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Which 3-manifold groups are Kähler groups?

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Abstract. The question in the title, first raised by Goldman and Donaldson, was partially answered by Reznikov. We give a complete answer, as follows: if $G$ can be realized as both the fundamental group of a closed 3-manifold and of a compact Kähler manifold, then $G$ must be finite—and thus belongs to the well-known list of finite subgroups of $O(4)$, acting freely on $S^3$.

Keywords. Kähler manifold, 3-manifold, fundamental group, cohomology ring, resonance variety, isotropic subspace

1. Introduction

1.1. As is well-known, every finitely presented group $G$ occurs as the fundamental group of a smooth, compact, connected, orientable 4-dimensional manifold $M$. As shown by Gompf [14], the manifold $M$ can be chosen to be symplectic. Requiring a complex structure on $M$ is no more restrictive, as long as one is willing to go up to complex dimension 3 (see Taubes [32]).

Suppose now $G$ is the fundamental group of a compact Kähler manifold $M$. Groups arising this way are called Kähler groups (or, projective groups, if $M$ is actually a smooth projective variety). The Kähler condition puts strong restrictions on what $G$ can be. For instance, the first Betti number, $b_1(G)$, must be even, by classical Hodge theory. Moreover, $G$ must be 1-formal, by work of Deligne, Griffiths, Morgan, and Sullivan [9]. Also, $G$ cannot split non-trivially as a free product, by a result of Gromov [17]. On the other hand, every finite group is a projective group, by a classical result of Serre [29]. We refer to [11] for a comprehensive survey of Kähler groups, and to the recent work of Delzant–Gromov [11], Napier–Ramachandran [25], and Delzant [10] for further geometric restrictions imposed by the Kähler condition on a group $G$. 

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Requiring that $M$ be a 3-dimensional compact, connected manifold also puts severe restrictions on $G = \pi_1(M)$. For example, if $G$ is abelian, then $G$ is either $\mathbb{Z}/n\mathbb{Z}$, $\mathbb{Z}$, $\mathbb{Z} \oplus \mathbb{Z}_2$, or $\mathbb{Z}^3$ (see [20]).

1.2. A natural question—raised by Goldman and Donaldson in 1989, and independently by Reznikov in 1993—is then: what are the 3-manifold groups which are Kähler groups?

In [28], Reznikov proved the following result, which Simpson [31] calls “one of the deepest restrictions” on the homotopy types that may occur for Kähler manifolds: Let $M$ be an irreducible, atoroidal 3-manifold, and suppose there is a homomorphism $\rho: \pi_1(M) \to \text{SL}(2, \mathbb{C})$ with Zariski dense image. Then $G = \pi_1(M)$ is not a Kähler group. The same conclusion was reached by Hernández-Lamoneda in [19], under the assumption that $M$ is a geometrizable 3-manifold, with all pieces hyperbolic.

In this note, we answer the above question for all 3-manifold groups, as follows.

**Theorem 1.1.** Let $G$ be the fundamental group of a compact, connected 3-manifold. If $G$ is a Kähler group, then $G$ is finite.

By the 3-dimensional spherical space-form conjecture, now established by Perelman [26, 27], a closed 3-manifold $M$ has finite fundamental group if and only if it admits a metric of constant positive curvature (for a detailed proof, see Morgan and Tian [24, Corollary 0.2]). Thus, $M = S^3/K$, where $G$ is a finite subgroup of $O(4)$, acting freely on $S^3$. The list of such finite groups (essentially due to Hopf) is given by Milnor in [23].

1.3. The paper is organized as follows. In §2 we discuss the characteristic and resonance varieties of a group $G$, and two notions of isotropy. In §3 we recall the Isotropic Subspace Theorem of Catanese, and a correspondence due to Beauville. In §4 we use these tools to prove a key result, tying the first resonance variety of a Kähler manifold to the rank of the cup-product map in low degrees. In §5 we investigate the first resonance variety of a closed, oriented 3-manifold; Poincaré duality and properties of Pfaffians yield a very different conclusion in this setting.

All this works quite well, provided the first Betti number of $G$ is positive. To deal with the remaining case, we need two theorems of Reznikov and Fujiwara, relating the Kähler, respectively the 3-manifold condition on a group to Kazhdan’s property $T$; we recall those in §6. Finally, we put everything together in §7 and give a proof of Theorem 1.1.

