Hardy-type inequalities

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1. Introduction

The well-known Hardy–Sobolev inequality states that for any given domain \( \Omega \subset \mathbb{R}^n \), \( n \geq 3 \) and any \( u \in C^\infty_c(\Omega) \),

\[
K^2 \int_\Omega \frac{u^2}{|x|^2} \leq \int_\Omega |\nabla u|^2,
\]

where \( K = (n - 2)/2 \). Though the constant \( K^2 \) is optimal, in the sense that

\[
K^2 = \inf_{u \in C^\infty_c(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2/|x|^2},
\]

equality in (1) is never achieved (by any \( u \in H^1_0(\Omega) \)). This fact has led to the improvement of the inequality in various ways: Brezis and Vázquez [BV] first showed that if \( \Omega \) is bounded then for some \( \gamma > 0 \),

\[
\gamma \left( \int_\Omega |u|^p \right)^{2/p} + K^2 \int_\Omega \frac{u^2}{|x|^2} \leq \int_\Omega |\nabla u|^2,
\]

with \( 1 \leq p < 2n/(n - 2) \). Vázquez and Zuazua [VZ] were then able to replace the \( L^p \) norm on the left hand side of (2) by a \( W^{1,q} \) norm for \( q < 2 \). Various improvements (involving e.g. weighted \( L^p \) or \( W^{1,p} \) norms) were also obtained and we refer the interested reader to [Da], [ACR], [FT], [BFT] and the references therein.

One of the consequences of inequality (2) is that the operator \( L_0 := -\Delta - \mu/|x|^2 \) has a positive first eigenvalue, in the sense that

\[
\inf_{\|u\|_{2^*(\Omega)} = 1} \int_\Omega \left( |\nabla u|^2 - \mu \frac{u^2}{|x|^2} \right) > 0,
\]

whenever \( \mu \leq K^2 \).

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J. Dávila: Departamento de Ingenieria Matematica, CMM (UMR CNRS), Universidad de Chile, Casilla 170/3 Correo 3, Santiago, Chile; e-mail: jdavila@dim.uchile.cl

L. Dupaigne: Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, Boîte courrier 187, 75252 Paris Cedex 05, France; e-mail: dupaigne@ann.jussieu.fr
In the first part of this work, given a compact smooth boundaryless manifold \( \Sigma \subset \Omega \) of codimension \( k \neq 2 \), we look at operators of the form

\[
L = -\Delta - \frac{\mu}{d(x)^2},
\]

where \( d(x) = \text{dist}(x, \Sigma) \) and \( \mu \in \mathbb{R} \), and wonder whether an inequality similar to (2) holds.

The first results in this direction are due to Marcus, Mizel and Pinchover [MMP] and Matskewich and Sobolevskii [MSo]. They showed that if \( \Omega \) is a convex domain and \( \Sigma = \partial \Omega \) then

\[
\frac{1}{4} \int_{\Omega} u^2 \frac{d(x)^2}{d} \leq \int_{\Omega} |\nabla u|^2.
\]

The same authors showed that (3) did not hold in a general domain \( \Omega \) and provided examples of smooth domains \( \Omega \) such that

\[
\inf_{u \neq 0} \frac{\int_{\Omega} |\nabla u|^2}{\int_{\Omega} u^2 d} < \frac{1}{4}.
\]

Alternatively, Brezis and Marcus showed in [BM] that the following inequality remains true on a general (smooth bounded) domain \( \Omega \):

\[
\frac{1}{4} \int_{\Omega} \frac{u^2}{d^2} \leq \int_{\Omega} |\nabla u|^2 + C \int_{\Omega} u^2,
\]

where \( C \) is some positive constant.

Finally, among many other results, Barbatis, Filippas and Tertikas [BFT] extended (3) to the case where \( \Sigma \subset \Omega \) is a smooth compact manifold of codimension \( k \), satisfying some geometric condition: they showed that if \( \Delta d^{2-k} \leq 0 \) in \( D'(\Omega \setminus \Sigma) \) then

\[
y \left( \int_{\Omega} |u|^p \right)^{2/p} + H^2 \int_{\Omega} \frac{u^2}{d^2} \leq \int_{\Omega} |\nabla u|^2,
\]

where \( H = (k-2)/2 \) and \( 1 \leq p < 2n/(n-2) \).

Our goal here is to drop the assumption \( \Delta d^{2-k} \leq 0 \). Our results are summarized in the following two theorems:

**Theorem 1.** Let \( \Omega \subset \mathbb{R}^n \) be an open bounded set and \( \Sigma \subset \Omega \) be a compact smooth manifold without boundary of codimension \( k \neq 2 \). Let \( H = (k-2)/2 \). Then there exist \( C > 0 \), \( \gamma > 0 \) independent of \( u \) such that for any \( u \in C^\infty_c(\Omega \setminus \Sigma) \),

\[
\gamma \left( \int_{\Omega} |u|^p \right)^{2/p} + H^2 \int_{\Omega} \frac{u^2}{d^2} \leq \int_{\Omega} |\nabla u|^2 + C \int_{\Omega} u^2,
\]

where \( d(x) = \text{dist}(x, \Sigma) \), \( 1 \leq p < p_k \) and \( p_k \) is given by

\[
\frac{1}{p_k} = \frac{1}{2} - \frac{2}{k(n-k+2)} \quad \text{for } k > 2, \quad \frac{1}{p_1} = \frac{1}{2} - \frac{1}{n+1} \quad \text{if } k = 1.
\]
Theorem 2. Under the assumptions of Theorem 1, there exist $\beta > 0$ and a neighborhood $\Omega_\beta := \{ x \in \Omega : d(x, \Sigma) < \beta \}$ of $\Sigma$ in $\Omega$ such that for any $u \in C_c^\infty (\Omega_\beta \setminus \Sigma)$,

$$H^2 \int_\Omega \frac{u^2}{d^2} \leq \int_\Omega |\nabla u|^2. \tag{5}$$

Remark 1.  

- If $k \geq 3$ it follows by density that (4) and (5) hold for all $u \in C_c^\infty (\Omega)$, respectively $u \in C_c^\infty (\Omega_\beta)$.
- The exponent $p_k$ appearing in Theorem 1 is probably not optimal and we expect that (4) holds for all $1 \leq p < \frac{2n}{n-2}$. In fact Maz'ja [Ma, Corollary 3, Section 2.1.6] proved this result when $\Sigma = \{ x \in \mathbb{R}^n : x_1 = x_2 = \cdots = x_k = 0 \}$.

As a direct consequence of Theorem 1, we see that the first eigenvalue of the operator $L = -\Delta - \mu \frac{d^2}{d}^2$ is finite, i.e.

$$\lambda_1 := \inf_{\|u\|_{L^2(\Omega)} = 1} \int_\Omega \left( |\nabla u|^2 - \mu \frac{u^2}{d^2} \right) > -\infty,$$

whenever $\mu \leq H^2$. We proved in [DD] that in such circumstances there exists an eigenfunction $\varphi_1$ associated to $\lambda_1$, i.e. a solution (in a sense which we shall make precise soon) of

$$\begin{cases}
-\Delta \varphi_1 - \mu \frac{\varphi_1}{d^2} = \lambda_1 \varphi_1 & \text{in } \Omega, \\
\varphi_1 = 0 & \text{on } \partial \Omega.
\end{cases}$$

Normalizing $\varphi_1$ by $\|\varphi_1\|_{L^2(\Omega)} = 1$ and $\varphi_1 > 0$, we then investigate the behavior of $\varphi_1$ near $\Sigma$ and show that in a neighborhood of $\Sigma$, there exist constants $C_1, C_2 > 0$ such that

$$C_1 d(x)^{-\alpha(\mu)} \leq \varphi_1 \leq C_2 d(x)^{-\alpha(\mu)}, \tag{6}$$

where $\alpha(\mu) = H - \sqrt{H^2 - \mu}$.

This result enables us to treat two model applications. First we consider the quantity

$$J_k := \inf_{u \neq 0} \frac{\int_\Omega |\nabla u|^2 - \lambda \int_\Omega u^2}{\int_\Omega u^2/d^2},$$

and extend a result of Brezis and Marcus [BM] stating that $J_k$ is achieved if and only if $J_k < H^2$.

Our second application is a nonexistence result for positive solutions of the equation

$$-\Delta u - \frac{\mu}{d^2} u = u^p + \lambda,$$

completing a study started in [DN]. See Section 4.2 for details.

The last purpose of this article is to extend some results in [DD]. This generalization is necessary to include the case of potentials $a(x) = \mu \mathrm{dist}(x, \Sigma)^{-2}$. More precisely, we shall derive estimates for solutions of the linear equation

$$\begin{cases}
-\Delta u - a(x) u = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases} \tag{7}$$
under the assumptions that \( a \in L^1_{\text{loc}}(\Omega) \), \( a \) is bounded below, i.e.

\[
\operatorname{ess} \inf_{\Omega} a > -\infty,
\]

and

\[
\gamma \left( \int_{\Omega} |u|^r \right)^{2/r} + \int_{\Omega} a(x) u^2 \leq \int_{\Omega} |\nabla u|^2 + M \int_{\Omega} u^2,
\]

(8)

for some \( r > 2 \), \( \gamma > 0 \), \( M > 0 \).

Let us now clarify what we mean by a solution of (7).

We first define the Hilbert space \( H \) as the completion of \( C^\infty_c(\Omega) \) with respect to the norm

\[
\|u\|_H^2 = (u|u)_H := \int_{\Omega} (|\nabla u|^2 - a(x)u^2 + Mu^2),
\]

where \( M \) is the same constant that appears in (8). Observe that the definition of \( H \) does not change if we replace \( M \) by any larger constant.

Given \( f \in H^* \), we then say that \( u \in H \) is a solution of (7) if

\[
(u|v)_H = (f,v)_{H^*} + M(u|v)_{L^2(\Omega)} \quad \forall v \in H.
\]

It is convenient at this point to recall some facts that were proved in [DD]. We start by mentioning that \( H \) embeds compactly in \( L^2(\Omega) \). In particular

\[
L = -\Delta - a(x)
\]

has a first eigenvalue \( \lambda_1 \), which is simple. \( \lambda_1 \) is not necessarily positive (Theorem 1 provides examples of potentials \( a(x) = H^2/d(x)^2 \) for which in general \( \lambda_1 \) can be nonpositive), but when it is, then for \( f \in H^* \) problem (7) has a unique solution \( u \in H \).

We note here that uniqueness fails if one considers other classes of solutions (see an example in [DD]).

The first eigenvalue \( \lambda_1 \) has an associated positive eigenfunction \( \varphi_1 \) (it is not only positive a.e. but it also satisfies \( \varphi_1 \geq c \operatorname{dist}(x, \partial \Omega) \) for some \( c > 0 \)).

