Regular elements in CAT(0) groups

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Abstract. Let $X$ be a locally compact geodesically complete CAT(0) space and $\Gamma$ be a discrete group acting properly and cocompactly on $X$. We show that $\Gamma$ contains an element acting as a hyperbolic isometry on each indecomposable de Rham factor of $X$. It follows that if $X$ is a product of $d$ factors, then $\Gamma$ contains $\mathbb{Z}^d$.

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Let $X$ be a proper CAT(0) space and $\Gamma$ be a discrete group acting properly and cocompactly by isometries on $X$. The flat closing conjecture predicts that if $X$ contains a $d$-dimensional flat, then $\Gamma$ contains a copy of $\mathbb{Z}^d$ (see [Gro93], Section 6.B.3). In the special case $d = 2$, this would imply that $\Gamma$ is hyperbolic if and only if it does not contain a copy of $\mathbb{Z}^2$. This notorious conjecture remains however open as of today. It holds when $X$ is a real analytic manifold of non-positive sectional curvature by the main result of [BS91]. In the classical case when $X$ is a non-positively curved symmetric space, it can be established with the following simpler and well known argument: by [BL93], Appendix, the group $\Gamma$ must contain a so called $\mathbb{R}$-regular semisimple element, i.e., a hyperbolic isometry $\gamma$ whose axes are contained in a unique maximal flat of $X$. By a lemma of Selberg [Sel60], the centraliser $Z_\Gamma(\gamma)$ is a lattice in the centraliser $\mathbb{Z}_{\text{Isom}(X)}(\gamma)$. Since the latter centraliser is virtually $\mathbb{R}^d$ with $d = \text{rank}(X)$, one concludes that $\Gamma$ contains $\mathbb{Z}^d$, as desired.

It is tempting to try and mimic that strategy of proof in the case of a general CAT(0) space $X$: if one shows that $\Gamma$ contains a hyperbolic isometry $\gamma$ which is maximally regular in the sense that its axes are contained in a unique flat of maximal possible dimension among all flats of $X$, then the flat closing conjecture will follow as above. The main result of this note provides hyperbolic isometries satisfying a weaker notion of regularity.

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Theorem. Assume that \( X \) is geodesically complete.

Then \( \Gamma \) contains a hyperbolic element which acts as a hyperbolic isometry on each indecomposable de Rham factor of \( X \).

Every CAT(0) space \( X \) as in the theorem admits a canonical de Rham decomposition, see [CM09a], Corollary 5.3 (ii). Notice that the number of indecomposable de Rham factors of \( X \) is a lower bound on the dimension of all maximal flats in \( X \), although two such maximal flats need not have the same dimension in general. As expected, we deduce a corresponding lower bound on the maximal rank of free abelian subgroups of \( \Gamma \).

Corollary 1. If \( X \) is a product of \( d \) factors, then \( \Gamma \) contains a copy of \( \mathbb{Z}^d \).

We believe that those results should hold without the assumption of geodesic completeness; in case \( X \) is a CAT(0) cube complex, this is indeed so, see [CS11], § 1.3.

The proof of the theorem and its corollary relies in an essential way on results from [CM09a] and [CM09b]. The first step consists in applying [CM09a], Theorem 1.1, which ensures that \( X \) splits as

\[
X \cong \mathbb{R}^d \times M \times Y_1 \times \cdots \times Y_q,
\]

where \( M \) is a symmetric space of non-compact type and the factors \( Y_i \) are geodesically complete indecomposable CAT(0) spaces whose full isometry group is totally disconnected. Moreover this decomposition is canonical, hence preserved by a finite index subgroup of \( \text{Isom}(X) \) (and thus of \( \Gamma \)). The next essential point is that, by [CM09b], Theorem 3.8, the group \( \Gamma \) virtually splits as \( \mathbb{Z}^d \times \Gamma' \), and the factor \( \Gamma' \) (resp. \( \mathbb{Z}^d \)) acts properly and cocompactly on \( M \times Y_1 \times \cdots \times Y_q \) (resp. \( \mathbb{R}^d \)). Therefore, our main theorem is a consequence of the following.

Proposition 2. Let \( X = M \times Y_1 \times \cdots \times Y_q \), where \( M \) is a symmetric space of non-compact type and \( Y_i \) is a geodesically complete locally compact CAT(0) space with totally disconnected isometry group.