A natural question arises out of this work: Which 3-manifold groups are quasi-Kähler? (A group $G$ is quasi-Kähler if $G = \pi_1(M \setminus D)$, where $M$ is a compact Kähler manifold and $D$ is a divisor with normal crossings.) We have some partial results in this direction; those results will be presented elsewhere.

2. Cohomology jumping loci and isotropic subspaces

2.1. Let $X$ be a connected CW-complex with finitely many cells in each dimension. Let $G = \pi_1(X)$ be the fundamental group of $X$, and $T = \text{Hom}(G, \mathbb{C}^*)$ its character variety.
Every character $\rho \in \mathbb{T}$ determines a rank 1 local system, $\mathbb{C}_\rho$, on $X$. The characteristic varieties of $X$ are the jumping loci for cohomology with coefficients in such local systems:

$$V^i_d(X) = \{\rho \in \mathbb{T} \mid \dim H^i(X, \mathbb{C}_\rho) \geq d\}. \quad (1)$$

The varieties $V_d(X) = V^1_d(X)$ depend only on $G = \pi_1(X)$, so we sometimes denote them as $V_d(G)$.

2.2. Consider now the cohomology algebra $A = H^*(X, \mathbb{C})$. Left multiplication by an element $x \in A^1$ yields a cochain complex $(A, x): A^0 \xrightarrow{x} A^1 \xrightarrow{x} A^2 \to \ldots$. The resonance varieties of $X$ are the jumping loci for the homology of this complex:

$$R^i_d(X) = \{x \in A^1 \mid \dim H^i(A, x) \geq d\}. \quad (2)$$

The varieties $R_d(X) = R^1_d(X)$ depend only on $G = \pi_1(X)$, so we sometimes denote them by $R_d(G)$. By definition, an element $x \in A^1$ belongs to $R_d(X)$ if and only if there exists a subspace $W \subset A^1$ of dimension $d + 1$ such that $x \cup y = 0$ for all $y \in W$.

Fix bases $\{e_1, \ldots, e_n\}$ for $A^1$ and $\{f_1, \ldots, f_m\}$ for $A^2$. Writing the cup-product as $e_i \cup e_j = \sum_{k=1}^m \mu_{i,j,k} f_k$, we may define an $m \times n$ matrix $\Delta$ of linear forms in variables $x_1, \ldots, x_n$, with entries

$$\Delta_{k,j} = \sum_{i=1}^n \mu_{i,j,k}x_i. \quad (3)$$

It is readily seen that $R_d(X) = V(E_d(\Delta))$, where $E_d$ denotes the ideal of $(n-d) \times (n-d)$ minors. Note also that $x \cup x = 0$ for all $x \in A^1$ implies $\Delta \cdot \vec{x} = 0$, where $\vec{x}$ is the column vector with entries $x_1, \ldots, x_n$.

2.3. Foundational results on the structure of the cohomology support loci for local systems on compact Kähler manifolds were obtained by Beauville [2], Green–Lazarsfeld [15], Simpson [30], and Campana [5]: if $G$ is the fundamental group of such a manifold, then $V_d(G)$ is a union of (possibly translated) subtori of the algebraic group $\mathbb{T}$.

In addition, Theorem A from [12] establishes a strong relationship between the characteristic and resonance varieties of a Kähler group $G$: the tangent cone to $V_d(G)$ at the identity of $\mathbb{T}$ equals $R_d(G)$ for all $d \geq 1$.

2.4. A non-zero subspace $E \subset H^1(X, \mathbb{C})$ is (totally) isotropic if the restriction of the cup-product map $\cup_X: H^1(X, \mathbb{C}) \wedge H^1(X, \mathbb{C}) \to H^2(X, \mathbb{C})$ to $E \wedge E$ is identically zero. By analogy, we say $E$ is 1-isotropic if the restriction of $\cup_X$ to $E \wedge E$ has 1-dimensional image.

