Some questions on quasinilpotent groups and related classes

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Abstract

In this paper we will prove that if $G$ is a finite group, $X$ a subnormal subgroup of $XF^*(G)$ such that $XF^*(G)$ is quasinilpotent and $Y$ is a quasinilpotent subgroup of $N_G(X)$, then $YF^*(N_G(X))$ is quasinilpotent if and only if $YF^*(G)$ is quasinilpotent. Also we will obtain that $F^*(G)$ controls its own fusion in $G$ if and only if $G = F^*(G)$.

The generalized Fitting subgroup $F^*(G)$ of a finite group $G$ is the product of the Fitting subgroup and the semisimple radical of $G$.

This generalized Fitting subgroup satisfies $C_G(F^*(G)) \leq F^*(G)$, for every finite group $G$. This property is similar to the corresponding one for the Fitting subgroup of a soluble group: $C_G(F(G)) \leq F(G)$. Quasinilpotent groups are those groups which coincide with their generalized Fitting subgroup. A group $G$ such that $F^*(G) = F(G)$ is a nilpotent-constrained group.

H. Bender stated that if $G$ is a nilpotent-constrained group, $X$ a subgroup of $G$ such that $XF(G)$ is nilpotent and $Y \leq N_G(X)$, then $YF(N_G(X))$ is nilpotent if and only if $YF(G)$ is nilpotent.

A well known theorem of Frobenius states that if a $p$-Sylow subgroup of $G$ controls its own fusion in $G$, then $G$ has a normal $p$-complement.

In this paper we will prove that if $G$ is a finite group, $X$ a subnormal subgroup of $XF^*(G)$ such that $XF^*(G)$ is quasinilpotent and $Y$ is a quasinilpotent subgroup of $N_G(X)$, then $YF^*(N_G(X))$ is quasinilpotent if and only if $YF^*(G)$ is quasinilpotent. Also we characterize when a nilpotent

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injector controls its own fusion in a nilpotent-constrained group or when a quasinilpotent injector controls its own fusion in a finite group.

**Notations.** All groups considered in this paper are assumed to be finite. The non-explicit notations are standard, see for instance [3]. We quote nevertheless the following:

- \( N \): class of nilpotent groups,
- \( S \): class of soluble groups,
- \( N^* \): class of quasinilpotent groups,
- \( F(G) \) is the Fitting subgroup of \( G \), i.e., the largest nilpotent normal subgroup of \( G \).

If \( \mathfrak{F} \) is a class of groups, \( S_n \mathfrak{F} = \{ G; G \triangleleft X \text{ for some } X \in \mathfrak{F} \} \), \( N_0 \mathfrak{F} = \{ G; G = \langle X_1, \ldots, X_n \rangle \text{ for some } X_i \trianglelefteq \trianglelefteq G, X_i \in \mathfrak{F}, 1 \leq i \leq n \} \).

A **Fitting class** \( \mathfrak{F} \) is an \( S_n \)- and \( N_0 \)-closed class, that is, a class such that \( \mathfrak{F} = S_n \mathfrak{F} = N_0 \mathfrak{F} \).

If \( \mathfrak{F} \) is a Fitting class, a subgroup \( H \) of \( G \) is an \( \mathfrak{F} \)-injector of \( G \) whenever \( H \cap N \) is an \( \mathfrak{F} \)-maximal subgroup of \( N \), for every subnormal subgroup \( N \) of \( G \). We denote by \( \text{Inj}_{\mathfrak{F}}(G) \) the set of all \( \mathfrak{F} \)-injectors of \( G \). The quasinilpotent injectors of a group \( G \) are characterized as the maximal quasinilpotent subgroups containing the generalized Fitting subgroup of \( G \) ([3]).

A group \( G \) is said to be **quasisimple** if \( G \) is perfect and \( G/Z(G) \) is simple.

The quasisimple subnormal subgroups of a (finite) group \( G \) are called the **components** of \( G \). The **semisimple radical** \( E(G) \) of \( G \) is the join of its components.