Any discrete cocompact group of isometries of \( X \) contains an element acting as an \( \mathbb{R} \)-regular hyperbolic element on \( M \), and as a hyperbolic element on \( Y_i \) for all \( i \).

As before, this yields a lower bound on the rank of maximal free abelian subgroups of \( \Gamma \), from which Corollary 1 follows.

Corollary 3. Let \( X = M \times Y_1 \times \cdots \times Y_q \) be as in the proposition. Then any discrete cocompact group of isometries of \( X \) contains a copy of \( \mathbb{Z}^{\text{rank}(M)+q} \).

Proof. Let \( \Gamma < \text{Isom}(X) \) be a discrete subgroup acting cocompactly. Upon replacing \( \Gamma \) by a subgroup of finite index, we may assume that \( \Gamma \) preserves the given product
decomposition of $X$ (see [CM09a], Corollary 5.3 (ii)). Let $\gamma \in \Gamma$ be as in Proposition 2 and let $\gamma_M$ (resp. $\gamma_i$) be its projection to Isom($M$) (resp. Isom($Y_i$)). Then $\text{Min}(\gamma_M) = \mathbb{R}^{\text{rank}(M)}$ and for all $i$ we have $\text{Min}(\gamma_i) \cong \mathbb{R} \times C_i$ for some CAT(0) space $C_i$, by [BH99], Theorem II.6.8 (5). Hence the desired conclusion follows from the following lemma.

\begin{proof}

By hypothesis, we have $\text{Min}(\gamma_M) \cong \mathbb{R}^{d_1} \times C_1$ for some CAT(0) space $C_1$. Therefore $\text{Min}(\gamma) \cong \mathbb{R}^{d_1+d_2+\cdots+d_p} \times C_1 \times \cdots \times C_p$. By [Rua01], Theorem 3.2, the centraliser $Z_{\gamma}(\gamma)$ acts cocompactly (and of course properly) on $\text{Min}(\gamma)$. Therefore, in view of [CM09b], Theorem 3.8, we infer that $Z_{\gamma}(\gamma)$ contains a subgroup isomorphic to $\mathbb{Z}^{d_1+d_2+\cdots+d_p}$.

It remains to prove Proposition 2. We proceed in three steps. The first one provides an element $\gamma_Y \in \Gamma$ acting as a hyperbolic isometry on each $Y_i$. This combines an argument of E. Swenson [Swe99], Theorem 11, with the phenomenon of Alexandrov angle rigidity, described in [CM09a], Proposition 6.8, and recalled below. The latter requires the hypothesis of geodesic completeness. The second step uses that $\Gamma$ has subgroups acting properly cocompactly on $M$, and thus contains an element $\gamma_M$ acting as an $\mathbb{R}$-regular isometry of $M$ by [BL93]. The last step uses a result from [PR72] ensuring that for all elements $\delta'$ in some Zariski open subset of Isom($M$) and all sufficiently large $n > 0$, the product $\gamma_M^n \delta'$ is $\mathbb{R}$-regular. Invoking the Borel density theorem, we finally find an appropriate element $\delta \in \Gamma$ such that the product $\gamma = \gamma_M^n \delta \gamma_Y$ has the requested properties. We now proceed to the details.

Proposition (Alexandrov angle rigidity). Let $Y$ be a locally compact geodesically complete CAT(0) space and $G$ be a totally disconnected locally compact group acting continuously, properly and cocompactly on $Y$ by isometries.

Then there is $\varepsilon > 0$ such that for any elliptic isometry $g \in G$ and any $x \in X$ not fixed by $g$, we have $\angle_c(gx, x) \geq \varepsilon$, where $c$ denotes the projection of $x$ on the set of $g$-fixed points.

\begin{proof}

See [CM09a], Proposition 6.8.
\end{proof}
Proposition 5. Let $Y = Y_1 \times \cdots \times Y_q$, where $Y_i$ is a geodesically complete locally compact CAT(0) space with totally disconnected isometry group, and $G$ be a locally compact group acting continuously, properly and cocompactly by isometries on $Y$.

Then $G$ contains an element acting on $Y_i$ as a hyperbolic isometry for all $i$.