Solutions in \( H \) of an equation like (7) are typically unbounded (see examples in [D, DD, DN]). In [DD] we showed that if \( \lambda_1 > 0 \) and \( f \geq 0 \), \( f \not= 0 \) then the solution \( u \in H \) of (7) is bounded below by a positive constant times \( \varphi_1 \). We also proved that if \( \lambda_1 > 0 \) and \( f = 1 \), then the solution \( u \) of (7) satisfies \( u \leq C\varphi_1 \) for some \( C > 0 \).

Our main result is the following:

**Theorem 3.** Let \( 0 < m < r \) and suppose that

\[
p > \frac{2r}{m(r-2)} \quad \text{and} \quad p \geq \frac{r}{r-m}.
\]

Assume that \( f \in H^* \) satisfies \( \|\varphi_1^{-m} f\|_{L^p(\Omega)} < \infty \) and that \( u \in H \) is a solution of (7). Then

\[
|u(x)| \leq C (\|\varphi_1^{-m} f\|_{L^p(\Omega)} + \|u\|_{L^2(\Omega)}) \varphi_1(x), \quad \text{a.e.} \ x \in \Omega.
\]
A simple corollary of this result, obtained by choosing \( m = 2 \), is the following:

**Corollary 1.** Assume \( f \in \mathcal{H}^s \) satisfies \( f/\phi_1 \in L^p(\Omega) \) for some \( p > r/(r - 2) \) and that \( u \in \mathcal{H} \) is a solution of \( (7) \). Then

\[
|u(x)| \leq C(\|f/\phi_1\|_{L^p(\Omega)} + \|u\|_{L^2(\Omega)})\phi_1(x), \quad \text{a.e. } x \in \Omega.
\]

The paper is organized as follows: Section 2 is devoted to the proof of Theorems 1 and 2. In Section 3 we derive (6) whereas Section 4 is dedicated to the aforementioned applications. Finally, we prove Theorem 3 in Section 5.

2. Hardy inequalities

2.1. Proof of Theorem 1

The object of this subsection is to prove Theorem 1. Our arguments are based on improvements of the one-dimensional Hardy inequality inspired by [BM] and a decomposition of \( L^2 \) functions in spherical harmonics taken from [VZ].

We start with a series of three lemmas, which yield a refined version of the classical Hardy inequality in \( \mathbb{R}^k \) (see (20)). The first lemma deals with radial functions:

**Lemma 1.** Let \( k \neq 2 \) and \( H = (k - 2)/2 \). There exists a constant \( C > 0 \) depending only on \( k \) such that

\[
\int_0^{1/2} \left[ \left( \frac{du}{dr} \right)^2 - H^2 \frac{u^2}{r^2} \right] r^{k-1} dr + C \int_0^{1/2} u^2 r^{k-1} dr \\
\geq \int_0^{1/2} \left[ \left( \frac{du}{dr} \right)^2 + H^2 \frac{u^2}{r^2} \right] r^k dr + \frac{1}{4} \int_0^{1/2} \left( \frac{d}{dr} (r^H u(r)) \right)^2 dr
\]

for all \( u \in C_\infty_c(0, 1/2) \).

*Proof.* Let \( u \in C_\infty_c(0, 1/2) \) and \( v(r) = r^H u(r) \). A standard computation yields

\[
\int_0^{1/2} \left[ \left( \frac{du}{dr} \right)^2 - H^2 \frac{u^2}{r^2} \right] r^{k-1} dr = r \left( \frac{dv}{dr} \right)^2 - H \frac{d(v^2)}{dr}.
\]

Integrating, it follows that

\[
A := \int_0^{1/2} \left[ \left( \frac{du}{dr} \right)^2 - H^2 \frac{u^2}{r^2} \right] r^{k-1} dr = \int_0^{1/2} \left( \frac{dv}{dr} \right)^2 dr.
\]

Similarly, by (10) and an integration by parts,

\[
B := \int_0^{1/2} \left[ \left( \frac{du}{dr} \right)^2 + H^2 \frac{u^2}{r^2} \right] r^k dr
\]

\[
= \int_0^{1/2} r^2 \left( \frac{dv}{dr} \right)^2 dr + 2H^2 \int_0^{1/2} u^2 r^k dr - H \int_0^{1/2} r \left( \frac{d(v^2)}{dr} \right) dr
\]

\[
= \int_0^{1/2} r^2 \left( \frac{dv}{dr} \right)^2 dr + (2H^2 + H) \int_0^{1/2} v^2 dr.
\]
Using integration by parts again, it follows that for given \( \varepsilon > 0 \), there exists \( C > 0 \) such that

\[
\int_0^{1/2} v^2 \, dr = -2 \int_0^{1/2} r v \frac{dv}{dr} \, dr \leq C \int_0^{1/2} v^2 r \, dr + \varepsilon \int_0^{1/2} \left( \frac{dv}{dr} \right)^2 r \, dr
\]

\[
= C \int_0^{1/2} u^2 r^{k-1} \, dr + \varepsilon \int_0^{1/2} \left( \frac{dv}{dr} \right)^2 r \, dr. \tag{13}
\]

Collecting (11), (12) and (13), we obtain for \( \varepsilon \) small enough

\[
A - B \geq \int_0^{1/2} r (1 - r - C \varepsilon) \left( \frac{dv}{dr} \right)^2 \, dr - C \int_0^{1/2} u^2 r^{k-1} \, dr
\]

\[
\geq \frac{1}{4} \int_0^{1/2} r \left( \frac{dv}{dr} \right)^2 \, dr - C \int_0^{1/2} u^2 r^{k-1} \, dr. \tag{14}
\]

The next lemma will help us deal with the nonradial part of a given function \( u : \mathbb{R}^k \to \mathbb{R} \).

**Lemma 2.** Let \( k \neq 2 \), \( H = (k - 2)/2 \) and \( c > \tilde{c} > 0 \). There exist constants \( C, \tau > 0 \) depending only on \( k \) and \( \tilde{c} \) such that

\[
\int_0^{1/2} \left( \left( \frac{du}{dr} \right)^2 - (H^2 - c) \frac{u^2}{r^2} \right) r^{k-1} \, dr + C \int_0^{1/2} u^2 r^{k-1} \, dr
\]

\[
\geq \int_0^{1/2} \left( \left( \frac{du}{dr} \right)^2 + (H^2 + c) \frac{u^2}{r^2} \right) r^k \, dr + \tau \int_0^{1/2} \left[ \left( \frac{du}{dr} \right)^2 + c \frac{u^2}{r^2} \right] r^{k-1} \, dr \tag{15}
\]

for all \( u \in C_c^\infty(0, 1/2) \).

**Proof.** It follows from (9) that if

\[
D := \int_0^{1/2} \left[ \left( \frac{du}{dr} \right)^2 - (H^2 - c) \frac{u^2}{r^2} \right] r^{k-1} \, dr + C \int_0^{1/2} u^2 r^{k-1} \, dr \tag{15}
\]

and

\[
E := \int_0^{1/2} \left[ \left( \frac{du}{dr} \right)^2 + (H^2 + c) \frac{u^2}{r^2} \right] r^k \, dr \tag{16}
\]

then

\[
D - E \geq c \int_0^{1/2} \frac{u^2}{r^2} (1 - r) r^{k-1} \, dr + \frac{1}{4} \int_0^{1/2} r \left( \frac{dv}{dr} \right)^2 \, dr
\]

\[
\geq c \frac{1}{2} \int_0^{1/2} \frac{u^2}{r^2} r^{k-1} \, dr + \frac{1}{4} \int_0^{1/2} r \left( \frac{dv}{dr} \right)^2 \, dr. \tag{17}
\]
We can also rewrite (10) as
\[ r^{k-1} \left( \frac{du}{dr} \right)^2 = H^2 \frac{u^2}{r^2} r^{k-1} + r \left( \frac{dv}{dr} \right)^2 - H \frac{d(v^2)}{dr} \]
so that if \( \tau = \min \left( \frac{r}{r^2}, \frac{1}{2} \right) \), then
\[ \tau \int_0^{1/2} r^{k-1} \left( \frac{du}{dr} \right)^2 dr \leq \frac{\tilde{c}}{4} \int_0^{1/2} u^2 r^{k-1} dr + \frac{1}{4} \int_0^{1/2} r \left( \frac{dv}{dr} \right)^2 dr. \]  
(18)

It then follows from (17) and (18) that
\[ D - E \geq \tau \int_0^{1/2} r^{k-1} \left( \frac{du}{dr} \right)^2 dr + \frac{c}{4} \int_0^{1/2} u^2 r^{k-1} dr. \]  
(19)

Hence (14) holds.

Finally, the following lemma yields the improved Hardy inequality (in \( \mathbb{R}^k \)) that we will be using in what follows.

**Lemma 3.** Let \( k \neq 2 \), \( H = (k - 2)/2 \) and \( \beta > 0 \). Let \( B^k_\beta \) denote the ball of \( \mathbb{R}^k \) centered at the origin and of radius \( \beta \). There exist positive constants \( C = C(\beta, k), \tau = \tau(k) \) and \( \alpha = \alpha(\beta, k) \) such that
\[ \int_{B^k_\beta} \left( |\nabla u|^2 - H^2 \frac{u^2}{|y|^2} \right) dy + C \int_{B^k_\beta} u^2 dy \]
\[ \geq \frac{1}{2\beta} \int_{B^k_\beta} |y| \left( |\nabla u|^2 + H^2 \frac{u^2}{|y|^2} \right) dy + \tau \int_{B^k_\beta} |\nabla(u - u_0)|^2 dy + \alpha \int_0^\beta r \left( \frac{dv_0}{dr} \right)^2 dr \]  
(20)

for all \( u \in C^\infty_c(B^k_\beta \setminus \{0\}) \), where \( u_0(r) = u_0(|y|) = f_{B^k_\beta} u d\sigma \) and \( v_0(r) = r^H u_0(r) \).

**Proof.** Let \( \{f_i\}_{i=0}^\infty \) be an orthonormal basis of \( L^2(S^{k-1}) \), composed of eigenvectors of the Laplace–Beltrami operator \( \Delta |_{S^{k-1}} \). The corresponding eigenvalues are given by \( c_i = i(k + i - 2), i = 0, 1, 2, \ldots \) (see e.g. [51]). Any \( u \in C^\infty_c(B^k_{1/2} \setminus \{0\}) \) can then be written as
\[ u(x) = \sum_{i=0}^\infty u_i(r) f_i(\theta) \]
where \( 1/2 > r > 0, \theta \in S^{k-1} \) and \( x = r \theta \).

