Mathematical analysis. — Local clustering of the non-zero set of functions in $W^{1,1}(E)$, by Emmanuele DiBenedetto, Ugo Gianazza and Vincenzo Vespri.

Abstract. — We extend to the $p=1$ case a measure-theoretic lemma previously proved by DiBenedetto and Vespri for functions $u \in W^{1,p}(K_\rho)$ where $K_\rho$ is an $N$-dimensional cube of edge $\rho$. It states that if the set where $u$ is bounded away from zero occupies a sizeable portion of $K_\rho$, then the set where $u$ is positive clusters about at least one point of $K_\rho$.

Key words: $W^{1,1}$ functions; measure theory; positivity set.

Mathematics Subject Classification (2000): Primary 46E35; Secondary 26B35.

1. Introduction and statement of the result

For $\rho > 0$, denote by $K_\rho(y) \subset \mathbb{R}^N$ a cube of edge $\rho$ centered at $y$. If $y$ is the origin of $\mathbb{R}^N$, we write $K_\rho(0) = K_\rho$. For any measurable set $A \subset \mathbb{R}^N$, by $|A|$ we denote its $N$-dimensional Lebesgue measure. We prove the following Measure-Theoretic Lemma.

**Lemma.** Let $u \in W^{1,1}(K_\rho)$ satisfy

$$\|u\|_{W^{1,1}(K_\rho)} \leq \gamma \rho^{N-1} \quad \text{and} \quad |\{u > 1\}| \geq \alpha |K_\rho|$$

for some $\gamma > 0$ and $\alpha \in (0, 1)$. Then for every $\delta \in (0, 1)$ and $0 < \lambda < 1$ there exist $x_0 \in K_\rho$ and $\eta = \eta(\alpha, \delta, \gamma, \lambda, N) \in (0, 1)$ such that

$$|\{u > \lambda\} \cap K_{\eta \rho}(x_0)| > (1 - \delta)|K_{\eta \rho}(x_0)|.$$  

Roughly speaking, the Lemma asserts that if the set where $u$ is bounded away from zero occupies a sizeable portion of $K_\rho$, then there exists at least one point $x_0$ and a neighborhood $K_{\eta \rho}(x_0)$ such that $u$ remains large in a large portion of $K_{\eta \rho}(x_0)$. Thus the set where $u$ is positive clusters about at least one point of $K_\rho$.

The Lemma was established in [1] for $u \in W^{1,p}(K_\rho)$ and $p > 1$. Such a limitation on $p$ was essential to the proof. We give a new proof which includes the case $p = 1$ and is simpler.

2. Proof

It suffices to establish the Lemma for $u$ continuous and $\rho = 1$. For $n \in \mathbb{N}$ partition $K_1$ into $n^N$ cubes, with pairwise disjoint interiors and each of edge $1/n$. Divide these cubes into two finite subcollections $Q^+$ and $Q^-$ by
\[ Q_j \in \mathbf{Q}^+ \Rightarrow |u > 1 \cap Q_j| > \frac{\alpha}{2}|Q_j|, \]
\[ Q_i \in \mathbf{Q}^- \Rightarrow |u > 1 \cap Q_i| \leq \frac{\alpha}{2}|Q_i|, \]

and denote by \(#(Q^+)\) the number of cubes in \(\mathbf{Q}^+\). By the assumption,

\[
\sum_{Q_j \in \mathbf{Q}^+} |u > 1 \cap Q_j| + \sum_{Q_i \in \mathbf{Q}^-} |u > 1 \cap Q_i| > \alpha |K_1| = \alpha n^N |Q|
\]

where \(|Q|\) is the common measure of the \(Q_i\). From the definitions of the classes \(\mathbf{Q}^\pm\),

\[
a \alpha n^N < \sum_{Q_j \in \mathbf{Q}^+} |u > 1 \cap Q_j| + \sum_{Q_i \in \mathbf{Q}^-} |u > 1 \cap Q_i| < #(Q^+) + \frac{\alpha}{2}(n^N - #(Q^+)).
\]

