^{-1})^f \varphi)^{1 \leq j \leq m g_i^\pi})^{1 \leq i \leq m g^\pi},$$

where $g_i = (x_i g \cdot x_i^{-1})^f$.

The kernel of $\varphi$ is precisely the $f$-core($H$), $G^\varphi$ is state-closed and $H^\varphi = \text{Fix}_{G^\varphi}(1)$.

### 5.1.1. Changing transversals.

We will show below that changing the transversal of $H$ in $G$ produces another representation of $G$, conjugate to the original one by an explicit automorphism of the $m$-ary tree.

**Proposition 3.** Let $(G, H, f)$ be a triple and

$$L = \{x_1, x_2, \ldots, x_m\}, L' = \{x'_1 = h_1 x_1, x'_2 = h_2 x_2, \ldots, x'_m = h_m x_m\}$$

right transversals of $H$ in $G$ where $h_i \in H$. Let $\varphi = \varphi_{x_i}$, $\varphi' = \varphi_{h_i x_i}: G \to \mathcal{A}(m)$ be the corresponding tree representations and define the following elements of $\mathcal{A}(m)$,

$$\gamma = \gamma_{h_i, \varphi'} = ((h_i f \varphi')^{1 \leq i \leq m},$$

$$\lambda = \lambda_{h_i, \varphi'} = \gamma^{(1)} \ldots \gamma^{(n)}.$$

Then

$$\varphi_{h_i x_i} = \varphi_{x_i}(\lambda_{h_i^{-1}, \varphi_{x_i}}).$$

**Proof.** The representations $\varphi, \varphi': G \to \mathcal{A}(m)$ are defined by

$$g^\varphi = ((x_i g \cdot (x(i)g)^{-1})^f \varphi)^{1 \leq i \leq m g^\pi},$$

$$g^{\varphi'} = ((x_i' g \cdot (x'(i)g)^{-1})^f \varphi')^{1 \leq i \leq m g^\pi}.$$  

The relationship between $\varphi'$ and $\varphi$ is established as follows,

$$g^{\varphi'} = ((h_i x_i g \cdot (h(i)g)^{-1} x(i) g^{-1})^f \varphi')^{1 \leq i \leq m g^\pi}$$

$$= ((h_i x_i g \cdot x(i) g^{-1} h(i) g^{-1})^f \varphi')^{1 \leq i \leq m g^\pi}$$

$$= ((h_i f \varphi')^{1 \leq i \leq m} \cdot (x_i g \cdot x(i) g^{-1} h(i) g^{-1})^f \varphi')^{1 \leq i \leq m g^\pi}$$

$$= ((h_i f \varphi')^{1 \leq i \leq m} \cdot (x_i g \cdot x(i) g^{-1} h(i) g^{-1})^f \varphi')^{1 \leq i \leq m g^\pi}.$$
Thus in the limit we obtain
\[ g_{\varphi'} = \gamma \cdot ((x_i g \cdot x_i^{-1})g_i)_{1 \leq i \leq m} \cdot \gamma^{-1}, \]
where \( \gamma = ((h_i)_{i \leq j < m}) \) is independent of \( g \). Repeating this development for each \( g_i = (x_i g \cdot x_i^{-1})g_i \), we find that
\[ g_{\varphi'} = \gamma \gamma^{(1)} \cdot ((x_j g_i \cdot x_i^{-1})g_i)_{1 \leq i \leq m} \cdot \gamma^{(1)} \gamma^{-1}. \]
Thus in the limit we obtain \( \lambda = \gamma \gamma^{(1)} \cdots \gamma^{(n)} \cdots \) such that
\[ g_{\varphi'} = \lambda g_{\varphi'} \lambda^{-1} \quad \text{for all} \quad g \in G, \]
\[ \varphi = \varphi' \lambda. \]

Introducing the explicit dependence of \( \varphi, \varphi', \lambda \) on the transversals, the previous equation becomes
\[ \varphi x_i = (\varphi_{h_i x_i})(\lambda_{h_i, \varphi_{h_i x_i}}). \]
On replacing \( h_i \) by \( h_i^{-1} \) and on denoting \( h_i^{-1} x_i \) by \( x_i' \), we obtain
\[ \varphi_{h_i x_i'} = (\varphi_{x_i'})(\lambda_{h_i^{-1}, \varphi_{x_i'}}). \]

\[ \square \]

**Example 1.** Let \( G = C = \langle a \rangle \) be the infinite cyclic group, let \( H = \langle a^2 \rangle \) and let \( f : H \to G \) be defined by \( a^2 \to a \). Given \( l, k \geq 0 \), then on choosing the transversal \( L_{k,l} = \{a^{2k}, a^{2l+1}\} \) for \( H \) in \( G \), we obtain the representation \( \varphi_{k,l} : G \to A(m), \) where \( \varphi_{k,l} : a \to \alpha = (a^{2k-l}, a^{-k+l+1}) \).