Furthermore, for \( g \in C(\mathbb{R}^+, \mathbb{R}), \)
\[ \int_{B^k_{1/2}} |\nabla u|^2 g(|y|) dy = \int_0^{1/2} r^{k-1} g(r) dr \int_{S^{k-1}} \left[ \left( \frac{du}{dr} \right)^2 + \frac{1}{r^2} |\nabla_\theta u|^2 \right] d\theta \]
\[ = \sum_{i=0}^\infty \int_0^{1/2} r^{k-1} g(r) \left[ \left( \frac{du_i}{dr} \right)^2 + \frac{c_i}{r^2} u_i^2 \right] dr. \]  
(21)
For $i = 0$, it follows from (20) that if $v_0(r) = r^H u_0(r)$, then
\[ \int_0^{1/2} \left[ \left( \frac{du_0}{dr} \right)^2 - H^2 u_0^2 \frac{2}{r^2} \right] r^{k-1} dr + C \int_0^{1/2} u_0^2 r^{k-1} dr \]
\[ \geq \int_0^{1/2} \left[ \left( \frac{du_0}{dr} \right)^2 + H^2 u_0^2 \frac{2}{r^2} \right] r^{k} dr + \frac{1}{4} \int_0^{1/2} r \left( \frac{dv_0}{dr} \right)^2 dr, \] (22)
while (14) implies that for $i \geq 1$,
\[ \int_0^{1/2} \left[ \left( \frac{du_i}{dr} \right)^2 - (H^2 - c_i) \frac{u_i^2}{r^2} \right] r^{k-1} dr + C \int_0^{1/2} u_i^2 r^{k-1} dr \]
\[ \geq \int_0^{1/2} \left[ \left( \frac{du_i}{dr} \right)^2 + (H^2 + c_i) \frac{u_i^2}{r^2} \right] r^{k} dr + \frac{1}{4} \int_0^{1/2} r \left( \frac{dv_i}{dr} \right)^2 r^{k-1} dr. \] (23)
Using (22), (23) and (21) with $g(r) = 1$ for terms involving $r^k$ and $g(r) = r$ for terms with $r^k$, we deduce (20) for $\beta = 1/2$. The general case is obtained by scaling.

Next, we introduce some geometric notation that will be needed in the proof of Theorem II. Define
\[ \Omega_{\beta} = \{ x \in \Omega \mid \text{dist}(x, \Sigma) < \beta \}. \]
We will work only with $\beta$ small enough so that the projection $\pi : \Omega_{\beta} \to \Sigma$ given by $|\pi(x) - x| = \text{dist}(x, \Sigma)$ is well defined and smooth.

Let $\{V_i\}_{i=1, \ldots, m}$ be a family of open disjoint subsets of $\Sigma$ such that
\[ \Sigma = \bigcup_{i=1}^m V_i, \quad |V_i \cap V_j| = 0 \quad \forall i \neq j. \]
We can also assume that:
(a) $\forall i = 1, \ldots, m$ there exists a smooth diffeomorphism
\[ p_i : B^n_{\alpha} \to U_i, \]
where $U_i \subset \Sigma$ is open and $\overline{V}_i \subset U_i$;
(b) $p_i^{-1}(V_i)$, which is an open set in $\mathbb{R}^{n-k}$, has a Lipschitz boundary; and
(c) there is a smooth choice of unit vectors $N^j_i(\sigma), \ldots, N^k_i(\sigma)$ for $\sigma \in U_i$ which form an orthonormal frame for $\Sigma$ on $U_i \subset \mathbb{R}^n$, i.e. for all $\sigma \in U_i$,
\[ N^j_i(\sigma) \in \mathbb{R}^n, \quad N^j_i(\sigma) \cdot N^k_i(\sigma) = \delta_{jk}, \quad N^j_i(\sigma) \cdot v = 0 \quad \forall v \in T_\sigma \Sigma. \]
Let $W_i = p^{-1}_i(V_i)$. For $z \in W_i$ we will also write (abusing the notation) $N^j_i(z) = N^j_i(p_i(z))$. Let
\[ F_i(y, z) = p_i(z) + \sum_{j=1}^k y_j N^j_i(z), \]
where $y = (y_1, \ldots, y_k) \in B^k_\rho$ and $z \in W_i$, so that $F_i$ is a smooth diffeomorphism between $B^k_\rho \times W_i$ and $T^i_\rho$, where
\[ T^i_\rho = \pi^{-1}(V_i) \cap \Omega_{\beta}. \] (24)
It follows from the condition $|V_i \cap V_j| = 0 \forall i \neq j$ that $|T^i_{\beta} \cap T^j_{\beta}| = 0 \forall i \neq j$, and hence, for any $f \in L^1(\Omega^1)$ we have

\[
\int_{\Omega^1} f = \sum_{i=1}^{m} \int_{T^i_{\beta}} f = \sum_{i=1}^{m} \int_{W_i \times B_{\beta}} f \circ F_i(y, z) J F_i(y, z) \, dy \, dz, \tag{25}
\]

where $J F_i(y, z)$ stands for the Jacobian of $F_i$ at $(y, z)$. We claim that

\[
J F_i(y, z) = G_i(z)(1 + O(|y|)), \tag{26}
\]

where $O(|y|)$ denotes a quantity bounded by $|y|$ (uniformly for $z \in W_i$) and $G_i(z)$ is a smooth function which is bounded away from zero. More precisely

\[
G_i(z) = J p_i(z) = \sqrt{(D p_i(z))^* D p_i(z)}.
\]

To prove (26) it suffices to observe that $J F_i(y, z)$ is smooth and to compute it at $y = 0$:

\[
J F_i(0, z)^2 = \det(D F_i(0, z))^* D F_i(0, z))
\]

\[= \det([D z \, p_i | N_i^1, \ldots, N_i^k]^* [D z \, p_i | N_i^1, \ldots, N_i^k])
\]

\[= \det\left[\begin{array}{cc}
(D z \, p_i)^* D z \, p_i & 0 \\
0 & I
\end{array}\right].
\]

**Proof of Theorem 1.** First, observe that it is sufficient to prove the theorem for $u$ with support near $\Sigma$. Indeed, following an idea of Vázquez and Zuazua [VZ], let $\eta \in C^\infty_c(\mathbb{R}^n)$ be such that $\eta \equiv 1$ in $\Omega^1/2$ and supp$(\eta) \subset \Omega^1$. Let $u \in C^\infty_c(\Omega \setminus \Sigma)$ and write $u = u_1 + u_2$ where $u_1 = \eta u$, $u_2 = (1 - \eta)u$. Suppose that the conclusion of the theorem holds for $u_1$. Then

\[
\int_{\Omega} \left(\nabla u)^2 - H^2 u_1^2 \frac{d^2}{d^2} \right) = \int_{\Omega} \left(\nabla u_1)^2 - H^2 u_1^2 \frac{d^2}{d^2} \right) + \int_{\Omega} \left(\nabla u_2)^2 - H^2 u_2^2 \frac{d^2}{d^2} \right) + 2 \int_{\Omega} \left(\nabla u_1 \cdot \nabla u_2 - H^2 u_1 u_2 \frac{d^2}{d^2} \right), \tag{27}
\]
Since $1/d$ is bounded away from $\Sigma$ we have
\[
\int_\Omega \left( \frac{u_1^2}{d^2} + \frac{u_1 u_2}{d^2} \right) \leq C \int_\Omega u^2.
\]

Also note that
\[
\int_\Omega \nabla u_1 \cdot \nabla u_2 = \int_\Omega \left[ \eta(1 - \eta) |\nabla u|^2 - |\nabla \eta|^2 u^2 + u \nabla \eta \cdot \nabla (1 - 2\eta) \right]
= \int_\Omega \left[ \eta(1 - \eta) |\nabla u|^2 - |\nabla \eta|^2 u^2 - \frac{1}{2} \int_{\Omega \cap \Omega_{\beta/2}} u^2 \nabla \cdot (\nabla \eta (1 - 2\eta)) \right]
\geq -C \int_\Omega u^2.
\]  

(28)

It follows from (27), (28) that
\[
\int_\Omega \left[ |\nabla u|^2 - H^2 \frac{u^2}{d^2} \right] + C \int_\Omega u^2 \geq \gamma \left( \int_\Omega |u_1|^p \right)^{2/p} + \int_\Omega |\nabla u_2|^2,
\]
for some $\gamma > 0$ independent of $u$. Hence the conclusion of the theorem for $u$ follows easily.

Let
\[
I_i = \int_{T_{\beta}} \left[ |\nabla u|^2 - H^2 \frac{u^2}{d^2} + u^2 \right],
\]  

(29)

where $T_{\beta}$ was defined in (24). In what follows we will fix $i$ and show that there are $p > 2$ and $C > 0$ independent of $u$ such that
\[
\left( \int_{T_{\beta}^i} |u|^p \right)^{2/p} \leq CI_i.
\]

For simplicity, and since $i$ is fixed, we will drop the index $i$ from all the notation that follows.

Let us introduce some additional notation:
\[
\tilde{u}(y, z) = u(F(y, z)),
\]
\[
\tilde{u}_0(r, z) = \int_{B_r} \tilde{u}(y, z) \, ds(y),
\]
\[
v_0(r, z) = r^H \tilde{u}_0(r, z).
\]

(30)

Let us write
\[
\nabla u = \nabla_N u + \nabla_T u
\]
where $\nabla_N u$ is the gradient of $u$ in the normal direction and $\nabla_T u$ is orthogonal to $\nabla_N u$. More precisely, for a point $x = F(y, z)$,

$$
\nabla_N u(x) = \sum_{j=1}^k \nabla u(x) \cdot N_j(z) N_j(z).
$$

**Step 1.** There exists $C > 0$ independent of $u$ such that

$$
CI \geq \int_{W \times B^k_\beta} \left[ |\nabla_N u(F(y, z))|^2 - H^2 \frac{\tilde{u}^2}{|y|^2} \right] G(z) \, dy \, dz
+ C \int_{W \times B^k_\beta} \left( |\nabla_N u(F(y, z))|^2 + H^2 \frac{\tilde{u}^2}{|y|^2} \right) G(z) |y| \, dy \, dz
+ \int_{W \times B^k_\beta} (|\nabla_T u(F(y, z))|^2 + \tilde{u}^2)(1 - C|y|) G(z) \, dy \, dz.
$$

(33)

First note that by (25), there is a constant $C > 0$ such that

$$
I \geq \int_{W \times B^k_\beta} \left( |\nabla_N u(F(y, z))|^2 - H^2 \frac{\tilde{u}^2}{|y|^2} \right) G(z) \, dy \, dz
- C \int_{W \times B^k_\beta} \left( |\nabla_N u(F(y, z))|^2 + H^2 \frac{\tilde{u}^2}{|y|^2} \right) G(z) |y| \, dy \, dz
+ \int_{W \times B^k_\beta} (|\nabla_T u(F(y, z))|^2 + \tilde{u}^2)(1 - C|y|) G(z) \, dy \, dz.
$$

(34)

For fixed $z$ we can apply Lemma 3 to the function $\tilde{u}(\cdot, z)$. Observe that

$$
\frac{\partial \tilde{u}(y, z)}{\partial y_j} = \nabla u(F(y, z)) \cdot N_j(z)
$$

and thus

$$
|\nabla_N \tilde{u}(y, z)|^2 = |\nabla_N u(F(y, z))|^2.
$$

Lemma 3 then yields

$$
\int_{B^k_\beta} \left( |\nabla_N u(F(y, z))|^2 - H^2 \frac{\tilde{u}^2}{|y|^2} \right) dy + C \int_{B^k_\beta} \tilde{u}^2 \, dy
\geq \frac{1}{2\beta} \int_{B^k_\beta} |y| \left( |\nabla_N u(F(y, z))|^2 + H^2 \frac{\tilde{u}^2}{|y|^2} \right) \, dy
+ \tau \int_{B^k_\beta} |\nabla \tilde{u}(\tilde{u} - \tilde{u}_0)|^2 \, dy + \alpha \int_0^\beta \left( \frac{dv_0}{dr} \right)^2 \, dr.
$$