We will need the description of some properties about the semisimple radical of a group. As we did not find any complete reference to it in the literature, for the sake of being self-contained, we include the following:

**Lemma 1** Let \( G \) be a group. Then:

1. If \( F^*(G) \leq H \leq G \) it follows that \( E(H) = E(G) \).
2. If \( H \) is a subnormal subgroup of \( G \) then \( E(H) \) is the product of all components \( Q \) of \( G \) such that \( [Q, H] \neq 1 \). In particular \( E(G) \leq N_G(H) \).
3. If \( H \trianglelefteq H F^*(G) \), then \( E(N_G(H)) = E(G) \).
Proof. (1) Clearly $E(G) \leq E(H)$. As $F^*(G) \leq N_G(E(H))$ then $E(H) \leq E(G)$ ([2] 4.25), thus $E(G) = E(H)$.

(2) If $Q$ is a component of $G$ and $H \leq G$, then either $[Q, H] = 1$ or $Q \leq [Q, H]$ ([7], X 13.18). When $H \trianglelefteq G$ the second alternative implies that $Q \leq H$. Therefore $E(H)$ is the product of all components $Q$ of $G$ such that $[Q, H] \neq 1$. In consequence $E(G) \leq N_G(H)$.

(3) By ([2], 4.26) $E(N_G(H)) \leq E(G)$. On the other hand, by (1) and (2) we have that $E(H F^*(G)) = E(G) \leq N_G(H)$, thus $E(G) \leq E(N_G(H))$. Therefore $E(G) = E(N_G(H))$.

In [8] we proved the following result:

Suppose that $N$ is a nilpotent normal subgroup of $G$ and let $X$ be a nilpotent subgroup of $G$ satisfying $C_G(N \cap X) \leq X$. Then $N X$ is nilpotent.

As a consequence of this result, it is easy to obtain:

If $X$ is a subgroup of $F(G)$ and $Y$ is a nilpotent subgroup of $N_G(X)$ containing $F(N_G(X))$, then $Y F(G)$ is nilpotent.

A generalization of this result would be:

If $X F(G)$ is a nilpotent subgroup of $G$ and $Y$ a subgroup of $N_G(X)$ satisfying $Y F(N_G(X))$ is nilpotent, then $Y F(G)$ is nilpotent.

In [1] H. Bender had given an affirmative answer when $G$ is a nilpotent-constrained group. Next we will prove that this result is true without any restriction:

Proposition 2 Let $X \leq G$ with $X F(G)$ nilpotent and let $Y \leq N_G(X)$ with $Y F(N_G(X))$ nilpotent, then $Y F(G)$ is nilpotent.

Proof. Work by induction on the order of $G$.

If $R = F(G) N_G(X) < G$ then $X F(G) \leq R$ thus $X F(G) \leq F(R)$ and $X F(R) = F(R)$. Therefore, since $N_G(X) = N_R(X)$, by the inductive hypothesis, it follows that $Y F(R)$ is nilpotent, so $Y F(G)$ is nilpotent.

Thus we can suppose that $G = F(G) N_G(X)$, so $X F(G) \leq G$, then $X \leq F(G)$ and by the consequence of ([8], 2.2), it follows that $Y F(N_G(X)) F(G)$ is nilpotent, so $Y F(G)$ is nilpotent.
Proposition 3 Let $X$ be a quasinilpotent subgroup of $G$ satisfying $X \cap F^*(G) \leq F^*(G)$. If $C_{F^*(G)}(X \cap F^*(G)) \leq X$ then $X F^*(G)$ is quasinilpotent.

Proof. Since $U = X \cap F^*(G) \leq F^*(G)$ and $C_{F^*(G)}(U) \leq U$, by ([7], X 15.1) it follows that $U = E(G)(U \cap F(G))$ and $C_{F^*(G)}(U \cap F(G)) \leq U$.

Then

$$C_{F^*(G)}(F(X) \cap F(G)) = C_{F^*(G)}((X \cap F(G)) \leq U \cap F(G) = F(X) \cap F(G).$$

Next, we will prove that $F(X) F(G)$ is nilpotent. It suffices to show that $F(X) O_p(G)$ is nilpotent for every prime $p$ in order of $F(G)$. Consider the action of $(O_p(G) \cap O_p(X)) \times O_p(F(X))$ on $O_p(G)$. Since

$$C_{O_p(G)}(O_p(G) \cap O_p(X)) \leq C_{F^*(G)}(F(G) \cap F(X)) \leq F(X),$$

we have $C_{O_p(G)}(O_p(G) \cap O_p(X)) \leq O_p(X)$ and $O_p(F(X))$ acts trivially on $C_{O_p(G)}(O_p(G) \cap O_p(X))$. The Thompson’s $P \times Q$-lemma implies that $O_p(F(X))$ also acts trivially on $O_p(G)$. Then $F(X) O_p(G)$ is nilpotent.

