
Abstract. — The classical Ostrowski inequality for functions on intervals is extended to functions on general domains in Euclidean space. For radial functions on balls the inequality is sharp.

Key words: Ostrowski inequality; sharp inequality; multivariate inequality.

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1. Introduction

The classical Ostrowski inequality (of 1938) [6] is

\[ \left| \frac{1}{b-a} \int_a^b f(y) \, dy - f(x) \right| \leq \left( \frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right) \frac{\|f'\|_{\infty}}{b-a}, \]

for \( f \in C^1([a, b]), \ x \in [a, b], \)

and it is sharp. It was extended from intervals to rectangles in \( \mathbb{R}^N, N \geq 1 \) (see [2, p. 507]). For other recent results related to Ostrowski’s inequality, see [3], [4] and [7], [8], [9].

The extension to general domains in \( \mathbb{R}^N \) has remained an open problem. Our purpose here is to solve this problem. We deduce Ostrowski type inequalities on general bounded domains in \( \mathbb{R}^N \), and the inequalities are shown to be sharp on balls.

2. Main results

Let \( N > 1, B(0, R) := \{x \in \mathbb{R}^N: |x| < R\} \) be the ball in \( \mathbb{R}^N \) centered at the origin and of radius \( R > 0 \). Let \( S^{N-1} := \{x \in \mathbb{R}^N: |x| = 1\} \) be the unit sphere in \( \mathbb{R}^N \).

Let \( d\omega \) be the element of surface measure on \( S^{N-1} \) and let \( \omega_N = \int_{S^{N-1}} d\omega = 2\pi^{N/2}/\Gamma(N/2) \). For \( x \in \mathbb{R}^N - \{0\} \) we can write \( x = r\omega \), where \( r = |x| > 0 \) and \( \omega = x/r \in S^{N-1} \). Note that \( \int_{B(0, R)} d\omega = \omega_NR^N/N \) is the Lebesgue measure of the ball.

For \( f \in C(B(0, R)) \) let

\[ \int_{B(0, R)} f(y) \, dy := \frac{1}{\text{Vol}(B(0, R))} \int_{B(0, R)} f(y) \, dy, \]

where \( \text{Vol}(B(0, R)) \) is the volume of the ball.
and
\[
\int_{S^{N-1}} f(r \omega) \, d\omega = \frac{1}{o_N} \int_{S^{N-1}} f(r \omega) \, d\omega
\]
be the averages of \( f \) over the ball and the sphere, respectively. Here \( f \) can be real or complex valued.

Let
\[
f'(r) := \int_{S^{N-1}} f(r \omega) \, d\omega
\]
be the average of \( f(x) \) as \( x \) ranges over \( \{ y \in \mathbb{R}^N : |y| = r \} \). Then
\[
(2.1) \quad \mathcal{N}(f) := \sup_{x \in B(0, R)} |f(x) - \tilde{f}(r)| = \| f - \tilde{f} \|_{\infty}
\]
measures how far \( f \) is from being a radial function. More precisely, \( \mathcal{N} \) is a seminorm on \( C(B(0, R)) \), and \( \mathcal{N}(f) = 0 \) if and only if \( f \) is a radial function, i.e. \( f(x) = g(r) \) for some function \( g \in C([0, R]) \).

We view how close \( f \) is to being radial by computing \( \mathcal{N}(f) \): the closer \( f \) is to being radial, the smaller \( \mathcal{N}(f) \) is, and conversely.

Let \( \Omega \) be a domain in \( \mathbb{R}^N \) and let
\[
(2.2) \quad \text{Lip}(\Omega) = \{ f \in C(\overline{\Omega}) : |f(x) - f(y)| \leq K|x - y| \text{ for some } K > 0 \text{ and all } x, y \in \Omega \}.
\]
The Lipschitz constant of \( f \in \text{Lip}(\Omega) \) is
\[
\| f \|_{\text{Lip}} = \inf \{ K : K \text{ as in (2.2)} \}.
\]
Then \( X := \text{Lip}(\Omega) \) is a Banach space under the norm
\[
f \mapsto \| f \|_{\infty} + \| f \|_{\text{Lip}} =: \| f \|_X.
\]
Equivalently, \( X \) is the Sobolev space \( W^{1, \infty}(\Omega) \) (cf. [5]).