Note that these properties of $E$ depend only on $G = \pi_1(X)$. Indeed, let $h: X \to K(G, 1)$ be a classifying map. Then $h_*: H_1(X, \mathbb{Z}) \to H_1(G, \mathbb{Z})$ is an isomorphism, and $h_*: H_2(X, \mathbb{Z}) \to H_2(G, \mathbb{Z})$ is an epimorphism. Using Kronecker duality and the functoriality of the cup-product, it is readily seen that $E$ is a (1-) isotropic subspace of $H^1(G, \mathbb{C})$ for $\cup_G$ if and only if $h^*(E)$ is a (1-) isotropic subspace of $H^1(X, \mathbb{C})$ for $\cup_X$. Which 3-manifold groups are Kähler groups?
3. The Isotropic Subspace Theorem

By a fibration we mean a surjective morphism \( f : M \to N \) with connected fibers between two compact complex manifolds \( M \) and \( N \). Two fibrations \( f : M \to C \) and \( f' : M \to C' \) over projective curves \( C \) and \( C' \) are said to be equivalent if there is an isomorphism \( \phi : C \to C' \) such that \( f' = \phi \circ f \). We denote by \( \mathcal{E}(M) \) the set of equivalence classes of fibrations \( f : M \to C \), with \( C \) a projective curve of genus \( g \geq 2 \).

Let \( M \) be a compact Kähler manifold. Beauville’s work [2] establishes a bijection between the set \( \mathcal{E}(M) \) and the set of irreducible components of the first characteristic variety \( V_1(M) \) passing through the identity of the algebraic group \( \mathbb{T} = \text{Hom}(\pi_1(M), \mathbb{C}^*) \).

In particular, the set \( \mathcal{E}(M) \) must be finite.

The Isotropic Subspace Theorem, due to Catanese [6, Theorem 1.10], establishes a relation between the set of equivalence classes of fibrations of a Kähler manifold \( M \) over curves of genus \( g \geq 2 \), and the maximal isotropic subspaces in \( H^1(M, \mathbb{C}) \).

**Theorem 3.1** (Catanese [6]). Let \( M \) be a compact Kähler manifold. Then, for any maximal isotropic subspace \( E \subset H^1(M, \mathbb{C}) \) of dimension \( g \geq 2 \), there is a fibration \( f : M \to C \) onto a smooth curve of genus \( g \) and a maximal isotropic subspace \( E' \subset H^1(C, \mathbb{C}) \) such that \( E = f^*E' \).

For more information on this correspondence, see [7].

4. The first resonance variety of a Kähler manifold

**Theorem 4.1.** Let \( M \) be a compact Kähler manifold with \( b_1(M) \neq 0 \). If \( R_1(M) = H^1(M, \mathbb{C}) \), then \( H^1(M, \mathbb{C}) \) is 1-isotropic.

**Proof.** By Hodge theory, we must have \( b_1(M) \geq 2 \). The equality \( R_1(M) = H^1(M, \mathbb{C}) \) says that, for any non-zero cohomology class \( x \in H^1(M, \mathbb{C}) \), there is a class \( y \in H^1(M, \mathbb{C}) \setminus \mathbb{C} : x \) such that \( x \cup y = 0 \). Consequently, the vector space spanned by \( x \) and \( y \) is a (2-dimensional) isotropic subspace containing \( x \).

Let \( U_x \) be a maximal isotropic subspace of \( H^1(M, \mathbb{C}) \) containing \( x \); we must then have \( \dim U_x \geq 2 \). Thus, by Theorem 3.1 there is a fibration \( f_x : M \to C_x \) onto a smooth projective curve \( C_x \) of genus \( g_x = \dim U_x \), with \( x \in f_x^*(H^1(C_x, \mathbb{C})) \).

Recall now that the set \( \mathcal{E}(M) \) of equivalence classes of fibrations of \( M \) over curves of genus at least 2 is finite. Thus, we may write the first cohomology group of \( M \) as a finite union of linear subspaces,

\[
H^1(M, \mathbb{C}) = \bigcup_{[f] \in \mathcal{E}(M)} f^*\left(H^1(C, \mathbb{C})\right),
\]

where \( f = f_x \) for some \( x \in H^1(M, \mathbb{C}) \), and \( C_f := C_x \). This is possible only if there is a fibration \( f_1 : M \to C_1 \) such that \( H^1(M, \mathbb{C}) = f_1^*(H^1(C_1, \mathbb{C})) \).