On the other hand, since $E(X)$ is a quasinilpotent perfect $U$-invariant subgroup, by ([7], X 15.2), it follows that $E(X) \leq E(G)$ so $E(X) = E(G)$, then $X F^*(G) = E(X)(F(X) F(G))$ that is quasinilpotent. ■

Remarks.

1. Notice that, as the following example shows, the condition of subnormality in the above result is necessary.

Let $G = \text{GL}(2, 5)$ and $Z = Z(G)$. By ([6], II 7.3) there exists $X \leq G$, $X \cong C_2$ satisfying $C_G(X) = X$. If $D = X \cap \text{SL}(2, 5)$ then $|D| = 6$ and if $\langle x \rangle \leq D$ such that $o(x)|4$ then by ([10], page 163) $C_{\text{SL}(2, 5)}(\langle x \rangle) = D$. Since $F^*(G) = \text{SL}(2, 5)Z$, then:

$$C_{F^*(G)}(F^*(G) \cap X) = Z C_{\text{SL}(2, 5)}(\text{SL}(2, 5)Z \cap X) = Z C_{\text{SL}(2, 5)}(\text{SL}(2, 5) \cap X) \leq Z C_{\text{SL}(2, 5)}(\langle x \rangle) = ZD \leq X.$$

As $|X \text{SL}(2, 5)| = |G|$, it follows that $G = X \text{SL}(2, 5) = X F^*(G)$, that is not quasinilpotent.

2. It is easy to prove that Proposition 3 is equivalent to the following:

Let $H \leq G$ such that $C_{F^*(G)}(H \cap F^*(G)) \leq H$ and $H \cap F^*(G) \leq F^*(G)$. If $X$ is a quasinilpotent subgroup of $G$, such that $F^*(H) \leq X \leq H$, then $X F^*(G)$ is quasinilpotent.
Next we will obtain a version for quasinilpotent groups of ([12], 2.1).

Recall that if $N$ is a normal subgroup of $G$ and $\theta \in \text{Irr}(N)$, then $I_G(\theta) = \{g \in G|\theta^g = \theta\}$ is the stabilizer of $\theta$ in $G$.

**Corollary 4** Let $N$ be a quasinilpotent normal subgroup of $G$. Let $\theta \in \text{Irr}(N)$ and let $T = I_G(\theta)$ the stabilizer of $\theta$ in $G$. If $T \cap F^*(G) \trianglelefteq F^*(G)$ and $X$ is a quasinilpotent subgroup of $G$ satisfying $F^*(T) \leq X \leq T$ then $X F^*(G)$ is quasinilpotent.

**Proof.** Since $N C_G(N) \leq T$ we have

$$C_{F^*(G)}(F^*(G) \cap T) \leq C_{F^*(G)}(N \cap T) = C_{F^*(G)}(N) \leq T.$$  

Now, by Remark 2, we obtain that $X F^*(G)$ is quasinilpotent.  

**Corollary 5** If $X \trianglelefteq F^*(G)$ and $Y$ is a quasinilpotent subgroup satisfying $F^*(N_G(X)) \leq Y \leq N_G(X)$, then $Y F^*(G)$ is quasinilpotent.

**Proof.** Since $X \trianglelefteq F^*(G)$, by Lemma 1 (3), it follows that $E(G) = E(N_G(X))$, thus

$$N_G(X) \cap F^*(G) = E(G)(N_G(X) \cap F(G)) \leq E(G) F(N_G(X))$$

$$= F^*(N_G(X)) \leq Y.$$

Hence,

$$Y \cap F^*(G) = N_G(X) \cap F^*(G) \trianglelefteq F^*(G)$$

Then

$$C_{F^*(G)}(Y \cap F^*(G)) = C_{F^*(G)}(N_G(X) \cap F^*(G)) \leq C_{F^*(G)}(X) \leq N_{F^*(G)}(X)$$

$$= F^*(G) \cap N_G(X) \leq Y.$$

Therefore, by Proposition 3, it follows that $Y F^*(G)$ is quasinilpotent.  