### 5.1.2. Subtriples, quotient triples.

Given a triple \( (G, H, f) \) and given subgroups \( V \leq G, U \leq H \cap V \) such that \((U)_{f} \leq V \), we call \((V, U, f|_{V})\) a sub-triple of \( G \). If \( N \) is a normal semi-invariant subgroup of \( G \), then \( \tilde{f} : \frac{H N}{N} \to \frac{G}{N} \) given by \( \tilde{f} : Nh \to Nh f \) is well defined and \((\frac{G}{N}, \frac{H N}{N}, \tilde{f})\) is a quotient triple.

Let \((G, H, f)\) be a simple triple where \( G \) is abelian and \([G : H] = m\). Then any sub-triple of \( G \) is simple. Let \( T = \text{tor}(G) \) denote the torsion subgroup of \( G \) and for \( l \geq 1 \) define \( G(l) = \{g \in T \mid o(g)|l\}, H(l) = G(l) \cap H \). Then, clearly, \( f : \text{tor}(H) \to \text{tor}(G) \) and \( f : H(l) \to G(l) \). Therefore, \( \text{tor}(G) \) and \( G(l) \) are semi-invariant and \((\text{tor}(G), \text{tor}(H), f|_{\text{tor}(H)})\) and \((G(l), H(l), f|_{H(l)})\) are simple sub-triples.

**Lemma 2.** Let \((G, H, f)\) be a simple triple. The triple \((\frac{G}{\text{tor}(G)}, \frac{H G(l)}{\text{tor}(H)}, \tilde{f})\) is also simple.

**Proof.** For suppose \( K \leq H \) is such that \( G(l) K F \leq G(l) K \). Then
\[ (G(l) K F)^l = (K F)^l = (K^l)^F \leq (G(l) K)^l = (K)^l; \]
that is, \( K^l \) is \( f \)-invariant. Since \( f \) is simple, \( K^l = \{e\} \), and so \( K \leq G(l) \). \( \square \)
5.2. The torsion subgroup

**Proposition 4.** Let $A$ be transitive state-closed abelian group of degree $m$. Then $\text{tor}(A)$ has finite exponent and is therefore a direct summand of $A$.

**Proof.** Let $T = \text{tor}(A)$, $A_1 = \text{Stab}_A(1)$, $T_1 = T \cap A_1$ and $[T : T_1] = m'$. Then the projection on the 1st coordinate of $T_1$ is a subgroup of $T$ and the triple $(T, T_1, \pi_1|T_1)$ is simple of degree $m'|m$; let $m = m'm''$. Hence, in this representation $T$ is a torsion transitive state-closed subgroup of $A_{m''}$, the automorphism group of the tree $T_{m'}$.

Fixing this last representation of $T$, let $Q = P(T)$ and let $\sigma_i (1 \leq i \leq k)$ be a minimal set of generators of $Q$ and as before, let $\beta_i = \beta(\sigma_i) \in T$ be such that $\sigma(\beta_i) = \sigma_i$. Let $r$ be the maximum order of the elements $\beta_1, \ldots, \beta_k$. As any $\alpha \in T$ can be written in the form

$$\alpha = \prod_{1 \leq i \leq k} \beta_i^{r_i} (\beta_i^{r_2})(l_2) \ldots (\beta_i^{r_j})(l_j) \ldots,$$

it follows that $\alpha^r = e$.

Since $T$ has finite exponent, it is a pure bounded subgroup of $A$ and therefore it is a direct summand of $A$ ([6], Theorem 4.3.8).

We recall a classic example of an abelian group $G$ which does not split over its torsion subgroup (see [6], p. 108).