Since \( f_1 \) is a fibration, the induced morphism \( f_1^* : H^1(C_1, \mathbb{C}) \to H^1(M, \mathbb{C}) \) is injective. The defining property of \( f_1 \) implies that \( f_1^* : H^1(C_1, \mathbb{C}) \to H^1(M, \mathbb{C}) \) is an isomorphism.
On the other hand, the induced morphism $f_1^* : H^2(C_1, \mathbb{C}) \to H^2(M, \mathbb{C})$ is also injective. To prove this claim, first note that any cohomology class in $H^1(M, \mathbb{C})$ is primitive. Using the Hodge–Riemann bilinear relations (see e.g. [16, p. 123]), it follows that, for any non-zero $(1, 0)$-class $a \in H^1(M, \mathbb{C})$, the product $\beta = \sqrt{-1} a \cup \overline{a}$ is a non-zero, real, $(1, 1)$-class in $H^2(M, \mathbb{C})$. Since $f_1^* : H^1(C_1, \mathbb{C}) \to H^1(M, \mathbb{C})$ is an isomorphism, there is an element $a \in H^1(C_1, \mathbb{C})$ such that $f_1^*(a) = \alpha$. Hence, $f_1^*(\sqrt{-1} a \wedge \overline{a}) = \beta$, and the claim is proved.

Consider now the commuting diagram

$$
\begin{array}{ccc}
H^1(M, \mathbb{C}) \wedge H^1(M, \mathbb{C}) & \xrightarrow{\cup_M} & H^2(M, \mathbb{C}) \\
\downarrow{f_1^* \wedge f_1^*} & & \downarrow{f_1^*} \\
H^1(C_1, \mathbb{C}) \wedge H^1(C_1, \mathbb{C}) & \xrightarrow{\cup_{C_1}} & H^2(C_1, \mathbb{C})
\end{array}
$$

As we saw above, the left arrow is an isomorphism, and the right one is an injection. Since $\cup_{C_1}$ surjects onto $H^2(C_1, \mathbb{C}) = \mathbb{C}$, we conclude that $\cup_{C_1}$ has 1-dimensional image. $\square$

Remark 4.2. An alternative way to prove Theorem 4.1 is by using the much more general Theorem C from [12], which guarantees that every positive-dimensional component of $R_1(M)$ is an 1-isotropic subspace of $H^1(M, \mathbb{C})$. This is the argument we had in an earlier version of this paper; at the urging of one of the referees, we came up with the above, more self-contained proof.

5. The first resonance variety of a 3-manifold

Let $M$ be a compact, connected, orientable 3-manifold. Fix an orientation on $M$, that is, pick a generator $[M] \in H^3(M, \mathbb{Z}) \cong \mathbb{Z}$. With this choice, the cup-product on $M$ determines an alternating 3-form $\mu = \mu_M$ on $H^1(M, \mathbb{Z})$, given by

$$\mu(x, y, z) = \langle x \cup y \cup z, [M] \rangle,$$

where $\langle , \rangle$ is the Kronecker pairing. In turn, the cup-product map $\cup_M : H^1(M, \mathbb{Z}) \wedge H^1(M, \mathbb{Z}) \to H^2(M, \mathbb{Z})$ is determined by $\mu$, via $\langle x \cup y, \gamma \rangle = \mu(x, y, z)$, where $z = \text{PD}(\gamma)$ is the Poincaré dual of $\gamma \in H_2(M, \mathbb{Z})$.

Now fix a basis $\{e_1, \ldots, e_n\}$ for $H^1(M, \mathbb{C})$, and choose as basis for $H^2(M, \mathbb{C})$ the set $\{e_1^\vee, \ldots, e_n^\vee\}$, where $e_i^\vee$ denotes the Kronecker dual of the Poincaré dual of $e_i$. Then

$$\mu(e_i, e_j, e_k) = \sum_{1 \leq m \leq n} \mu_{i,j,m} e_m^\vee \cdot \text{PD}(e_k) = \mu_{i,j,k}.$$  

(7)

Recall from (3) the $n \times n$ matrix with entries $\Delta_{k,j} = \sum_{i=1}^n \mu_{i,j,k} x_i$. Since $\mu$ is an alternating form, $\Delta$ is a skew-symmetric matrix.

Proposition 5.1. Let $M$ be a closed, orientable 3-manifold. Then:

1. $H^1(M, \mathbb{C})$ is not 1-isotropic.
2. If $b_1(M)$ is even, then $R_1(M) = H^1(M, \mathbb{C})$. 