Our first main result is the following:

**Theorem 2.1.** Let \( f \in \text{Lip}(B(0, R)) = W^{1, \infty}(B(0, R)) \). Then for \( x = r \omega \) as above,
\[
(2.3) \quad \left| f(x) - \int_{B(0, R)} f(y) \, dy \right| \leq \mathcal{N}(f) + \frac{N}{R^N} \| \nabla f \|_{\infty} \left[ \frac{2|x|^{N+1}}{N(N+1)} + R^N \left( \frac{R}{N+1} - \frac{|x|}{N} \right) \right].
\]
The constants in (2.3) are best possible, and equality can be attained for nontrivial radial functions at any \( r \in [0, R] \).
PROOF. Let \( f \in \text{Lip}(B(0, R)) \). Then

\[
\tag{2.4} \left| f(x) - \int_{B(0, R)} f(y) \, dy \right| \\
\leq |f(x) - \bar{f}(r)| + \left| \int_{S^{N-1}} f(r\omega') \, d\omega' - \frac{N}{R^N} \int_{S^{N-1}} \int_0^R f(s\omega') s^{N-1} \, ds \, d\omega' \right| \\
\leq N(f) + \frac{N}{R^N} \int_{S^{N-1}} \left| \int_0^R |f(r\omega') - f(s\omega')| s^{N-1} \, ds \right| \, d\omega' \\
\leq N(f) + \frac{N}{R^N} \int_{S^{N-1}} \left| \frac{\partial f}{\partial r}(\omega') \right|_{L^\infty([0, R])} |s-r| s^{N-1} \, ds \, d\omega' \\
= N(f) + \frac{1}{N} \left[ \frac{2R^{N+1}}{N(N+1)} + R^N \left( \frac{R}{N+1} - \frac{r}{N} \right) \right].
\]

Then (2.3) follows since

\[
\left\| \frac{\partial f}{\partial r} \right\|_{L^\infty([0, R])} \leq \| \nabla f \|_{\infty}.
\]

In particular, a stronger form of (2.3) actually holds in all cases, with \( \| \nabla f \|_{\infty} \) replaced by \( \| \partial f/\partial r \|_{\infty} \). Let \( r \in [0, R] \) and \( g^*(z) = |z - r| \). We can view \( g^* \) as a radial function on \( B(0, R) \). Then

\[
g^{*r}(z) = \begin{cases} 
\text{sign}(z-r), & z \neq r, \\
1, & r = 0, \\
-1, & r = R.
\end{cases}
\]

Thus \( \| g^{*r} \|_{\infty} = 1 \). Therefore

\[
\text{L.H.S.}(2.3) = \left| g^*(z) - \frac{N}{R^N} \int_0^R g^*(s) s^{N-1} \, ds \right| = \left| |z-r| - \frac{N}{R^N} \int_0^R |s-r| s^{N-1} \, ds \right| \\
= \left| |z-r| - \frac{N}{R^N} \left[ \frac{2R^{N+1}}{N(N+1)} + R^N \left( \frac{R}{N+1} - \frac{r}{N} \right) \right] \right|.
\]

Also

\[
\text{R.H.S.}(2.3) = \frac{N}{R^N} \left[ \frac{2R^{N+1}}{N(N+1)} + R^N \left( \frac{R}{N+1} - \frac{r}{N} \right) \right].
\]

Hence equality holds in (2.3) at \( z = r \).

Note that the function \( g^*(z) = |z - r| \) is in \( C^1([0, R]) \) only for \( r = 0 \) and \( r = R \); for \( 0 < r < R \), \( g^* \in \text{Lip}([0, R]) - C^1([0, R]) \). Of course for \( 0 < r < R \), \( g^* \) can be approximated by \( C^1 \) functions, namely \( g_n(z) = |z - r|^{1+1/n} \).

REMARK 1. A key step in the proof is the fact that we can evaluate exactly

\[
Q(r) = \int_0^R |s-r| p(s) \, ds
\]
for $0 \leq r \leq R$, where $p$ is a nonnegative continuous function satisfying $\int_0^R p(s) \, ds = 1$. In the Ostrowski case ($N = 1$), $p(s) = 1/R$, while in our $N$-dimensional case, $p(s) = Ns^{N-1}/R^N$.