The following example shows that, in the above result, the subnormality condition is necessary:

**Example.** Let $\Sigma_7 = A_7\langle(6,7)\rangle$ and $A_5 \leq A_7 = E(\Sigma_7) = F^*(\Sigma_7)$ (where $A_5$ is considered as the group of all even permutations of the set $\{1, 2, 3, 4, 5\}$). Clearly $N_{A_5}(A_5) = A_5\langle(6,7)\rangle$ and $F^*(N_{A_5}(A_5)) = A_5\langle(6,7)\rangle$ however $F^*(N_{A_5}(A_5)) F^*(\Sigma_7)$ coincides with $\Sigma_7$, that is not quasinilpotent.
**Theorem 6** Let $X \trianglelefteq X F^*(G)$ where $X F^*(G)$ is quasinilpotent and let $Y$ be a quasinilpotent subgroup of $N_G(X)$. Then $Y F^*(N_G(X))$ is quasinilpotent if and only if $Y F^*(G)$ is quasinilpotent.

**Proof.** Suppose that $Y F^*(N_G(X))$ is quasinilpotent. We argue by induction on $|G|$.

If $R = N_G(X) F^*(G) < G$, then $X F^*(G) \trianglelefteq R$ and $X F^*(G) \leq F^*(R)$, thus $X \trianglelefteq X F^*(G) \trianglelefteq F^*(R)$. Since $N_G(X) = N_R(X)$, by induction we obtain that $Y F^*(R)$ is quasinilpotent. Since $F^*(G) \leq Y F^*(R)$, by Lemma 1 (1) we have $E(Y F^*(R)) = E(G)$, thus $Y F^*(G) / E(G) \leq Y F^*(R) / E(G)$ is nilpotent so $Y F^*(G) \trianglelefteq Y F^*(R)$, then $Y F^*(G)$ is quasinilpotent.

Thus we can suppose that $G = N_G(X) F^*(G)$. Then $X F^*(G) \leq G$ and $G \trianglelefteq X F^*(G)$. Using Corollary 5 it follows that $Y F^*(N_G(X)) F^*(G)$ is quasinilpotent. Since $E(Y F^*(N_G(X)) F^*(G)) = E(G)$ we have $Y F^*(G)$ is a subnormal subgroup of $Y F^*(N_G(X)) F^*(G)$ and $Y F^*(G)$ is quasinilpotent as desired.

Assume now that $Y F^*(G)$ is quasinilpotent. As $Y F^*(G) / E(G)$ is nilpotent, then $Y E(G)$ is a subnormal quasinilpotent subgroup of $Y F^*(G)$. Write $Y_1 = Y E(G)$. Then $Y_1 \leq N_G(X)$ by Lemma 1 (3). Notice that $F(X), F(Y_1)$, $F(N_G(X))$ are subgroups of $C = C_G(E(G))$, that is the nilpotent-constrained radical of $G$. As $F(X) F(G), F(Y_1) F(G)$ are nilpotent subgroups of $C$ and $F(Y_1) \leq N_G(F(X))$ it follows from ([1]) that $F(Y_1) F(N_G(F(X)))$ is nilpotent.

On the other hand, as $X = F(X) E(X)$ and $E(X) \leq E(X F^*(G)) = E(G)$ it follows that $N_C(F(X)) = N_C(X)$. Moreover, $N_C(X) = C \cap N_G(X) \leq N_G(X)$, thus $F(N_C(X)) \leq F(N_G(X)) \leq C$, hence $F(N_C(X)) = F(N_G(X))$ and $F(Y_1) F(N_G(X))$ is nilpotent. As $Y_1 F^*(G) / E(G)$ is nilpotent, it follows that $E(Y_1) \leq E(G)$. Therefore $Y F^*(N_G(X)) = Y_1 F^*(N_G(X)) = F(Y_1) F(N_G(X)) E(G)$ is quasinilpotent.  