Proof. Upon replacing $G$ by a finite index subgroup, we may assume that $G$ preserves the given product decomposition of $Y$, see [CM09a], Corollary 5.3 (ii). Let $\rho: [0, \infty) \to Y$ be a geodesic ray which is regular, in the sense that its projection to each $Y_i$ is a ray (in other words the end point $\rho(\infty)$ does not belong to the boundary of a subproduct).

Since $G$ is cocompact, we can find a sequence $(g_n)$ in $G$ and a strictly increasing sequence $(t_n)$ in $\mathbb{Z}_+$ such that the sequence of maps

$$\rho_n: [-t_n, \infty) \to Y, \quad t \mapsto g_n \cdot \rho(t + t_n),$$

converges uniformly on compact subsets of $\mathbb{R}$ to a geodesic line $\ell: \mathbb{R} \to Y$. Set $h_{i,j} = g_i^{-1}g_j \in G$ and consider the angle

$$\theta = \angle_{\rho(t_i)}(h_{i,j}^{-1} \cdot \rho(t_i), h_{i,j} \cdot \rho(t_i)).$$

As in [Swe99], Theorem 11, observe that $\theta$ is arbitrarily close to $\pi$ for $i < j$ large enough.

We shall prove that for all $i < j$ large enough, the isometry $h_{i,j}$ is regular hyperbolic, in the sense that its projection to each factor $Y_k$ is hyperbolic. We argue by contradiction and assume that this is not the case. Notice that $\text{Isom}(Y_k)$ does not contain any parabolic isometry by [CM09a], Corollary 6.3 (iii). Therefore, upon extracting and reordering the factors, we may then assume that there is some $s \leq q$ such that for all $i < j$, the projection of $h_{i,j}$ on $\text{Isom}(Y_1), \ldots, \text{Isom}(Y_s)$ is elliptic, and the projection of $h_{i,j}$ on $\text{Isom}(Y_{s+1}), \ldots, \text{Isom}(Y_q)$ is hyperbolic. We set $Y' = Y_1 \times \cdots \times Y_s$ and $Y'' = Y_{s+1} \times \cdots \times Y_q$. We shall prove that for $i < j$ large enough, the projections of $(h_{i,j})$ on $\text{Isom}(Y')$ forms a sequence of elliptic isometries which contradict Alexandrov angle rigidity.

Fix some small $\delta > 0$. Let $x_i$ (resp. $y_i$) be the point at distance $\delta$ from $\rho(t_i)$ and lying on the geodesic segment $[h_{i,j}^{-1} \cdot \rho(t_i), \rho(t_i)]$ (resp. $[\rho(t_i), h_{i,j} \cdot \rho(t_i)]$). By construction, for $i < j$ large enough, the union of the two geodesic segments $[x_i, \rho(t_i)] \cup [\rho(t_i), y_i]$ lies in an arbitrary small tubular neighbourhood of the geodesic ray $\rho$. Since the projection $Y \to Y'$ is $1$-Lipschitz, it follows that the $Y'$-component of $[x_i, \rho(t_i)] \cup [\rho(t_i), y_i]$, which we denote by $[x_i', \rho'(t_i)] \cup [\rho'(t_i), y_i']$, is uniformly close to the $Y'$-component of $\rho$, say $\rho'$. Since $\rho$ is a regular ray, its projection $\rho'$ is also a geodesic ray. Therefore, the angle

$$\theta' = \angle_{\rho'(t_i)}(x_i', y_i')$$

is arbitrarily close to $\pi$ for $i < j$ large enough. Pick $i < j$ so large that $\theta' > \pi - \epsilon$, where $\epsilon > 0$ is the constant from Alexandrov angle rigidity for $Y'$. Set $h = h_{i,j}$ and
let $h'$ be the projection of $h$ on $\text{Isom}(Y_i)$. By assumption $h'$ is elliptic. Let $c$ denote the projection of $\rho'(t_i)$ on the set of $h'$-fixed points. Then the isosceles triangles $\Delta(c, (h')^{-1} \cdot \rho'(t_i), \rho'(t_i))$ and $\Delta(c, \rho'(t_i), h' \cdot \rho'(t_i))$ are congruent, and we deduce

$$
\angle_c (\rho'(t_i), h' \cdot \rho'(t_i)) \leq \pi - \angle_{\rho'(t_i)}(c, h' \cdot \rho'(t_i)) \leq \pi - \angle_{\rho'(t_i)}((h')^{-1} \cdot \rho'(t_i), h' \cdot \rho'(t_i)) = \pi - \theta' < \varepsilon.
$$

This contradicts Alexandrov angle rigidity.

\begin{proof}[Proof of Proposition 2] Let $\Gamma$ be a discrete group acting properly and cocompactly on $X$. First observe that (after passing to a finite index subgroup) we may assume that $\Gamma$ preserves the given product decomposition of $X$, see [CM09a], Corollary 5.3 (ii).