This works for many other cases including: linear combinations $p(s) = \sum_{j=1}^m a_j q_j$, where $a_j > 0$, $q_j \geq 0$ (not necessarily an integer) and

$$\sum_{j=1}^m a_j e^{q_j |x|} = 1;$$

and sums $p(s)$ of the form

$$a_j \sin(b_j s + c_j) + d_j \cos(e_j s + f_j),$$

where the coefficients are such that $p(s) \geq 0$ and $\int_0^R p(s) \, ds = 1$.

The space $\text{Lip}(\Omega) \cap C_0(\Omega)$ consists of all Lipschitz continuous functions on $\Omega$ vanishing on the boundary $\partial \Omega$ of $\Omega$. Note that

$$\text{Lip}(\Omega) \cap C_0(\Omega) = \{ f \in W^{1,\infty}(\Omega) \cap C(\overline{\Omega}) : f = 0 \text{ on } \partial \Omega \}$$

(cf. [5]).

Next comes our more general result where we consider functions over general domains.

**Theorem 2.2.** Let $f \in \text{Lip}(\Omega) \cap C_0(\Omega)$, where $\Omega$ is a bounded domain in $\mathbb{R}^N$. Extend $f$ by zero to $F$ on $B(0, R)$, the smallest ball centered at the origin and containing $\Omega$. Then for all $x \in \Omega$,

$$\left| f(x) - \int_\Omega f(y) \, dy \right| \leq N(F) + \left( 1 - \frac{\text{Vol}(\Omega)}{\text{Vol}(B(0, R))} \right) \left| \int_\Omega f(y) \, dy \right| + \frac{N}{R^N} \| \nabla f \|_{L^\infty(\Omega)} \left[ \frac{2|x|^{N+1}}{N(N+1)} + R^N \left( \frac{R}{N+1} - \frac{|x|}{N} \right) \right].$$

**Proof.** Let $R := \inf \{ R_0 > 0 : \Omega \subset B(0, R_0) \}$. Then

$$F(x) := \begin{cases} f(x), & x \in \Omega, \\ 0, & x \in B(0, R) - \Omega, \end{cases}$$

satisfies

$$F \in \text{Lip}(B(0, R)) \cap C_0(B(0, R)).$$

Then for $x \in \Omega$,

$$\left| f(x) - \int_\Omega f(y) \, dy \right| \leq \left| f(x) - \int_{B(0, R)} F(y) \, dy \right| + \left| \int_{B(0, R)} F(y) \, dy - \int_\Omega f(y) \, dy \right| = J_1 + J_2,$$
where
\[ J_1 := \left| F(x) - \int_{B(0, R)} F(y) \, dy \right|, \]
\[ J_2 := \left| \left( \frac{1}{\text{Vol}(B(0, R))} - \frac{1}{\text{Vol}(\Omega)} \right) \int_{\Omega} f(y) \, dy \right|. \]

By Theorem 2.1,
\[ J_1 \leq N(F) + \frac{N}{R^N} \| \nabla f \|_{\infty} \left[\frac{2|x|^{N+1}}{N(N + 1)} + R^N \left( \frac{R}{N + 1} - \frac{|x|}{N} \right)\right]. \]

and
\[ J_2 = \left[ 1 - \frac{\text{Vol}(\Omega)}{\text{Vol}(B(0, R))} \right] \left| \int_{\Omega} f(y) \, dy \right|. \]

This completes the proof of Theorem 2.2.

**Remark 2.** Note that \( \mathcal{N}(F) \) appears in (2.5). In this context, \( \mathcal{N}(f) \) does not make sense. Also, \( \mathcal{N}(F) \) need not be small (of course, it is small if \( f \) is approximately spherically symmetric).

Here is a simple example to illustrate that \( \mathcal{N}(F) \) can be large. Let \( x_0 \in \Omega \) and choose \( \varepsilon > 0 \) small enough so that \( B(x_0, \varepsilon) \subset \Omega \). Let \( f \in C^\infty(\Omega) \) have support in \( B(x_0, \varepsilon) \) and satisfy \( f(y) = g(\rho) \) where \( \rho = |y - x_0| \) for \( 0 \leq \rho \leq \varepsilon \). Assume further that \( g \) is nonincreasing and \( g(0) = f(x_0) = M > 0 \). Fix \( M \). Then

\[ 0 < f(x_0) - \int_{\Omega} f(y) \, dy \to M = \| f \|_{\infty} \quad \text{as} \ \varepsilon \to 0+. \]

**References**


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