**Corollary 7** Let $X \trianglelefteq X F^*(G)$, where $X F^*(G)$ is a quasinilpotent subgroup of $G$ and let $Y$ be a quasinilpotent injector of $N_G(X)$. Then there exists a quasinilpotent injector $K$ of $G$ satisfying $K \cap N_G(X) = Y$.

**Proof.** By Theorem 6, $Y F^*(G)$ is quasinilpotent. Let $K$ be a maximal quasinilpotent subgroup of $G$ containing $Y F^*(G)$, then $K$ is a quasinilpotent injector of $G$. Thus $K = E(G) I$, where $I$ is a nilpotent injector of $C_G(E(G))$; hence $Y \leq K \cap N_G(X) = E(G) (I \cap N_G(X))$, that is quasinilpotent. Therefore $Y = K \cap N_G(X)$.  

Recall that, if $H \leq G$, it is said that $H$ controls its own $G$-fusion (briefly $H$ is c-closed in $G$), if any two elements of $H$, that are $G$-conjugate, are
already $H$-conjugate. It is well known the Frobenius theorem, that states that in a finite group $G$, a Sylow $p$-subgroup of $G$ is c-closed in $G$ if and only if $G$ has a normal $p$-complement. Also, C. Sah proved, in $\pi$-separable groups, an analogous result for Hall $\pi$-subgroups. We will prove corresponding results for nilpotent injectors in nilpotent-constrained groups and for quasinilpotent injectors in finite groups.

Lemma 8 Let $H$ be c-closed in $G$. Then:

(i) $H \leq K \leq G$ implies that $H$ is c-closed in $K$.

(ii) If $K \leq H \leq G$ and $K$ is c-closed in $H$ then $K$ is c-closed in $G$.

(iii) If $K \leq H$ and $K \leq G$ then $H/K$ is c-closed in $G/K$.

(iv) If $N \leq G$ and $(|N|, |H|) = 1$ then $HN/N$ c-closed $G/N$.

Proof. See ([13], 2.2)

Theorem 9 Let $G$ be a nilpotent-constrained group and let $I$ be a nilpotent injector of $G$. The following conditions are equivalent:

(i) $G$ is nilpotent.

(ii) $I$ is c-closed in $G$.

(iii) $F(G)$ is c-closed in $G$.

Proof. Clearly (i) implies (ii).

(ii) $\Rightarrow$ (iii) Let $p \in \pi(|I|)$. As $I$ is c-closed in $G$ it follows that $I_p$ is c-closed in $G$. Since $I_p \in \text{Syl}_p(C_G(O_p'(F(G))))$ by ([11], 1), then $I_p$ is c-closed in $C_p = C_G(O_p'(F(G)))$, thus $C_p$ is $p$-nilpotent $C_p = I_pO_p'(C_p) = I_pZ(O_p'(F(G)))$, therefore $I_p \leq C_p$ and then $I_p = O_p(C_p) = O_p(G)$.

Hence $F(G) = I$ and $F(G)$ is c-closed in $G$.

(iii) $\Rightarrow$ (i) Suppose that there exists $p \in \pi(|G|) \setminus \pi(|F(G)|)$. Let $P \in \text{Syl}_p(G)$, then $F(G)$ is a Hall $p'$-subgroup of $F(G)P$ and $F(G)$ is c-closed in $F(G)P$. Then, by ([13], 1), we obtain that $P \leq F(G)P$ so $P \leq C_G(F(G)) \leq F(G)$, that is a contradiction. Consequently, $\pi(|F(G)|) = \pi(|G|)$.

As $F(G)$ is c-closed in $G$, it follows that $O_p'(F(G))$ is c-closed in $G$, for every $p \in \pi(|G|)$. Take $P \in \text{Syl}_p(G)$, then $O_p'(F(G))$ is c-closed in $PO_p'(F(G))$, thus $P \leq PO_p'(F(G))$ by ([13], 1). Then $P \leq C_G(O_p'(F(G)))$ and by ([11], 1), we conclude that $G$ is nilpotent.  ■
Theorem 10 Let $I$ be a quasinilpotent injector of $G$. The following conditions are equivalent:

(i) $G$ is quasinilpotent.

(ii) $I$ is c-closed in $G$.

(iii) $F^*(G)$ is c-closed in $G$.