Let $G$ be the closure of the projection of $\Gamma$ to $\text{Isom}(Y_1) \times \cdots \times \text{Isom}(Y_q)$. Then $G$ acts properly cocompactly on $Y = Y_1 \times \cdots \times Y_q$. Therefore it contains an element $g$ acting as a hyperbolic isometry on $Y_i$ for all $i$ by Proposition 5. Since $\Gamma$ maps densely to $G$ and since the stabiliser of each point of $Y$ in $G$ is open by [CM09a], Theorem 1.2, it follows that $\Gamma$-orbits on $Y \times Y$ coincide with the $G$-orbits. In particular, given $y \in \text{Min}(g)$, we can find $\gamma_Y \in \Gamma$ such that $\gamma_Y(y, g^{-1}y) = (gy, y)$. Since $\angle_y(g^{-1}y, gy) = \pi$, we infer that $\gamma_Y$ is hyperbolic and has an axis containing the segment $[g^{-1}y, gy]$. In particular $\gamma_Y$ acts as a hyperbolic isometry on $Y_i$ for all $i$.

Let $\gamma_Y = (\alpha, h)$ be the decomposition of $\gamma_Y$ along the splitting $\text{Isom}(X) = \text{Isom}(M) \times \text{Isom}(Y)$. By construction $h$ acts as a hyperbolic isometry on $Y_i$ for all $i$.

Let $U \leq \text{Isom}(Y)$ be the pointwise stabiliser of a ball containing $y$, $\gamma_Y y$ and $\gamma_Y^{-1} y$. Notice that every element of $\text{Isom}(Y)$ contained in the coset $Ug$ maps $y$ to $hy$ and $h^{-1}y$ to $y$, and therefore acts also as a hyperbolic isometry on $Y_i$ for all $i$.

On the other hand $U$ is a compact open subgroup of $\text{Isom}(Y)$ by [CM09a], Theorem 1.2. Set $\Gamma_U = \Gamma \cap (\text{Isom}(M) \times U)$. Notice that $\Gamma_U$ acts properly and cocompactly on $M$ by [CM09b], Lemma 3.2. In other words the projection of $\Gamma_U$ to $\text{Isom}(M)$ is a cocompact lattice. Abusing notation slightly, we shall denote this projection equally by $\Gamma_U$.

By the appendix from [BL93] (see also [Pra94] for an alternative argument), the group $\Gamma_U$ contains an element $\gamma_M$ acting as an $\mathbb{R}$-regular element on $M$. By [PR72], Lemma 3.5, there is a Zariski open set $V = V(\gamma_M)$ in $\text{Isom}(M)$ with the following property. For any $\delta \in V$ there exists $n_\delta$ such that an element $\gamma_M^n \delta$ is $\mathbb{R}$-regular for any $n \geq n_\delta$. By the Borel density theorem, the intersection $\Gamma_U \cap V \alpha^{-1}$ is nonempty. Pick an element $\delta \in \Gamma_U \cap V \alpha^{-1}$. Then $\delta \alpha \in V$ which means by definition that $\gamma_M^n \delta \alpha$ is $\mathbb{R}$-regular for all $n \geq n_0$ for some integer $n_0$.

Pick an element $\gamma'_M \in \Gamma$ (resp. $\delta' \in \Gamma$) which lifts $\gamma_M$ (resp. $\delta$). Set

$$
\gamma = (\gamma'_M)^{n_0} \delta' \gamma_Y \in \Gamma_U.
$$

\end{proof}
The projection of $\gamma$ to $\text{Isom}(M)$ is $\gamma^0_M \delta \alpha$ and is thus $\mathbb{R}$-regular. The projection of $\gamma$ to $\text{Isom}(Y)$ belongs to the coset $Uh$, and therefore acts as a hyperbolic isometry on $Y_i$ for all $i$.

References


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