**Example 2.** Let $G$ be the direct product of groups $\prod_{i \geq 1} C_i$, where $C_i = \langle c_i \rangle$ is cyclic of order $p^i$ and let $H$ be the direct sum $\sum_{i \geq 1} C_i$. Then $H \leq \text{tor}(G) = \bigcup_{i \geq 1} G(p^i)$. Moreover, $H$ is a basic subgroup of $G$ and in particular, $G/H$ is $p$-divisible. This observation leads directly to a proof that $G$ does not split over $\text{tor}(G)$.

The proof of the previous proposition did not establish the exponent of $\text{tor}(A)$. This we do in the next two lemmas.

**Lemma 3.** Let $m$ be a prime number and $A$ an abelian transitive state-closed torsion group of degree $m$. Then $A$ is conjugate by a tree automorphism to a subgroup of the diagonal-topological closure of $\langle \sigma \rangle$ and so has exponent $m$.

**Proof.** We observe that $A(m)$ is not contained in $A_1 = \text{Stab}_A(1)$. For otherwise, $A(m)$ would be invariant under the projection on the first coordinate. Choose $a \in A \setminus A_1$ of order $m$. Therefore, $A = A_1 \langle a \rangle$. On choosing $\{a^i \mid 0 \leq i \leq m - 1\}$ as a transversal of $A_1$ in $A$, the image of $a$ acquires the form $\sigma = (1, \ldots, m)$ in this tree representation of $A$. Thus, we may suppose by Proposition 3 that $\sigma \in A$. Therefore, $\tilde{A}$ contains the subgroup $\langle \tilde{\sigma} \rangle = \langle \sigma^i \mid i \geq 0 \rangle$. By Proposition 2, we have $C_{\tilde{A}}(\tilde{\sigma}) = \langle \sigma \rangle^*$ and thus, $A \leq C_{\tilde{A}}(A) \leq \langle \sigma \rangle^*$.
Lemma 4. Suppose that $A$ is an abelian transitive state-closed torsion group of degree $m$. Then the exponent of $A$ is equal to the exponent of $P(A)$.  

**Proof.** By induction on $|P(A)| = m$. The exponent of $A$ is a multiple of the exponent of $P(A)$. By the previous lemma, we may assume $m$ to be composite. Let $p$ be a prime divisor of $m$ and $A(p) = \{a \in A \mid a^p = e\}$. Then $A(p)$ is a nontrivial subgroup and $P(A(p)) \leq \{\sigma \in P \mid \sigma^p = e\}$. By Lemma 2, $(\frac{A}{A(p)}, \frac{A(A(p))}{A(p)}, \pi_0)$ is simple; also, $P(A) = P(A(p)).$ The proof follows by induction.  

**Theorem 7.** Suppose that $A$ is an abelian transitive state-closed torsion group of degree $m$. Then $A$ is conjugate to a subgroup of the topological closure of $\widehat{P(A)} = \langle \sigma^i \mid \sigma \in P(A), i \geq 0 \rangle$.  

**Proof.** Let $P = P(A)$ have exponent $r$ and let $B$ be a maximal homogeneous subgroup of $P$ of exponent $r$ (that is, $B$ is a direct sum of cyclic groups of order $r$), minimally generated by $\{\sigma_i \mid 1 \leq i \leq s\}$. Choose for each $\sigma_i$ an element $\beta_i = \beta(\sigma_i) \in A$ and let $\hat{B} = \langle \beta_i \mid 1 \leq i \leq s \rangle$. Then, as the order of each $\beta_i$ is a multiple of $r$, while the exponent of $A$ is $r$, we conclude from the previous lemma that $o(\beta_i) = o(\sigma_i) = r$ for $1 \leq i \leq s$. Since $\beta_i \rightarrow \sigma_i$ defines a projection of $\hat{B}$ onto $B$ we conclude that $\hat{B} \cong B$ and $\hat{B} \cap A_1 = \{e\}$, where $A_1 = \text{Stab}_A(1)$. 