Proof. Clearly (i) implies (ii).

(ii) $\Rightarrow$ (iii) We know that $I = E(G) V$ where $V$ is a nilpotent injector of $C_G(E(G))$. Since $V$ is c-closed in $C_G(E(G))$, by Theorem 9, it follows that $C_G(E(G))$ is nilpotent. Therefore $C_G(E(G)) = F(G)$, and $I = E(G) F(G) = F^*(G)$.

(iii) $\Rightarrow$ (i) By induction on order of $G$. Suppose that $Z = Z(G) \neq 1$. Then, by Lemma 8 (iii) and the inductive hypothesis, we obtain that $G/Z = F^*(G)/Z = F^*(G)/Z$ so $G = F^*(G)$. Therefore, we can suppose that $Z = 1$. Since $G = F^*(G) C_G(x)$, for every $x \in F^*(G)$, we can conclude that $Z(F(G)) \leq Z(G) = 1$, thus $F(G) = 1$ and $F^*(G) = E(G)$.

Suppose that $E(G) \leq L$, where $L$ is a maximal subgroup of $G$. By Lemma 1 (1) it follows that $E(G) = E(L)$, and as $F(L) \leq C_G(E(G)) = Z(E(G)) = 1$ we conclude, by induction, that $E(G) = L$. Then $E(G)$ is a maximal subgroup of $G$, so there exists a prime $p$ such that $|G/E(G)| = p$.

Let $Q$ be a component of $G$. Since $G = E(G) C_G(x)$ for all $x \in E(G)$, it follows that $Z(Q) = 1$. Therefore $E(G) = Q_1 \times \ldots \times Q_r$, where $Q_1, \ldots, Q_r$ are the components of $G$ which are nonabelian simple groups. Also they are c-closed in $G$.

Let $i \in \{1, \ldots, r\}$ $g \in G$ and let $\alpha_g$ be the inner automorphism of $G$ determined by $g$. Since $Q_i \trianglelefteq G$, one has that the restriction $\alpha_g|_{Q_i}$ is an automorphism of $Q_i$. Note that $\alpha_g(C) = C$ for any conjugacy class $C$ of $Q_i$; hence, by ([4], Theorem C), there exists $z_i \in Q_i$ such that $\alpha_g(x) = x^{z_i}$ for every $x \in Q_i$. If $x \in E(G)$, then $x = x_1 \ldots x_r$, $x_i \in Q_i$, $1 \leq i \leq r$. Thus, $x^g = x_1^g \ldots x_r^g = x_1^{z_1} \ldots x_r^{z_r} = x_1^z \ldots x_r^z = x^z$, where $z = z_1 \ldots z_r \in E(G)$. Therefore $\alpha_g|_{E(G)}$ is the inner automorphism of $E(G)$ of $G$ determined by $z = z_1 \ldots z_r$. It follows from ([7], X 13.1) that $G$ is quasinilpotent. ■

Corollary 11 If $G$ is a group then

$$F^*(G) = \bigcap_{\theta \in \text{Irr}(F^*(G))} I_G(\theta).$$
Proof. Work by induction on $|G|$. We know that

$$F^*(G) \leq \bigcap_{\theta \in \text{Irr}(F^*(G))} I_G(\theta) = N \leq G.$$  

Suppose that $N < G$; since $F^*(G) = F^*(N)$, by induction, it follows that

$$F^*(G) = F^*(N) = \bigcap_{\theta \in \text{Irr}(F^*(N))} I_G(\theta) = N.$$  

Therefore, we can suppose that $N = G$. Then $I_G(\theta) = G$, for all $\theta \in \text{Irr}(F^*(G))$. Let $\text{Irr}(F^*(G)) = \{\theta_1, \theta_2, ..., \theta_m\}$. Suppose that $x, y \in F^*(G)$ with $x^g = y$, for some $g \in G$, then

$$\theta_i(x) = \theta_i(x^g) = \theta_i(y), \quad 1 \leq i \leq m.$$  

Thus $\sum \theta_i(x)\theta_i(y^{-1}) = \sum \theta_i(x)\theta_i(y^{-1}) \neq 0$. Then $x$ and $y$ are conjugate in $F^*(G)$ and, in consequence, $F^*(G)$ is c-closed in $G$. Then, using Theorem 10, it follows that $G = F^*(G)$ as desired.  