(35)

We choose (and fix once for all) $\beta > 0$ small enough so that $1/(2\beta) \geq C + 1$. Then multiplying (35) by $G(z)$, integrating over $W$ and combining the result with (34) we conclude that (33) holds.
Step 2.

\[ \|\nabla v_0\|^2_{L^2(W \times B^k_\beta)} \leq CI. \]  

(36)

By (33) the partial derivative \( \partial v_0/\partial r \) is bounded in \( L^2(W \times B^2_\beta) \) by \( CI \). We just have to control the derivatives \( \partial v_0/\partial z_i, i = 1, \ldots, n-k \). But

\[ \frac{\partial v_0}{\partial z_i}(r, z) = r^H \int_{\partial B_r} \frac{\partial \tilde{u}}{\partial z_i}(y, z) \, ds(y) \]

and

\[ \frac{\partial \tilde{u}}{\partial z_i}(y, z) = \nabla u(F(y, z)) \cdot \left[ \frac{\partial p}{\partial z_i} + \sum_{j=1}^{k} y_j \frac{\partial N_j}{\partial z_i} \right]. \]

But note that \( \partial p/\partial z_j \) is a tangent vector, hence

\[ |\nabla z \tilde{u}(y, z)| \leq C |\nabla_T u(F(y, z))| + C |y| |\nabla N u(F(y, z))|. \]

Integrating over \( W \times B^k_\beta \) we have

\[ \int_{W \times B^k_\beta} |\nabla z \tilde{u}(y, z)|^2 \, dy \, dz \leq CI, \]

(37)

for some \( C \) independent of \( u \) by (33). It follows that

\[ \int_{W \times B^k_\beta} |\nabla z \tilde{u}(y, z)|^2 \, dy \, dz = \int_{W} \int_{0}^{\beta} r^{2H+1} \int_{\partial B_r} \nabla z \tilde{u}(y, z) \, ds(y) \, dr \, dz \]

\[ \leq \int_{W} \int_{0}^{\beta} r^{k-1} \int_{\partial B_r} |\nabla z \tilde{u}(y, z)|^2 \, ds(y) \, dr \, dz \]

\[ \leq C \int_{W \times B^k_\beta} |\nabla z \tilde{u}(y, z)|^2 \, dy \, dz \leq CI \]

(38)

by (37).

Step 3. There is \( p > 2 \) such that

\[ \|\tilde{u}_0\|^2_{L^p(W \times B^k_\beta)} \leq CI. \]

(39)

More precisely, for \( k \geq 3 \) one can take any \( 2 < p < p_k \) where \( p_k \) is given by

\[ \frac{1}{p_k} = 1 - \frac{2}{k(n-k+2)}. \]

and for \( k = 1 \) one can take \( 2 < p \leq p_1 \) where \( p_1 \) is given by

\[ \frac{1}{p_1} = 1 - \frac{1}{n+1}. \]
Using Sobolev’s inequality (on $W \times B^2$) combined with (36) we obtain
\[ \int_W \int_0^\beta |v_0|^q r \, dr \, dz \leq C I^{q/2}. \]

with $q$ given by $1/q = 1/2 - 1/(n - k + 2)$. That is, in terms of $\tilde{u}_0$ we have
\[ \int_W \int_0^\beta |\tilde{u}_0|^q r^{qH+1} \, dr \, dz \leq C I^{q/2}. \] (40)

We want an estimate for $\int |\tilde{u}_0|^p r^{k-1} \, dr \, dz$ for some suitable $2 < p < q$ and for this we use Hölder’s inequality, distinguishing two cases:

**Case $k \geq 3$.** We have
\[
\int_W \int_0^\beta |\tilde{u}_0|^p r^{k-1} \, dr \, dz = \int_W \int_0^\beta |\tilde{u}_0|^p r^{q+1} r^{\alpha} \, dr \, dz \\
\leq C \left( \int_W \int_0^\beta |\tilde{u}_0|^q r^{q+1} \, dr \, dz \right)^{p/q} \left( \int_0^\beta r^{(1-\frac{\alpha}{p})+1} \, dr \right)^{1-p/q}. \] (41)

We then choose $\alpha$ so that
\[ \frac{\alpha}{p} = H = \frac{k-2}{2}. \]

In order to have the second factor on the right hand side of (41) finite we need to impose
\[ \frac{k-2-\alpha}{1-p/q} > -2, \]

which is equivalent to the condition
\[ \alpha < \frac{k}{1 + \frac{4}{q(k-2)}}. \]

Thus we need $p = \alpha/H < p_k$, where $p_k$ is given by
\[ p_k = \frac{2k}{(k-2)(1 + \frac{4}{q(k-2)})}, \]
i.e.
\[ \frac{1}{p_k} = 2 - \frac{2}{k(n-k+2)}. \]

Observe that $p_k > 2$. Combining then (40) and (41) finishes this case.
Case \( k = 1 \). In this case \( q \) is given by \( 1/q = 1/2 - 1/n + 1 \), and we can choose \( p = q \):

\[
\int_W \int_0^\beta |\tilde{u}_0|^q r^{k-1} \, dr \, dz = \int_W \int_0^\beta |\tilde{u}_0|^q \, dr \, dz \\
\leq \int_W \int_0^\beta |\tilde{u}_0|^{q-r^{-q/2+1}} \, dr \, dz \\
= \int_W \int_0^\beta |\tilde{u}_0|^{q} \, r^{Hq+1} \, dr \, dz
\]

because \(-q/2 + 1 < 0\).

Step 4.

\[\|\tilde{u} - \tilde{u}_0\|_{L^2(W \times B^k_\beta)}^2 \leq CI. \tag{42}\]

This is a consequence of Sobolev’s inequality applied to the function \( \tilde{u} - \tilde{u}_0 \) on the domain \( W \times B^k_\beta \). (33) already provides a bound in \( L^2(W \times B^k_\beta) \) for \( \nabla_y (\tilde{u} - \tilde{u}_0) \). Hence we only need to obtain a bound for the derivative of \( \tilde{u} - \tilde{u}_0 \) with respect to \( z \). In the case of the function \( \tilde{u} \) we have it already in (37). For \( \tilde{u}_0 \) it is derived by a computation very similar to that at the end of Step 2. Indeed,

\[
\int_{W \times B^k_\beta} |\nabla_z \tilde{u}_0|^2 \, dy \, dz = \int_W \int_0^\beta r^{k-1} \left( \left| \int_{\partial B^r} \nabla_z \tilde{u}(y, z) \, ds(y) \right|^2 \right) \, dr \, dz \\
\leq CI,
\]

which we obtain as in (38).

Conclusion. By (39) and (42) we see that

\[\|\tilde{u}\|_{L^2(W \times B^k_\beta)}^2 \leq CI\]

for some \( C \) independent of \( u \). Changing variables and reintroducing the index \( i \) we have

\[\|u\|_{L^2(T^i_\beta)}^2 \leq C \int_{T^i_\beta} \left( |\nabla u|^2 - H^2 \frac{u^2}{r^2} + u^2 \right). \]

Adding these inequalities over \( i \) proves the statement of the theorem. \( \square \)

2.2. A local version of the Hardy inequality

In this section, we show how to adapt the proof of Theorem 1 to obtain Theorem 2. We first derive variants of Lemmas 1, 2, 3.

Lemma 4. Let \( k \neq 2 \) and \( H = (k-2)/2 \). There exist constants \( C, \beta_0 > 0 \) such that for \( 0 < \beta \leq \beta_0 \),

\[
\int_0^\beta \left( \left( \frac{du}{dr} \right)^2 - H^2 \frac{u^2}{r^2} \right) r^{k-1} \, dr \geq \int_0^\beta \left( \left( \frac{du}{dr} \right)^2 + H^2 \frac{u^2}{r^2} \right) r^{k} \, dr \tag{43}\]

for all \( u \in C^\infty_c(0, \beta) \).
Proof. Given \( v \in C^\infty_c(0, 1/2) \), we have
\[
\int_0^{1/2} v^2 \, dr = -2 \int_0^{1/2} r v \frac{dv}{dr} \, dr \leq C \int_0^{1/2} r^2 \left( \frac{dv}{dr} \right)^2 \, dr + \frac{1}{2} \int_0^{1/2} v^2 \, dr.
\]
Using this and (12) we obtain
\[
\int_0^{1/2} \left[ \left( \frac{du}{dr} \right)^2 + H^2 \frac{u^2}{r^2} \right] r^k \, dr \leq C \int_0^{1/2} r^2 \left( \frac{dv}{dr} \right)^2 \, dr.
\]
Changing variables, it then follows that for \( u \in C^\infty_c(0, \beta) \),
\[
\int_0^\beta \left[ \left( \frac{du}{dr} \right)^2 + H^2 \frac{u^2}{r^2} \right] r^k \, dr \leq C \beta \int_0^\beta \left( \frac{dv}{dr} \right)^2 \, dr.
\]
while (11) becomes
\[
\int_0^\beta \left[ \left( \frac{du}{dr} \right)^2 - H^2 \frac{u^2}{r^2} \right] r^{k-1} \, dr = \beta^{-k-2} \int_0^\beta r \left( \frac{dv}{dr} \right)^2 \, dr.
\]
If we pick \( \beta \) small, (43) follows from (44) and (45).
□

A straightforward corollary of the above lemma is:

**Lemma 5.** Let \( k \neq 2, H = (k - 2)/2 \) and \( c > 0 \). There exist constants \( C, \beta_0 > 0 \) such that for \( 0 < \beta \leq \beta_0 \),
\[
\int_0^\beta \left[ \left( \frac{du}{dr} \right)^2 - (H^2 - c) \frac{u^2}{r^2} \right] r^{k-1} \, dr \geq \int_0^\beta \left[ \left( \frac{du}{dr} \right)^2 + (H^2 + c) \frac{u^2}{r^2} \right] r^k \, dr
\]
for all \( u \in C^\infty_c(0, \beta) \).

Combining these two lemmas, we then obtain:

**Lemma 6.** Let \( k \neq 2, H = (k - 2)/2 \) and \( \beta > 0 \). Let \( B_\beta^k \) denote the ball of \( \mathbb{R}^k \) centered at the origin and of radius \( \beta \). There exist positive constants \( C, \beta_0 \) such that for \( \beta \leq \beta_0 \),
\[
\int_{B_\beta^k} \left| \nabla u \right|^2 - H^2 \frac{u^2}{|y|^2} \, dy \geq C \beta \int_{B_\beta^k} \left| \nabla u \right|^2 + H^2 \frac{u^2}{|y|^2} \, dy
\]
for all \( u \in C^\infty_c(B_\beta^k \setminus \{0\}) \), where \( u_0(r) = u_0(|y|) = \int_{S_{\beta}^k} u \, d\sigma \) and \( v_0(r) = r^H u_0(r) \).