Clearly $\hat{B}$ is a pure bounded subgroup and so it has a complement $L$ in $A$, which may be chosen to contain $A_1$. Choose a right transversal $W$ of $A_1$ in $L$. Then the set $W\hat{B}$ is a right transversal of $A_1$ in $A$. With respect to this transversal, the triple $(A, A_1, \pi_1)$ produces a transitive state-closed representation $\varphi$ where $B^\varphi = B$. By Proposition 3, we may rewrite $A^\varphi$ as $A$. Then the diagonal-topological closure $A^*$ contains $B^*$. Let $V$ be a complement of $B$ in $P$. Each $\alpha \in A^*$ can be factored as $\alpha = \beta \gamma$, where $\beta \in B^*$ and $\gamma$ is such that each of its states $\gamma_u$ have activity $\sigma(\gamma_u) \in V$. Therefore, the set of these $\gamma$’s is a group $\Gamma$ such that $\Gamma = \Gamma^* \oplus B^*$. Then $(\Gamma, \Gamma \cap A_1, \pi_1)$ is a simple triple with $P(\Gamma)$ having exponent smaller than $r$. The proof is finished by induction on the exponent.  

The example below illustrates some of the ideas developed so far.  

**Example 3.** Let $m = 4$, $Y = \{1, 2, 3, 4\}$ and let $\sigma$ be the cycle $(1, 2, 3, 4)$. Furthermore, let $\alpha = (e, e, e, \alpha^2) \sigma \in A(4)$ and let $A = \langle \alpha \rangle$. Then 

$$\alpha^2 = (\alpha^2, e, e, \alpha^2)(1, 3)(2, 4),$$

$$\alpha^4 = (\alpha^2)(1) = \alpha^{2x}, \quad (\alpha^{2-x})^2 = e.$$ 

Thus $A$ is cyclic, torsion-free, transitive and state-closed; it is, however, not diagonally closed because $\alpha^x \not\in A$. Even though $A$ is torsion-free, its diagonal closure $\bar{A} = \langle \alpha^{x^i} \mid i \geq 0 \rangle$ is not; for $\kappa = \alpha^{2-x}$ has order 2. Let $K = \langle \kappa^{x^i} \mid i \geq 0 \rangle$.  


Then $K \leq \text{tor}(A)$ and it is direct to check that $\tilde{A} = \langle \alpha, K \rangle$. Therefore, $K = \text{tor}(\tilde{A})$ and
\[ \tilde{A} = \text{tor}(\tilde{A}) \oplus A. \]

Let $Y_1 = \{1, 3\}$, $Y_2 = \{2, 4\}$. Then $\{Y_1, Y_2\}$ is a complete block system for the action of $\alpha$ on $Y$. Also, $\alpha^2$ induces the binary adding machine on both $\mathcal{T}(Y_1)$ and $\mathcal{T}(Y_2)$. The topological closure $\tilde{A}$ of $A$ is torsion-free and
\[ \text{tor}(A^*) = \text{tor}(\tilde{A}), \quad A^* = \text{tor}(A^*) \oplus \tilde{A}. \]

Moreover, $\text{tor}(A^*)$ induces a faithful state-closed, diagonally and topologically closed actions on the binary tree $\mathcal{T}(Y_1)$. Therefore, $\text{tor}(A^*)$ is isomorphic to $\mathbb{Z}_2[[x]]$. Furthermore, $\alpha$ is represented as the binary adding machine on $\mathcal{T}(\{Y_1, Y_2\})$ and $\tilde{A}$ is represented on this tree as the topological closure of the image of $A$.

6. Cyclic $\mathbb{Z}_m[[x]]$-modules

Cyclic automorphism groups $\langle \alpha \rangle$ of the tree, for which their state-diagonal-topological closure is isomorphic to a cyclic $\mathbb{Z}_m$-module have the form
\[ \alpha = (\alpha^{q_1}, \ldots, \alpha^{q_m})\sigma, \]
where $q_i \in \mathbb{Z}_m[[x]]$ for $1 \leq i \leq m$; here
\[ q_i = \sum_{j \geq 0} q_{ij} x^j, \quad q_{ij} = \sum_{u \geq 0} q_{ij,u} m^u \in \mathbb{Z}_m. \]

We prove

Theorem 8. (i) The expression
\[ \alpha = (\alpha^{q_1}, \ldots, \alpha^{q_m})\sigma \]

is a well-defined automorphism of the $m$-ary tree.