Proof. To prove (1), suppose \( \dim \text{im}(\bigcup M) = 1 \). This means there is a hyperplane \( E \subset H := H^1(M, \mathbb{C}) \) such that \( x \cup y \cup z = 0 \) for all \( x, y \in H \) and \( z \in E \). Hence, the skew 3-form \( \mu : \bigwedge^3 H \to \mathbb{C} \) factors through a skew 3-form \( \tilde{\mu} : \bigwedge^3 (H/E) \to \mathbb{C} \). But \( \dim H/E = 1 \) forces \( \tilde{\mu} = 0 \), and so \( \mu = 0 \), a contradiction.

To prove (2), recall \( R_1(M) = V(E_1(\Delta)) \). Since \( \Delta \) is a skew-symmetric matrix of even size, it follows from Buchsbaum–Eisenbud [4, Corollary 2.6] that \( V(E_1(\Delta)) = V(E_0(\Delta)) \) (see [8, eq. (6.9)]). But \( \Delta \cdot \vec{x} = 0 \) implies \( \det \Delta = 0 \), and so \( V(E_0(\Delta)) = H \).

\( \square \)

Remark 5.2. As noted by S. Papadima, the following holds. Suppose \( M \) is a closed, orientable 3-manifold, with \( b_1(M) \) odd. Then \( R_1(M) \neq H^1(M, \mathbb{C}) \) if and only if \( \mu_M \) is generic, in the sense of [3].

6. Kazhdan’s property \( T \)

The following question is due to J. Carlson and D. Toledo (see J. Kollár [22]): For a Kähler group \( G \), is \( b_2(G) \neq 0 \)? This question was answered in the affirmative by A. Reznikov in [28], under an additional assumption, as follows.

Theorem 6.1 (Reznikov [28]). Let \( G \) be a Kähler group. If \( G \) does not satisfy Kazhdan’s property \( T \), then \( b_2(G) \neq 0 \).

Recall that a discrete group \( G \) satisfies Kazhdan’s property \( T \) (for short, \( G \) is a Kazhdan group) if and only if \( H^1(G, \mathcal{H}) = 0 \) for all orthogonal or unitary representations of \( G \) on a Hilbert space \( \mathcal{H} \) (see de la Harpe and Valette [18, p. 47]). In particular, if \( b_1(G) \neq 0 \), then \( G \) is not Kazhdan. (For a simple proof of Theorem 6.1 in this case, see [21].)

We will also need the following relationship between 3-manifold groups and Kazhdan’s property \( T \), established by K. Fujiwara in [13].

Theorem 6.2 (Fujiwara [13]). Let \( G \) be the fundamental group of a closed, orientable 3-manifold. If \( G \) satisfies Kazhdan’s property \( T \), then \( G \) is finite.

In fact, the theorem is valid for any subgroup \( G < \pi_1(M) \), where \( M \) is a compact (not necessarily boundaryless), connected, orientable 3-manifold. Fujiwara further assumes that each piece of the canonical decomposition of \( M \) along embedded spheres, disks and tori admits one of the eight geometric structures in the sense of Thurston, but this is now guaranteed by the work of Perelman [26, 27].

7. Kähler 3-manifold groups

We are now in a position to prove Theorem 1.1 from the introduction.

Let \( G \) be the fundamental group of a compact, connected 3-manifold \( M \). Suppose \( G \) is a Kähler group, and \( G \) is not finite.
Which 3-manifold groups are Kähler groups?

Step 1. A finite-index subgroup of a Kähler group is again a Kähler group (see [1, Example 1.10]). Passing to the orientation double cover of $M$ if necessary, we may as well assume $M$ is orientable.

Step 2. Since $G$ is an infinite, orientable 3-manifold group, $G$ is not Kazhdan, by Fujiiwara’s Theorem 6.2. Since $G$ is Kähler and not Kazhdan, $b_2(G) \neq 0$, by Reznikov’s Theorem 6.1.

Step 3. Since $b_2(M) \geq b_2(G)$, we must also have $b_2(M) \neq 0$. By Poincaré duality, $b_1(M) = b_2(M)$. Hence, $b_1(G) = b_1(M)$ is not zero.

Step 4. Since $G$ is Kähler, $b_1(G)$ must be even. Since $M$ is a closed, orientable 3-manifold with $G = \pi_1(M)$, Proposition 5.1 tells us that $H_1(G) = H^1(G, \mathbb{C})$ and $H^1(G, \mathbb{C})$ is not 1-isotropic. Since, on the other hand, $G$ is Kähler, Theorem 4.1 tells us that $b_1(G) = 0$.

Our assumptions have led us to a contradiction. Thus, the theorem is proved.

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