As in Lemma 3 for a fixed value \( \beta = \beta_0 > 0 \) the proof is an application of the decomposition of a function in spherical harmonics. A simple scaling then yields the \( \beta \)-dependence of the constant appearing in (47).

**Proof of Theorem 2.** Instead of (29), we now consider
\[
J_i := \int_{T_i^k} \left| \nabla u \right|^2 - H^2 \frac{u^2}{d^2}.
\]
Using the notation of (30) we then have, by (25) and (26),
\[ J_i \geq \int_{W \times B_k} \left( |\nabla u(F_i(y, z))|^2 - H^2 \frac{u^2}{|y|^2} \right) G(z) \, dy \, dz \]
\[ - C \int_{W \times B_k} |y| \left( |\nabla u(F_i(y, z))|^2 + H^2 \frac{u^2}{|y|^2} \right) G(z) \, dy \, dz \geq 0, \]
where we used Lemma 6 with \( \beta > 0 \) small in the last inequality. Adding the above estimates over \( i \) yields the desired result. \( \square \)

3. Remarks on the potential \( a(x) = \mu \, \text{dist}(x, \Sigma)^{-2} \)

For \( 0 < \mu \leq H^2 \) we consider the potential
\[ a(x) = \mu / d(x)^2 \]
and let \( L \) denote the operator
\[ Lu = -\Delta u - a(x)u. \]

Note that \( a(x) \) and \( L \) depend on \( \mu \) but we will omit this dependence from the notation.

Recall that we defined the Hilbert space \( \mathcal{H} \) as the completion of \( C_c^\infty(\Omega) \) with respect to the norm
\[ \|u\|_{\mathcal{H}}^2 = \int_\Omega (|\nabla u|^2 - a(x)u^2 + Mu^2), \quad (49) \]
where \( M \) is the constant that appears in (8). If \( \mu < H^2 \) then by Theorem 1 \( \mathcal{H} \) coincides with \( H^1_0(\Omega) \).

The main concern in this section is to obtain a precise description of the behavior near \( \Sigma \) of the first eigenfunction \( \varphi_1 \) of the operator \( L \). Indeed, we shall prove:

Lemma 7. There are positive constants \( C_1, C_2 \) such that
\[ C_1 d(x)^{-\alpha(\mu)} \leq \varphi_1(x) \leq C_2 d(x)^{-\alpha(\mu)} \quad (50) \]
for \( x \) in a neighborhood of \( \Sigma \), where \( \alpha(\mu) \) is given by
\[ \alpha(\mu) = H - \sqrt{H^2 - \mu}. \quad (51) \]

Note that when \( \mu = H^2 \) we have \( -\alpha(\mu) = 1 - k/2 \). Thus \( \varphi_1 \notin H^1_0(\Omega) \) in this case. Before proving the above lemma it will be necessary to show that if \( \mu = H^2 \) then \( d^{1-k/2} \) (appropriately modified so that it is zero on \( \partial \Omega \)) belongs to \( \mathcal{H} \). We prove this and a little more next.
Lemma 8. Let $\mu = H^2$ and define

$$v_\varepsilon(x) = \eta(x)d(x)^{1-k/2}(- \log d(x))^{-s},$$

where $\eta \in C^\infty_c(\Omega)$ is a cut-off function such that $\eta \equiv 1$ in a neighborhood of $\Sigma$ and $\eta(x) = 0$ for $d(x) \geq \text{dist}(\Sigma, \partial \Omega)/2$. Then $v_\varepsilon \in \mathcal{H}$ if and only if $s > -1/2$.

Remark 2. This lemma was stated in [VZ] in the case where $\Sigma$ is a point.

Proof. Let us recall and also introduce some notation:

$$\Omega_r = \{ x \in \mathbb{R}^N | d(x) < r \}, \quad \Sigma_r = \partial \Omega_r = \{ x \in \mathbb{R}^N | d(x) = r \}.$$  

By the Pappus theorems, the $(N-1)$-dimensional area of $\Sigma_r$ is given by

$$|\Sigma_r|_{n-1} = \omega_{k-1} r^{k-1} |\Sigma|_{n-k},$$

where $\omega_{k-1}$ is the area of the unit sphere in $\mathbb{R}^k$ and $| \cdot |_j$ denotes the $j$-dimensional Lebesgue measure.

First we prove that $v_\varepsilon \in \mathcal{H}$ for $s > -1/2$. For this purpose it is enough to exhibit a sequence $f_\varepsilon \in \mathcal{H}$ such that $\|f_\varepsilon\|_{\mathcal{H}} \leq C$ with $C$ independent of $\varepsilon$ and such that $f_\varepsilon \rightarrow v_\varepsilon$ a.e. as $\varepsilon \rightarrow 0$; we take

$$f_\varepsilon = \eta d^{1-k/2+\varepsilon}(- \log d)^{-s}, \quad \varepsilon > 0.$$  

Clearly $f_\varepsilon \in H^1_0(\Omega) \subset \mathcal{H}$, $\int \Omega f_\varepsilon^2 \leq C$ and $f_\varepsilon$ is smooth away from $\Sigma$. Thus to estimate $\|f_\varepsilon\|_{\mathcal{H}}$ it is sufficient to verify that for a fixed $R > 0$ small

$$\int |\nabla f_\varepsilon|^2 - a(x) f_\varepsilon^2 \leq C$$  

with $C$ independent of $\varepsilon$.

Near $\Sigma$, $\eta \equiv 1$ and

$$|\nabla f_\varepsilon|^2 = d^{-k+2\varepsilon} \left( (1 - k/2 + \varepsilon)^2(- \log d)^{-2s} + (2 - k + 2\varepsilon)(- \log d)^{-2s-1} + s( - \log d)^{-2s-2} \right),$$

so that

$$\frac{1}{\omega_{k-1}|\Sigma|_{n-k}} \int_{\Omega_R} |\nabla f_\varepsilon|^2 = (1 - k/2 + \varepsilon)^2 \int_0^R r^{2s-1} (- \log r)^{-2s} dr$$

$$+ s(2 - k + 2\varepsilon) \int_0^R r^{2s-1} (- \log r)^{-2s-1} dr$$

$$+ s^2 \int_0^R r^{2s-1} (- \log r)^{-2s-2} dr.$$
Note that the last integral on the right hand side above is bounded independently of $\varepsilon$ for $s > -1/2$, that is,
\[
\int_0^R r^{2s-1} (-\log r)^{-2s} \, dr = O(1).
\]
Therefore
\[
\frac{1}{\omega_{k-1}|\Sigma|^{n-k}} \int_{\Omega_R} \left( |\nabla f_\varepsilon|^2 - H^2 \frac{f_\varepsilon^2}{d^2} \right) = \varepsilon (2 - k + \varepsilon) \int_0^R r^{2s-1} (-\log r)^{-2s} \, dr + s(2 - k + 2\varepsilon) \int_0^R r^{2s-1} (-\log r)^{-2s-1} \, dr + O(1). \tag{53}
\]
Integrating by parts gives
\[
\int_0^R r^{2s-1} (-\log r)^{-2s} \, dr = \frac{1}{2\varepsilon} R^{2s} (-\log R)^{-2s} - \frac{s}{\varepsilon} \int_0^R r^{2s-1} (-\log r)^{-2s-1} \, dr
\]
and substituting in (53) yields
\[
\frac{1}{\omega_{k-1}|\Sigma|^{n-k}} \int_{\Omega_R} \left( |\nabla f_\varepsilon|^2 - H^2 \frac{f_\varepsilon^2}{d^2} \right) = \frac{2 - k + \varepsilon}{2} R^{2s} (-\log R)^{-2s} + \varepsilon s \int_0^R r^{2s-1} (-\log r)^{-2s-1} \, dr + O(1)
\]
\[
= \varepsilon s \int_0^R r^{2s-1} (-\log r)^{-2s-1} \, dr + O(1). \tag{54}
\]
Integrating by parts again shows that
\[
\int_0^R r^{2s-1} (-\log r)^{-2s-1} \, dr = \frac{1}{2\varepsilon} R^{2s} (-\log R)^{-2s-1} - \frac{s+1}{2\varepsilon} \int_0^R r^{2s-1} (-\log r)^{-2s-2} \, dr = O\left(\frac{1}{\varepsilon}\right).
\]
After substitution in (54) we finally obtain the estimate (52). Hence $v_s \in \mathcal{H}$ for $s > -1/2$.

Our argument to show that $v_s \not\in \mathcal{H}$ for $s \leq -1/2$ relies on the intuitive idea that
\[
\int (-\Delta v - a(x)v + Mv)v = \|v\|^2_{L^1}. \tag{*}
\]
To exploit this idea, let us first compute $\Delta v_s$ near $\Sigma$, where $\eta \equiv 1$. Write
\[
y(t) = t^{1-k/2} (-\log t)^{-s}.
\]
Then near $\Sigma$, since $|\nabla d|^2 = 1$,
\[
\Delta v_s = y''(d) + y'(d) \Delta d.
\]
We recall here the fact (see [DN]) that
\[
\Delta d = \frac{k - 1}{d} + g,
\]
where \( g \in L^\infty \). Hence,
\[
\Delta v_s = -H^2 d^{-k/2-1}(-\log d)^{-s} + s(s + 1)d^{-k/2-1}(-\log d)^{-s-2}
+ (1 - k/2)gd^{-k/2}(-\log d)^{-s} + sgd^{-k/2}(-\log d)^{-s-1}
\]
and so that
\[
\Delta v_s + H^2 \frac{v_s}{d^2} = s(s + 1)d^{-k/2-1}(-\log d)^{-s-2}
+ (1 - k/2)gd^{-k/2}(-\log d)^{-s} + sgd^{-k/2}(-\log d)^{-s-1}.
\] (55)

Observe that \( \Delta v_s, v_s/d^2 \in L^1(\Omega) \) and that equation (55) holds in the sense of distributions. Since we also have \( \nabla v_s \in L^1(\Omega) \), it follows that for any \( \phi \in C_0^\infty(\Omega) \),
\[
(v_s | \phi)_H = \int_{\Omega} \left( -\Delta v_s - H^2 \frac{v_s}{d^2} \right) \phi + M \int_{\Omega} v_s \phi
= -\int_{\Omega} \left( s(s + 1)d^{-k/2-1}(-\log d)^{-s-2}
+ (1 - k/2)gd^{-k/2}(-\log d)^{-s} + sgd^{-k/2}(-\log d)^{-s-1} \right) \phi
+ \int_{\Omega} \left( -\Delta v_s - H^2 \frac{v_s}{d^2} \right) \phi + M \int_{\Omega} v_s \phi.
\] (56)

By density (56) also holds if \( \phi \) is Lipschitz and \( \phi = 0 \) on \( \partial \Omega \).