(ii) Let $A$ be the state closure of $\langle \alpha \rangle$. Then $A^*$ is abelian, isomorphic to the quotient ring $\mathbb{Z}_m[[x]]/(r)$, where
\[ r = m - qx \quad \text{and} \quad q = q_1 + \cdots + q_m. \]

Proof. (1) Let $\sigma(l)$ denote the permutation induced by $\alpha$ on the $l$-th level. Then the expression $\alpha = (\alpha^{q_1}, \ldots, \alpha^{q_m})\sigma$ represents
\[ \sigma(1) = \sigma, \quad \sigma(l) = (\sigma(l - 1)\overline{q_l}, \ldots, \sigma(l - 1)\overline{q_m})\sigma, \]
where $\overline{q_l} = \overline{q_{l,0}} + \overline{q_{l,1}} x + \cdots + \overline{q_{l,(l-1)}} x^{l-1}$ and $\overline{q_{ij}} = q_{ij,0} + q_{ij,1} m + \cdots + q_{ij,l-1} m^{l-1}$. 
(2.1) The states of $\alpha$ are words in $\alpha^p$ for $p \in \mathbb{Z}_m[[x]]$. Let $v = \alpha^{l_1} \ldots \alpha^{l_n}$, $w = \alpha^{n_1} \ldots \alpha^{n_m} \in \mathbb{A}^n$. Then clearly $[v, w] \in \text{Stab}_4(1)$. We will prove that the entries of $[v, w]$ are products of conjugates of words in elements of the form $[\alpha^s, \alpha^t]$ where $s, t \in \mathbb{Z}_m[[x]]$.

Clearly $[v, w]$ can be developed into a word in conjugates of $[\alpha^{l_i}, \alpha^{n_j}]$.

Write $p = p_0 + p'x, n = n_0 + n'x$. We compute

$$[\alpha^p, \alpha^n] = ([\alpha^{p_0}, \alpha^{n_0}] \alpha^{p'x} \alpha^{n'x})$$

Therefore, we have to check $[\alpha^x, \alpha^n]$ where $\xi \in \mathbb{Z}_m, n \in \mathbb{Z}_m[[x]]$. Write $\xi = \xi_0 + m\xi'$. Then

$$[\alpha^x, \alpha^n] = [\alpha^{\xi_0 + m\xi'}, \alpha^n] = [\alpha^{\xi_0}, \alpha^n] \alpha^{m\xi'} = [\alpha^x, \alpha^{m\xi'}].$$

Now

$$\alpha^{\xi_0} = (v_1, v_2, \ldots, v_m)\sigma^{\xi_0},$$

where $v_i$ are words in $\alpha^{q_1}, \ldots, \alpha^{q_m}$ and

$$\alpha^{m} = (\alpha^{q_1} \ldots \alpha^{q_m}, \alpha^{q_2} \ldots \alpha^{q_m}, \alpha^{q_1}, \ldots, \alpha^{q_m} \alpha^{q_1} \ldots \alpha^{q_{m-1}}).$$

Therefore,

$$[\alpha^{\xi_0}, \alpha^n] = ([v_1, \alpha^n], \ldots, [v_m, \alpha^n])$$

and similarly

$$[\alpha^{m\xi'}, \alpha^n] = ([\alpha^{q_1} \ldots \alpha^{q_m} \alpha^{q_1} \ldots \alpha^{q_{m-1}} \alpha^{\xi'}, \alpha^n]).$$

Now we write $\beta = \alpha^{q_1} \ldots \alpha^{q_m}$. Then $[\beta^{\xi'}, \alpha^n]$ can be developed further, as asserted. The same applies to the other entries.

(2.2) First, clearly $r\alpha = 0$. Now let $u = u(x)$ annul $\alpha$; write $u = u_0 + u'x$ where $u_0 = u(0)$. Then $m|u_0$ and so

$$u = m\frac{u_0}{m} + u'x = (xq)\frac{u_0}{m} + u'x + vr = xw_1 + vr$$

for some $v = v(x)$ and $w_1 = q\frac{u_0}{m} + u'$. Then $xw_1$ annuls $\alpha$ and so does $w_1$. On repeating, we find $w_i$ such that $u \equiv x^i w_i \mod r$ and $w_i$ annuls $\alpha$ for all $i \geq 1$.