Therefore

(2.1) \[
#(Q^+) > \frac{\alpha}{2} n^N.
\]

Fix \(\delta, \lambda \in (0, 1)\). The integer \(n\) can be chosen depending upon \(\alpha, \delta, \lambda, \gamma, \eta\) and \(N\), such that

(2.2) \[
|u > \lambda \cap Q_j| \geq (1 - \delta)|Q_j| \quad \text{for some } Q_j \in \mathbf{Q}^+.
\]

This would establish the Lemma for \(\eta = 1/n\). Let \(Q \in \mathbf{Q}^+\) satisfy

(2.3) \[
|u > \lambda \cap Q| < (1 - \delta)|Q|.
\]

Then there exists a constant \(c = c(\alpha, \delta, \gamma, \eta, N)\) such that

(2.4) \[
\|u\|_{W^{1,1}(Q)} \geq c(\alpha, \delta, \gamma, \eta, N) \frac{1}{n^{N-1}}.
\]

From the assumptions,

\[ |u \leq \lambda \cap Q| \geq \delta|Q| \quad \text{and} \quad \left|\left[ u \geq \frac{1 + \lambda}{2} \right] \cap Q \right| > \frac{\alpha}{2}|Q|.
\]

For fixed \(x \in [u \leq \lambda \cap Q] \) and \(y \in [u > (1 + \lambda)/2] \cap Q\),

\[
1 - \lambda \leq u(y) - u(x) = \int_0^{y-x} Du(x + t\omega) \cdot \omega \, dt \quad \text{where} \quad \omega = \frac{y-x}{|y-x|}.
\]

Let \(R(x, \omega)\) be the polar representation of \(\partial Q\) with pole at \(x\), for the solid angle \(\omega\). Integrate the previous relation with respect to \(y\) over \([u > (1 + \lambda)/2] \cap Q\). Minorize the resulting left hand side, by using the lower bound on the measure of such a set, and majorize the resulting integral on the right hand side by extending the integration over \(Q\). Expressing such integration in polar coordinates with pole at \(x \in [u \leq \lambda] \cap Q\) gives

\[
\frac{\alpha(1 - \lambda)}{4} |Q| \leq \int_{|\omega| = 1} \int_0^{R(x, \omega)} \int_0^{R(x, \omega)} \int_0^{z-x} |Du(x + t\omega)| \, dt \, dr \, d\omega
\]
\[
\leq N^{N/2} |Q| \int_{|\omega| = 1} \int_0^{R(x, \omega)} |Du(x + t\omega)| \, dt \, d\omega
\]
\[
= N^{N/2} |Q| \int_{Q} \frac{|Du(z)|}{|z-x|^{N-1}} \, dz.
\]
Integrate now with respect to $x$ over $[u \leq \lambda] \cap Q$. Minorize the resulting left hand side by using the lower bound on the measure of such a set, and majorize the resulting right hand side by extending the integration to $Q$. This gives

$$\frac{\alpha \delta (1 - \lambda)}{4N^{N/2}} |Q| \leq \|u\|_{W^{1,1}(Q)} \sup_{z \in Q} \int_Q \frac{1}{|z - x|^{N-1}} \, dx \leq C(N) \frac{|Q|^{1/N}}{\|u\|_{W^{1,1}(Q)}}$$

for a constant $C(N)$ depending only upon $N$.

If (2.2) does not hold for any cube $Q_j \in Q^+$, then (2.4) is satisfied for all such $Q_j$. Adding over such cubes and taking into account (2.1) gives

$$\frac{\alpha}{2 - \alpha} c(\alpha, \delta, \gamma, N)n \leq \|u\|_{W^{1,1}(K_1)} \leq \gamma.$$

REFERENCES


Received 15 September 2005, and in revised form 20 September 2005.