Let us consider first the case \( s \neq -1 \), so that \( s(s + 1) \neq 0 \), and suppose that \( s < -1/2 \) and \( v_s \in H \). Then there exist \( v_n \in C_c^\infty(\Omega) \) such that \( v_n \to v_s \) in \( H \). Note that since the injection \( H \subset L^2(\Omega) \) is continuous, by passing to a subsequence we also have \( v_n \to v_s \) a.e. Recall from [DN] inequality (1.4) of Lemma 1.1 that for \( u \in H \), we have \( u^+ \in H \) and \( \|u^+\|_H \leq \|u\|_H \). As a consequence \( v_n^+ \to v_s \) in \( H \) and a.e. Using \( v_n^+ \) in (56) we conclude that
\[
\int_{\Omega \cap R} d^{-k/2-1}(-\log d)^{-s-2} v_n^+ \leq C
\]
with \( C \) independent of \( n \). But then Fatou’s lemma implies that
\[
\int_{\Omega \cap R} d^{-k}(-\log d)^{-2s-2} < \infty,
\]
which is impossible for \( s < -1/2 \).

For the case \( s = -1 \) the argument above does not work. We see that in this case, if \( \Sigma \) is flat and \( \eta \equiv 1 \) in an open set then actually
\[
\Delta w + H^2 \frac{w}{d^2} = 0
\]
in that open set, where
\[ w := v_{(−1)} = ηd^{1−k/2}(− \log d). \]
So we argue as follows: let \(-1/2 < s < 0\). We are going to show that \((L + M)v_s \geq (L + M)w\) near \(Σ\). If we assume that \(w \in \mathcal{H}\), then we can apply the maximum principle and deduce that \(v_s \geq εw\) near \(Σ\), which is impossible. Indeed, by formula (55),
\[ (L + M)w = -(1 - k/2)gd^{-k/2}(− \log d) + gd^{-k/2} + Md^{1−k/2}(− \log d), \]
and
\[ (L + M)v_s = -s(s + 1)d^{-k/2−1}(− \log d)^{-s−2} - (1 - k/2)gd^{-k/2}(− \log d)^{-s} - sgd^{-k/2}(− \log d)^{−s−1} + Md^{1−k/2}(− \log d)^{−s}. \]
Thus, there is a neighborhood \(Ω_R\) of \(Σ\) such that for any \(ε \in (0, 1),\)
\[ (L + M)(εw - v_s) ≤ 0 \quad \text{in } Ω_R. \tag{57} \]
Pick \(ε > 0\) such that \(εw - v_s ≤ 0\) in \(∂Ω_R\). Under the hypothesis \(w ∈ \mathcal{H}\) we can use a version of the maximum principle to deduce that
\[ εw - v_s ≤ 0 \quad \text{in } Ω_R. \]
Indeed, assuming \(w ∈ \mathcal{H}\), we have \((εw - v_s)^+ ∈ \mathcal{H}\). Hence the function
\[ z = \begin{cases} (εw - v_s)^+ & \text{in } Ω_R, \\ 0 & \text{in } Ω \setminus Ω_R, \end{cases} \]
also belongs to \(\mathcal{H}\). Let \(z_n ∈ C^∞_c(Ω)\) be such that \(z_n \to z\) in \(\mathcal{H}\). Note that (57) holds in the sense of distributions and hence testing (57) with \(z_n^+\) we see that
\[ (εw - v_s|z_n^+)_{\mathcal{H}} ≤ 0. \]
Letting \(n \to ∞\) we get
\[ ||z||_{\mathcal{H}} = (εw - v_s|z)_{\mathcal{H}} ≤ 0. \]
Thus \(z ≡ 0\), which implies that \(εw ≤ v_s\) in \(Ω_R\), concluding the proof of Lemma 8. \(\square\)

**Remark 3.** To show that \(v_s ∈ \mathcal{H}\) for \(s > -1/2\) one may be tempted to use other approximating sequences, and a very natural one is
\[ f_i = \min(v_s, i), \quad i = 1, 2, \ldots. \]
Again it would be sufficient to establish that for a fixed \(R > 0\) small
\[ \int_{Ω_R} (|∇ f_i|^2 - a(x)f_i^2) ≤ C \]
with \(C\) independent of \(i\). For \(i\) large let \(r_i > 0\) be such that
\[ r_i^{1−k/2}(− \log r_i)^{−s} = i \]
so that \( r_i \to 0 \) as \( i \to \infty \). A computation (that we omit) shows that

\[
\frac{1}{w_{k-1}} \int_{\Omega_R} (|\nabla f_i|^2 - a(x) f_i^2) = \frac{k-2}{4} (-\log r_i)^{-2s} - \frac{k-2}{2} (-\log R)^{-2s} \\
+ \frac{s^2}{2s+1} ((-\log R)^{2s-1} - (-\log r_i)^{2s-1}).
\]

We see that the above quantity remains bounded as \( i \to \infty \) only for \( s \geq 0 \).

Remark 4. The above example shows that for \( m \in \mathbb{N} \) there exists \( v_m \in \mathcal{H} \) with \( \|v_m\|_{\mathcal{H}} = 1 \) and

\[
\|\min(v_m, m)\|_{\mathcal{H}} \to \infty \quad \text{as} \quad m \to \infty.
\]

Since \( v = \min(v, m) + (v - m)^+ \) we also have

\[
\|v_m - m^+\|_{\mathcal{H}} \to \infty \quad \text{as} \quad m \to \infty.
\]

\( \mathcal{H} \) is thus quite different from \( H^1_0(\Omega) \), in the sense that truncation operators like the one above are not uniformly bounded in \( \mathcal{H} \) whereas it is always true that for any \( v \in H^1_0(\Omega) \), \( \min(v, m) \to v \) in the \( H^1 \) topology.

Proof of Lemma [7] We will give a proof using a comparison argument with a suitable function. First let us recall that in a neighborhood of \( \Sigma \),

\[
\Delta d = \frac{k-1}{d} + g,
\]

where \( g \) is a bounded function. Hence

\[
Ld^{-\alpha} = -\Delta d^{-\alpha} - \mu d^{-\alpha} \frac{d^{-\alpha}}{d^2} = -d^{-\alpha-2}(\alpha^2 - \alpha(k-2) + \mu - \alpha gd).
\]

Let \( \alpha = \alpha(\mu) \) as given by (51). This implies that \( \alpha^2 - \alpha(k-2) + \mu = 0 \). Then

\[
L(d^{-\alpha} + C_1 d^{-\alpha+1}) = -d^{-\alpha-1}[-\alpha g + C_1((\alpha - 1)^2 - (\alpha - 1)(k-2) + \mu - (\alpha - 1)gd)].
\]

Instead of working with the operator \( L = -\Delta - a(x) \) consider \( L + M \), where \( M \) is so large that (8) holds (this is the same \( M \) that we use in the definition of the space \( \mathcal{H} \)). Then, since \( (\alpha - 1)^2 - (\alpha - 1)(k-2) + \mu > 0 \) we conclude that for \( C_1 > 0 \) large enough

\[
(L + M)(d^{-\alpha} + C_1 d^{-\alpha+1})
\]

\[
= -d^{-\alpha-1}[-\alpha g + C_1((\alpha - 1)^2 - (\alpha - 1)(k-2) + \mu - (\alpha - 1)gd)] \\
+ M(d^{-\alpha} + C_1 d^{-\alpha+1}) \leq 0
\]

in some fixed neighborhood \( \Omega_R, R > 0, \) of \( \Sigma \). On the other hand, the first eigenfunction \( \psi_1 \) of \( L \) satisfies

\[
(L + M)\psi_1 = (\lambda_1 + M)\psi_1 \geq 0.
\]
Now, both functions $\varphi_1$ and $d^{-\alpha} + C_1d^{-\alpha+1}$ are smooth away from $\Sigma$ so that one can find $\varepsilon > 0$ such that $\varepsilon(d^{-\alpha} + C_1d^{-\alpha+1}) \leq \varphi_1$ in $\partial\Omega_R$. We can now use the same version of the maximum principle as in the previous lemma to deduce that

$$\varepsilon(d^{-\alpha} + C_1d^{-\alpha+1}) \leq \varphi_1 \quad \text{in } \Omega_R.$$  

For the estimate $\varphi_1 \leq C_2d^{-\alpha(\mu)}$ we need a result from [DD].

**Theorem 4.** Let $\Omega$ be a bounded smooth domain. Assume that $\tilde{a} \in L^1_{\text{loc}}(\Omega)$, $\tilde{a}$ is bounded below (i.e. $\inf_{\Omega} \tilde{a} > -\infty$) and that it satisfies

$$\gamma \left( \int_{\Omega} |u|^r \right)^{2/r} + \int_{\Omega} \tilde{a}(x)u^2 \leq \int_{\Omega} |\nabla u|^2$$

for some $\gamma > 0$ and $r > 2$. Let $\varphi_1 > 0$ denote the first eigenfunction for the operator $L = -\Delta - \tilde{a}(x)$ with zero Dirichlet boundary condition, normalized by $\|\varphi_1\|_{L^2(\Omega)} = 1$, and let $\zeta_0$ denote the solution of

$$\begin{cases}
-\Delta \zeta_0 - \tilde{a}(x)\zeta_0 = 1 & \text{in } \Omega, \\
\zeta_0 = 0 & \text{on } \partial\Omega.
\end{cases}$$

Then there exists $C = C(\Omega, \gamma(\alpha), r) > 0$ such that

$$C^{-1}\zeta_0 \leq \varphi_1 \leq C\zeta_0.$$  

**Proof of Lemma 7 continued.** We use the above theorem with $\tilde{a} = a - M$. In view of this result it suffices to show that

$$\zeta_0 \leq C d^{-\alpha(\mu)}.$$  

Using (58) and taking $\alpha = \alpha(\mu)$ we have

$$(L + M)(d^{-\alpha} - Cd^{-\alpha+1})$$

$$= -d^{-\alpha-1}\left[ -\alpha g - C((\alpha - 1)^2 - (\alpha - 1)(k - 2) + \mu - (\alpha - 1)g) \right]$$

$$+ M(d^{-\alpha} - Cd^{-\alpha+1})$$

$$\geq 1$$

in $\Omega_R$ if we choose $R > 0$ small and $C > 0$ large enough. Now take $C_1$ so large that $\zeta_0 \leq C_1(d^{-\alpha} - Cd^{-\alpha+1})$ in $\partial\Omega_R$. Using the maximum principle as before we deduce that $\zeta_0 \leq C_1(d^{-\alpha} - Cd^{-\alpha+1})$, which finishes the proof. □

**Remark 5.** The fact that $d^{1-k/2} \in \mathcal{H}$ for $\mu = H^2$ was used in the proof above at the point where the maximum principle was applied. That argument requires that both functions that one would like to compare are in $\mathcal{H}$. In general, if one of these functions does not belong to $\mathcal{H}$ then the maximum principle cannot be applied; see [DD] for an example.
4. Some applications

4.1. Minimizers for the Hardy inequality

We start this section by extending a result of Brezis and Marcus [BM] regarding the quantity

$$J_\lambda = \inf_{u \in C_0^\infty(\Omega)} \frac{\int_\Omega (|\nabla u|^2 - \lambda u^2)}{\int_\Omega u^2/d(x)^2},$$

(61)

where as usual $d(x) = \text{dist}(x, \Sigma)$.