In other words, $u \in \bigcap_{n \geq 1} (x\mathbb{Z})^n + (r) = (r).$

The group $D_m(j)$. Recall $\alpha = (e, \ldots, e, \alpha^{x^{j-1}}) \in \mathbb{A}_m$. Then $\alpha^m = \alpha^{x^j}$; that is, $\alpha^r = e$ where $r = m - x^j$. The states of $\alpha$ are $\alpha^x, \ldots, \alpha^{x^{j-1}}$ and

$$D_m(j) = \{\alpha, \alpha^x, \ldots, \alpha^{x^{j-1}}\};$$

therefore $D_m(j)$ is diagonally closed. The topological closure $D_m(j)$ is isomorphic to the quotient ring $S = \mathbb{Z}_m[[x]]/(r)$, which is clearly a free $\mathbb{Z}_m$-module of rank $j$. 

6.1. The case \( P(A) \) cyclic of prime order

**Theorem 9.** Let \( m \) be a prime number. Let \( A \) be a torsion-free abelian transitive state-closed subgroup of \( A_m \). Let \( \beta \in A \setminus \text{Stab}_A(j) \). Then \( A^* = (\beta)^* \) and is topologically finitely generated. Furthermore, \( A^* \) is conjugate to \( D_{m(j)} \) for some \( j \geq 1 \).

The proof is developed in four steps.

**Step 1.** For \( z \in A \), define \( \zeta(z) = j \) such that \( z^m \in \text{Stab}(j) \setminus \text{Stab}(j + 1) \). As \( A \) is torsion-free, \( \zeta(z) \) is finite for all nontrivial \( z \) and \( z^m = (v)^{(j)} \), \( v \in A \setminus \text{Stab}_A(1) \).

Choose \( \beta = (\beta_1, \beta_2, \ldots, \beta_m)\sigma \in A \setminus \text{Stab}_A(1) \) having minimum \( \zeta(\beta) = j \). If \( z \in \text{Stab}_A(1) \), \( z \neq e \), then there exists \( l > 0 \) such that \( z^m = (c)^{(l)} \) and \( c \in A \setminus \text{Stab}_A(1) \). Therefore, by minimality of \( \beta \) we have \( \zeta(c) \geq \zeta(\beta) \) and \( \zeta(z) > \zeta(\beta) \).

**Lemma 5** (Uniform gap). Let \( z \in \text{Stab}_A(1) \). Then \( \zeta(z\beta) = \zeta(\beta) \).

**Proof.** First note that

\[
\beta^m = (\beta_1\beta_2 \cdots \beta_m)^{(1)},
\beta_1\beta_2 \cdots \beta_m = (\gamma)^{(j-1)}, \quad \gamma \in A \setminus \text{Stab}_A(1).
\]

We have \( z = c^{(1)} \) and \( z\beta = (c\beta_1, c\beta_2, \ldots, c\beta_m)\sigma \), \( (z\beta)^m = (u)^{(1)} \), where \( u = c^m\beta_1 \ldots \beta_m = c^m(\gamma)^{(j-1)} \). If \( c \in A \setminus \text{Stab}_A(1) \) then \( \zeta(c) = n \geq j \), \( c^m \in \text{Stab}(n) \setminus \text{Stab}(n + 1) \), and so, \( u \in \text{Stab}_A(j - 1) \setminus \text{Stab}(j) \). If \( c \in \text{Stab}_A(1) \) then \( \zeta(c) > j \) and so \( c^m \in \text{Stab}(k) \), where \( k > j \) and again \( u \in \text{Stab}(j - 1) \setminus \text{Stab}(j) \).

**Step 2.** Note that

\[
\beta^m = (\gamma)^{(j)}, \quad \gamma^m = (\lambda)^{(j)},
\beta^m = (\lambda)^{(2j)},
\]

where, by the uniform gap lemma above, \( \gamma, \lambda \in A \setminus \text{Stab}_A(1) \). Therefore, repeating this process, we find that \( \beta^{m^{t}} \) induces \( \sigma^{(sj)} \) on the \((sj)\)-th level of the tree for all \( s \geq 0 \). Now given a level \( t \geq 0 \), dividing \( t \) by \( j \), we get \( t = sj + i \) with \( 0 \leq i \leq j - 1 \), and then \( (\beta^{(i)})^{m^t} = (\beta^{m^t})^{(i)} \) induces \( \sigma^{(sj)}(i) = \sigma^{(sj+i)} = \sigma^{(i)} \) on the \( t \)-th level of the tree. It follows that the group \( A \) is a subgroup of the topological closure of \( \langle \beta, \beta^{(1)}, \ldots, \beta^{(j-1)} \rangle \).

**Step 3.** We have for \( \beta = (\beta_1, \beta_2, \ldots, \beta_m)\sigma \),

\[
\beta_i = \beta^{p_i}, \quad p_i = r_{i0} + r_{i1}x + \cdots + r_{i(j-1)}x^{j-1} \in \mathbb{Z}_m[x],
\]

and

\[
\beta^m = (\beta_1\beta_2 \cdots \beta_m)^{(1)},
\beta_1\beta_2 \cdots \beta_m = \beta^{p_1 + \cdots + p_m},
\]

\[
p_1 + \cdots + p_m = q \cdot x^{j-1},
\]
where \( q \) is an invertible element of \( \mathbb{Z}_m[[x]] \).

**Proposition 5.** The element \( \beta = (\beta_1, \beta_2, \ldots, \beta_m) \sigma \) is conjugate in \( \mathbb{A}_m \) to \( \alpha = (e, \ldots, e, \alpha^{x^{j-1}}) \sigma \).

**Proof.** Let \( h = (h_1, h_2, \ldots, h_m) \) be an automorphism of the tree. Then
\[
\beta^h = (h_1^{-1} \beta_1 h_2, h_2^{-1} \beta_2 h_3, \ldots, h_m^{-1} \beta_m h_1) \sigma.
\]
Therefore \( \beta^h = \alpha \) holds if and only if
\[
h_2 = \beta_1^{-1} h_1, \quad h_3 = \beta_2^{-1} h_2, \quad \ldots, \quad h_m = \beta_m^{-1} h_{m-1}, \quad h_1 = \beta_m^{-1} h_m \alpha^{x^{j-1}}.
\]
These conditions can be rewritten as
\[
h_2 = \beta_1^{-1} h_1, \quad h_3 = \beta_2^{-1} \beta_1^{-1} h_1, \quad \ldots, \quad h_m = \beta_m^{-1} \ldots \beta_1^{-1} h_1, \quad h_1 = \beta_m^{-1} \beta_{m-1}^{-1} \ldots \beta_1^{-1} h_1 \alpha^{x^{j-1}},
\]
or as
\[
h = (h_1, \beta_1^{-1} h_1, \beta_2^{-1} \beta_1^{-1} h_1, \ldots, \beta_m^{-1} \ldots \beta_1^{-1} h_1) = (e, \beta_1^{-1}, \beta_2^{-1} \beta_1^{-1}, \ldots, \beta_m^{-1} \ldots \beta_1^{-1})(h_1)^{(1)},
\]
and
\[
(\beta_1 \beta_2 \ldots \beta_m)^{h_1} = \alpha^{x^{j-1}}.
\]
Since
\[
\beta_1 \beta_2 \ldots \beta_m = \beta^q x^{j-1},
\]
we repeat the above procedure replacing \( \beta \) by \( \beta^q \) and replacing \( h_1 \) by \( (h_1')^{x^{j-1}} \). This leads to the conjugation equation
\[
(\beta^q)^{h_1'} = \alpha.
\]
In this manner, we determine an automorphism \( h \) of the tree which effects the required conjugation
\[
\beta^h = \alpha.
\]

**Example 4.** Let \( \beta = (e, \beta^q) \sigma \), where \( q = 1 + x \). Then \( \beta \) is conjugate to the adding machine \( \alpha = (e, \alpha) \sigma \). Note that from Example 1, \( \beta \) is not obtainable from \( \alpha \) by simply choosing a different transversal. To exhibit the conjugator \( h: \beta \to \alpha \) constructed in the proof, define the polynomial sequences
\[
c_0 = 1, \quad c_1 = q, \quad c_n = 2c_{n-2} + c_{n-1};
\]
\[
c'_{n-1} = 0, \quad c'_0 = 0, \quad c'_n = c_{n-1} + c'_{n-1}.
\]
Then
\[
h = (e, e^{(0)})(e, \beta^{-1})^{(1)}(e, \beta^{-(1+q)})^{(2)} \ldots (e, \beta^{-c'_n})^{(n)} \ldots.
\]
Step 4. By Proposition 2, we have $A \leq \tilde{\mathcal{A}} = C_A(\alpha)$ and

$$A^h \leq C_A(\alpha^h) = C_A(\beta) = D_m(j).$$

This finishes the proof of the theorem.

References


Received July 20, 2009; revised October 2, 2009

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