The case studied in [BM] corresponds to $\Sigma = \partial \Omega$, and an interesting feature that the authors found in that work is the following, which we state in our situation:

**Theorem 5.** Fix $\lambda \in \mathbb{R}$. Then the infimum in (61) is achieved (in $H^1_0(\Omega)$) if and only if

$$J_\lambda < H^2.$$

**Proof.** To prove that the condition $J_\lambda < H^2$ is sufficient for the infimum in (61) to be achieved, one just needs to mimic the arguments in [BM] so we skip this step.

We prove the converse, that is, the claim that if $J_\lambda = H^2$ then the infimum is not achieved, with an argument similar in spirit to that of [BM]. Suppose that the infimum is achieved by a function $u \in H^1_0(\Omega)$, which we can assume to be nonnegative and not identically zero. Assume also that $J_\lambda = H^2$. Then $u$ satisfies

$$-\Delta u - \frac{\mu}{d(x)^2} u = \lambda u.$$

It follows that $\lambda$ is the first eigenvalue for the operator $-\Delta - \frac{\mu}{d(x)^2}$ and that $u > 0$. Moreover $u$ has to be a multiple of $\varphi_1$ (for this result see e.g. [DD, Lemma 2.3]). But by (50) we know that $\varphi_1 \sim d^{1-k/2}$. This shows on the one hand that $\int_\Omega u^2/d^2 = \infty$. But Hardy’s inequality (4) implies on the other hand that $\int_\Omega u^2/d^2 < \infty$. □

4.2. Study of a semilinear problem

In this section, we return to the study of a semilinear problem studied in [DN]. For $p > 1$, $0 < \mu \leq H^2$ and $\lambda > 0$ consider the equation

$$\begin{cases}
-\Delta u - \frac{\mu}{d(x)^2} u = u^p + \lambda & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}$$

(62)

where as usual $d(x) = \text{dist}(x, \Sigma)$. We showed in [DN] that (at least for small values of $\mu > 0$) there exists a critical exponent

$$p_0 = 1 + \frac{2}{\alpha(\mu)} \quad \text{with} \quad \alpha(\mu) = H - \sqrt{H^2 - \mu}.$$
such that (62) admits no solution (in any reasonable sense) for \( p > p_0 \) and \( \lambda > 0 \), whereas for some \( \lambda^* = \lambda^*(p) \) solutions exist when \( p < p_0 \) and \( 0 < \lambda \leq \lambda^* \) (and again no solution exists when \( \lambda > \lambda^* \)). However, the critical case \( p = p_0 \) remained open. Using Lemma 7 in combination with Theorem 4, and following the proof of Proposition 6.1 of [DN], one can prove the following:

**Theorem 6.** Given any \( \lambda > 0 \), Problem (62) with \( p = p_0 \) admits no solution.

5. Estimate for solutions of some singular equations

In what follows we will use the method developed in [DD] to prove Theorem 8. The idea is to work with \( w = u/\phi_1 \), which satisfies an elliptic equation to which Moser’s iteration technique can be applied. In the argument it is desirable to approximate the potential \( a(x) \) by bounded ones. In order to get the convergence of the corresponding solutions, it is convenient to rewrite the equation (7) as

\[-\Delta u - \tilde{a}(x)u = C_0 u + f,\]

where

\[\tilde{a} = a - C_0\]

and \( C_0 \) is chosen large enough, larger than \( M \) in (8) (although it will be taken even larger at one point below). We observe that now for any \( h \in H^1 \) the equation

\[
\begin{align*}
-\Delta v - \tilde{a}v &= h & \text{in } \Omega, \\
v &= 0 & \text{on } \partial \Omega,
\end{align*}
\]

has a unique solution \( v \in H^1 \). Let us also note that the first eigenfunction for the operator \(-\Delta - \tilde{a}\) is still \( \phi_1 \).

Let us state a result which is a kind of Sobolev inequality with weight (see a proof in [DD]).

**Lemma 9.** Assume that \( a \) satisfies (8). Then for any \( 2 \leq q \leq r \) there is a constant \( C \) depending only \( \Omega, r \) and \( \gamma(a) \) such that

\[
\left( \int_{\Omega} \phi_1^q |w|^q \right)^{2/q} \leq C \int_{\Omega} \phi_1^2 (|\nabla w|^2 + w^2)
\]

for all \( w \in C^1(\Omega) \), where \( s \) is given by the relation

\[
\frac{s}{r} = \frac{q - 2}{r - 2}.
\]
Lemma 10. Let $0 < m < r$ and suppose that
\begin{equation}
    p > \frac{2r}{m(r-2)}
\end{equation}
and
\begin{equation}
    p \geq \frac{r}{r-m}.
\end{equation}
Then for $f \in \mathcal{H}^*$, the unique solution $v$ to (63) satisfies
\begin{equation}
    |v(x)| \leq C \|\psi_1^{1-m}h\|_p \psi_1(x), \quad \text{a.e. } x \in \Omega.
\end{equation}

Remark 6. If $m \geq 1$, the assumption $h \in \mathcal{H}^*$ can be dropped since one can prove that $\|h\|_{\mathcal{H}} \leq C \|\psi_1^{1-m}h\|_p$.

Proof of Remark 6. If $\|\psi_1^{1-m}h\|_p = +\infty$, there is nothing to prove. Otherwise $h$ is locally integrable and for $\varphi \in C_c^\infty()\Omega_1)$,
\begin{equation}
    \left| \int_{\Omega} h\varphi \right| \leq \|h\|_p \|\psi_1^{1-m}\|_p \|\varphi\|_{\mathcal{H}} \leq \|h\|_p \|\psi_1^{1-m}\|_p \|\psi\|_{\mathcal{H}},
\end{equation}
where we used Hölder’s inequality twice. Now (66) implies that $mp' \leq r$, so we end up with
\begin{equation}
    \left| \int_{\Omega} h\varphi \right| \leq C \|h\|_p \|\psi\|_{\mathcal{H}} \quad \forall \varphi \in C_c^\infty(\Omega),
\end{equation}
which is the desired result. \(\square\)

Proof of Lemma 10. First we note that it is sufficient to prove this result for a bounded potential $a$, as long as the constants that appear in the estimates only depend on the constants $r, \gamma, C$ appearing in (8) and $\Omega$. This is the same argument employed in \[DD\] and we will just sketch it here. Consider $\tilde{a}_k = \min(\tilde{a}, k)$, and the first eigenfunction $\varphi_k^1$ and solution $v_k$ of (63) with the potential $a$ replaced by $\tilde{a}_k$. Then $\varphi_k^1 \to \varphi_1$ in $\mathcal{H}$ and $v_k \to v$.

Furthermore, $\tilde{a}_k$ satisfies
\begin{equation}
    \gamma \left( \int_{\Omega} |u|^2 \right)^{2/r} + \int_{\Omega} \tilde{a}_k(x)u^2 \leq \int_{\Omega} |\nabla u|^2, \quad \forall u \in C_c^\infty(\Omega).
\end{equation}
So it is enough to establish the results for $\tilde{a}_k$. We will assume then that $\tilde{a}$ is bounded. Then all functions involved belong to $C^{1,a}(\Omega)$.

By working with $h^+$ and $h^-$ we can assume that $h \geq 0$ and hence $v \geq 0$. Set $w = v/\varphi_1$.

Then $w$ satisfies the equation
\begin{equation}
    -\nabla \cdot (\varphi_1^2 \nabla w) = \varphi_1 h - (C_0 + \lambda_1)\varphi_1 v.
\end{equation}
Multiplying the equation by $w^{2j-1}$ and integrating in $\Omega$ we find
\[
\frac{2j - 1}{j^2} \int_{\Omega} \psi_1^2 |\nabla w|^2 = \int_{\Omega} \psi_1 w^{2j-1} h - (C_0 + \lambda_1) \int_{\Omega} \psi_1 u w^{2j-1} = \int_{\Omega} \psi_1 w^{2j-1} h - (C_0 + \lambda_1) \int_{\Omega} \psi_1^2 w^{2j}.
\] (67)

Using the variant of Sobolev’s inequality \((64)\) applied to $w^j$ with $s = mp'$ we obtain
\[
\left( \int_{\Omega} \psi_1^{mp'} w^{qj} \right)^{2/q} \leq C \int_{\Omega} \psi_1^2 (|\nabla w|^2 + w^{2j}),
\] (68)

where $q$ is given by
\[
q = 2 + mp' r - \frac{2}{r}.
\]

We note that by \((66)\) we have $mp' \leq r$ and therefore we can indeed apply Lemma \([9]\). Combining (67) with (68) we get
\[
\left( \int_{\Omega} \psi_1^{mp'} w^{qj} \right)^{2/q} \leq C j \int_{\Omega} \psi_1 w^{2j-1} h.
\]

By Hölder’s inequality
\[
\int_{\Omega} \psi_1 w^{2j-1} h \leq \left( \int_{\Omega} \psi_1^{mp'} w^{(2j-1)p'} \right)^{1/p'} \|\psi_1^{-m} h\|_p
\]
and therefore
\[
\left( \int_{\Omega} \psi_1^{mp'} w^{qj} \right)^{1/(qj)} \leq (C j)^{1/(2j)} \left( \int_{\Omega} \psi_1^{mp'} w^{(2j-1)p'} \right)^{1/(2j p')} \|\psi_1^{-m} h\|_p^{1/(2j)}. \tag{69}
\]

Observe now that condition \((65)\) is equivalent to $q > 2p'$ and therefore a standard iteration argument yields the result. Let indeed $j_0 = 1/2 + m/2$ and for $k \geq 1$, define $j_k$ inductively by
\[
(2j_k - 1) p' = q j_{k-1}.
\] (70)

One can easily show that $\{j_k\}$ is increasing and converges to $+\infty$ as $k \to +\infty$, so that if
\[
\theta_k = \left( \int_{\Omega} \psi_1^{mp'} w^{qj_k} \right)^{1/(q j_k)} \left( \int_{\Omega} \psi_1^{mp'} \right)^{-(1/q j_k)},
\]
then \( \{\theta_k\} \) is increasing and converges to \( \|w\|_\infty \) as \( k \to \infty \). Observe in passing that since \( \varphi_1 \in \mathcal{H} \) and \( mp' \leq r \), we have
\[
\left( \int_{\Omega} \varphi_1^{mp'} \right) < \infty.
\]

Equation (69) then yields
\[
\theta_k \leq (Cj_k)^{1/(2j_k)} \left( \int_{\Omega} \varphi_1^{mp'} \right)^{-1/(qj_k)+1/(2j_k)p'} \theta_{k-1} \left( \int_{\Omega} \varphi_1^{(1-m)p} h^p \right)^{1/(2j_k)p}.
\]

Now, either \( \{\theta_k\} \) remains bounded by \( \left( \int_{\Omega} \varphi_1^{m} h^p \right)^{1/p} \) for all \( k \), in which case passing to the limit provides the desired inequality, or there exists a smallest integer \( k_0 \) such that for \( k \geq k_0 - 1 \),
\[
\theta_k \geq \left( \int_{\Omega} \varphi_1^{(1-m)p} h^p \right)^{1/p}.
\]

Using this inequality in (71), we obtain for \( k \geq k_0 \),
\[
\theta_k \leq (Cj_k)^{1/(2j_k)} \left( \int_{\Omega} \varphi_1^{mp'} \right)^{1/q (1 + 1/p')} \theta_{k-1}.
\]

Applying (72) inductively, it follows that
\[
\|w\|_\infty \leq \prod_{k=k_0}^{\infty} (Cj_k)^{1/(2j_k)} \left( \int_{\Omega} \varphi_1^{mp'} \right)^{1/q (1 + 1/p')} \theta_{k_0-1}.
\]

Starting from (70), a straightforward computation shows that for some \( c > 0 \),
\[
j_k = \left( \frac{q}{2p'} \right)^k j_0 + \frac{\left( \frac{q}{2p'} \right)^k - 1}{\frac{q}{2p'} - 1} \sim c \left( \frac{q}{2p'} \right)^k \quad \text{as} \quad k \to \infty.
\]

Since \( q/(2p') > 1 \), we then conclude that the infinite product on the right hand side of (73) converges to some finite constant.

If \( k_0 \geq 2 \), applying again (71) for \( k = k_0 - 1 \), we also have
\[
\theta_{k_0-1} \leq Cq^{k_0-2/(2j_{k_0-1})} \left( \int_{\Omega} \varphi_1^{(1-m)p} h^p \right)^{1/(2j_{k_0-1})} \leq C \left( \int_{\Omega} \varphi_1^{(1-m)p} h^p \right)^{1/p},
\]
where we used the minimality of \( k_0 \) in the last inequality. Combining (74) and (73) yields the desired result.

If \( k_0 = 1 \) then by (69),
\[
\left( \int_{\Omega} \varphi_1^{mp'} w^{qj_0} \right)^{1/(qj_0)} \leq C \left( \int_{\Omega} v^r \right)^{m/(2j_0 r)} \left( \int_{\Omega} \varphi_1^{(1-m)p} h^p \right)^{1/(2j_0 p)},
\]

(75)
where we used H"older’s inequality and the fact that $mp' \leq r$, which follows from (66). Now,

$$\gamma \| v \|_{2}^{2} \leq \| v \|_{H}^{2} = \int_{\Omega} hv \leq \| h \|_{L^{p}} \| \varphi_{1}^{1-m} \|_{p} \| \varphi_{1}^{m-1} v \|_{p'}$$

$$\leq \| h \|_{L^{p}} \| w \|_{\infty} \| \varphi_{1}^{m} \|_{p'} \leq C \| h \|_{L^{p}} \| w \|_{\infty}.$$  

(76)

Using (73), (75) and (76), we obtain

$$\| w \|_{\infty} \leq C \| h \|_{L^{p}} \| \varphi_{1}^{1-m} \|_{p} \| w \|_{\infty}^{m+1/(m+1)} \| w \|_{\infty},$$

which after simplification yields the desired result. □

**Lemma 11.** Let $0 < m < r$ and suppose that

$$p < \frac{2r}{m(r-2)} \quad \text{and} \quad p \geq \frac{r}{r-m}.$$  

Then given $h \in \mathcal{H}^{*}$, the unique solution $v$ to

$$\begin{cases}
-\Delta v - \tilde{a}v = h & \text{in} \; \Omega, \\
v = 0 & \text{on} \; \partial \Omega,
\end{cases}$$  

(77)

satisfies

$$\left( \int_{\Omega} \varphi_{1}^{mp} \left| \frac{v}{\varphi_{1}} \right|^{w} \right)^{1/\alpha} \leq C \| \varphi_{1}^{1-m} h \|_{p}. $$  

(78)

for any $\alpha \geq 1$ such that

$$\frac{1}{\alpha} \geq \frac{1}{p} - \left(1 - \frac{2}{q}\right),$$  

(79)

where

$$q = 2 + mp'r - 2.$$  

Proof. The computations of the previous lemma are valid up to (69). Now observe that (69) yields an estimate for

$$\left( \int_{\Omega} \varphi_{1}^{mp'} u^{qj} \right)^{1/(qj)}$$

if $qj \geq (2j-1)p'$, which is equivalent to

$$j \leq \frac{p'}{2p' - q}.$$  

Take $\alpha$ satisfying (79) and $j = \alpha/q$. By H"older’s inequality

$$\int_{\Omega} \varphi_{1}^{mp'} u^{(2j-1)p'} \leq \left( \int_{\Omega} \varphi_{1}^{mp'} u^{\alpha} \right)^{(2j-1)p'/\alpha} \left( \int_{\Omega} \varphi_{1}^{mp'} \right)^{1-(2j-1)p'/\alpha},$$

but observe that $\int_{\Omega} \varphi_{1}^{mp'} < \infty$ because $mp' \leq r$ and $\varphi_{1} \in L^{r}$. The previous inequality together with (69) yields the result. □
Remark 7. A direct consequence of the above lemma is that if instead of assuming
\[ p < \frac{2r}{m(r-2)} \]
we assume that
\[ p = \frac{2r}{m(r-2)}, \]
then the conclusion is that for all \( 1 \leq \alpha < \infty \),
\[ \left( \int_{\Omega} \frac{|v|}{\varphi_1}^{\alpha} \right)^{\frac{1}{\alpha}} \leq C \| \varphi_1^{1-m} h \|_p, \]
where the constant \( C \) may depend on \( \alpha \).

Remark 8. In contrast with what we observed in Remark 6, \( h \) need not be in \( H^s \) for \( \| h \varphi_1^{1-m} \|_p \) to be finite. Hence, in light of inequality (78), one can define by density an operator
\[ T = (-\Delta - \tilde{a}(x))^{-1} : L^p(\Omega, \varphi_1^{1-m} \, dx) \rightarrow L^a(\Omega, \varphi_1^{mp'/\alpha-1} \, dx), \]
which restricted to \( h \in H^s \) assigns the corresponding solution \( v =: T(h) \in H \) of (77).

On the other hand, given \( h \in L^1(\Omega) \), one can consider a weak solution \( u \in L^1(\Omega) \) of equation (77) in the sense that
\[ \int_{\Omega} a(x) |u| (\text{dist}(x, \partial \Omega)) < \infty \]
and
\[ \int_{\Omega} u (-\Delta v + \tilde{a}(x) v) = \int_{\Omega} f \varphi \]
for all \( \varphi \in C^2(\bar{\Omega}) \) with \( \varphi |_{\partial \Omega} \equiv 0 \). If \( h \in L^p(\Omega, \varphi_1^{1-m} \, dx) \) and \( u \in L^a(\Omega, \varphi_1^{mp'/\alpha-1} \, dx) \), is it true that \( u = T(h) \)?

Proof of Theorem 3. Consider now \( u \in H \) satisfying (7) and let \( u_1 \) be the solution of
\[
\begin{cases}
-\Delta u_1 - \tilde{a} u_1 = f & \text{in } \Omega, \\
u_1 = 0 & \text{on } \partial \Omega.
\end{cases}
\]
We remark that Lemma 10 implies that \( \| u_1 / \varphi_1 \|_\infty \leq C \| \varphi_1^{1-m} f \|_p \). Thus
\[
\left( \int_{\Omega} \frac{|u_1|}{\varphi_1} \right)^{\frac{1}{l}} \leq C \| \varphi_1^{1-m} f \|_p \tag{80}
\]
for any \( l \geq 1 \). Define \( u_2 = u - u_1 \) so that \( u = u_1 + u_2 \) and \( u_2 \in H \) is the unique solution of
\[
\begin{cases}
-\Delta u_2 - \tilde{a} u_2 = C_0 u & \text{in } \Omega, \\
u_2 = 0 & \text{on } \partial \Omega.
\end{cases}
\]
Starting with $p_1 = 2$ we shall construct a finite increasing sequence $p_k$ which will stop at some $\bar{k}$ such that

$$p_{\bar{k}} \geq \frac{r}{r-2}$$

and such that for each $k = 1, \ldots, \bar{k}$ the following inequality holds:

$$\left( \int_{\Omega} \varphi_1^2 \frac{|u|}{\varphi_1} \right)^{1/p_k} \leq C(\|\varphi_1^{-m} f\|_p + \|u\|_2). \quad (81)$$

Indeed, Lemma 11 applied to $u_2$ with $p = p_1 = 2$ and $m_1 = 1$ implies

$$\left( \int_{\Omega} \varphi_1^2 \frac{|u_2|}{\varphi_1} \right)^{1/p_2} \leq C\|u\|_{L^2}, \quad (82)$$

where $p_2$ is given by

$$\frac{1}{p_2} = \frac{1}{p_1} - \left(1 - \frac{2}{q}\right)$$

and $q = 2 + 2(r-2)/r$. Inequality (80) combined with (82) shows that (81) holds for $p_2$.

We continue this process using Lemma 11 repeatedly with $p$ and $m$ in that lemma given by

$$\frac{1}{p_{k+1}} = \frac{1}{p_k} - \left(1 - \frac{2}{q}\right), \quad m_k = \frac{2(p_k - 1)}{p_k}$$

and $q = 2 + 2(r-2)/r$. At each step we obtain (inductively)

$$\left( \int_{\Omega} \varphi_1^2 \frac{|u_{k+1}|}{\varphi_1} \right)^{1/p_{k+1}} \leq C\left( \int_{\Omega} \varphi_1^2 \frac{|u|}{\varphi_1} \right)^{1/p_k} \leq C(\|\varphi_1^{-m} f\|_p + \|u\|_2).$$

This together with (80) proves that (81) holds for $p_{k+1}$. We can continue in this way provided

$$\frac{1}{p_k} = \left(1 - \frac{2}{q}\right) > 0,$$

or equivalently

$$p_k < \frac{r}{r-2}.$$

Let $\hat{k}$ be the first time that we find

$$p_{\hat{k}} \geq \frac{r}{r-2}$$

so that (81) still holds for $\hat{k}$. If $p_{\hat{k}} > r/(r-2)$ then we can apply Lemma 10 directly and conclude that

$$\|u_2/\varphi_1\|_{\infty} \leq C(\|\varphi_1^{-m} f\|_p + \|u\|_2),$$

which would finish the proof of the theorem.

In the case $p_{\hat{k}} = r/(r-2)$ we first use Remark 7 and then Lemma 10. □
References