\[\square\]

Corollary 12 If $G$ is a nilpotent-constrained group then

$$F(G) = \bigcap_{\theta \in \text{Irr}(F(G))} I_G(\theta).$$  

Proof. Since $G$ is a nilpotent-constrained group, we have $F^*(G) = F(G)$. Now apply the above result.  

\[\square\]

Corollary 13 Let $\mathfrak{F}$ be a Fitting class such that $\mathfrak{N} \subseteq \mathfrak{F} \subseteq \mathfrak{N}^*$ and let $G$ be an $\mathfrak{F}$-constrained group (i.e. $C_G(G_{\mathfrak{F}}) \leq G_{\mathfrak{F}}$). If $I \in \text{Inj}_{\mathfrak{F}}(G)$, the following statements are equivalent:

(i) $G \in \mathfrak{F}$.

(ii) $I$ is c-closed in $G$.

(iii) $G_{\mathfrak{F}}$ is c-closed in $G$.

Proof. Note that, as $G$ is an $\mathfrak{F}$-constrained group, by ([9], 2), we have $F^*(G) = G_{\mathfrak{F}}$ and, by ([9], 8), $\text{Inj}_{\mathfrak{F}}(G) = \text{Inj}_{\mathfrak{F}^*}(G)$. Now the result follows from Theorem 10.  

\[\square\]
Remarks.

1. The last results suggest that, perhaps, one can obtain a general result for any Fitting class, but there exist Fitting classes of full characteristic and finite groups, do not belong to the corresponding Fitting class, but whose injectors are c-closed:

Consider $G = A_5$ and $\mathfrak{F} = \mathfrak{S}$. If $S \in \text{Syl}_5(G)$ then $N_G(S) \cong D_{10}$ is an $\mathfrak{F}$-injector of $G$. Moreover $N_G(S)$ is c-closed in $A_5$. Indeed, let $x \in N_G(S) \setminus \{1\}$ and $g \in G$ such that $x^g \in N_G(S)$.

If $\text{o}(x) = 5$, then $\langle x \rangle = S$ and we obtain that $g \in N_G(S)$.

If $\text{o}(x) = 2$, then $\langle x \rangle, \langle x^g \rangle$ are Sylow 2-subgroups of $N_G(S)$, thus there exists $h \in N_G(S)$ such that $\{1, x^g\} = \langle x^g \rangle = \langle x^h \rangle = \{1, x^h \}$ and so $x^g = x^h$.

2. Even more, there exist Fitting classes of soluble groups with full characteristic and soluble groups, do not belong to the corresponding Fitting class, but whose injectors are c-closed:

If $G$ is a soluble group, we define an homomorphism $d_G : G \rightarrow GF(5)^*$ as follows: let $M_1, M_2, \ldots, M_r$ the 5-chief factors of a prefixed chief series of $G$. If $g \in G$ and $d_i(g)$ denotes the determinant of the linear map which $g$ induces on $M_i$, then

$$d_G(g) = \prod_{i=1}^{r} d_i(g)$$

The class $\mathfrak{F} = \{G \in \mathfrak{S} | d_G(G) = 1\}$ is a normal Fitting class ([3], IX 2.14).

Let

$$A = \langle \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix} \rangle \leq GL(2, 5).$$

Consider $A$ acting in the natural way on $C_5 \times C_5$. Let $G$ be the semidirect product of $C_5 \times C_5$ by $A$:

$$G = [C_5 \times C_5] \langle \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix} \rangle$$

and let

$$S = [C_5 \times C_5] \langle \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \rangle.$$

Observe that $|G| = 2^4 \cdot 5^2 = 400$ and $|S| = 2^3 \cdot 5^2$.

We will see that $S$ is c-closed in $G$.

We have that $G = S \langle \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \rangle$ and if $S_2 = \langle \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \rangle \in \text{Syl}_2(S)$ then $\langle \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \rangle \leq C_G(S_2)$. 

Hence, if \( x \in S_2 \) and \( g \in G \), we have \( g = cs \), where \( c \in \langle \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \rangle \), so \( x^g = x^{cs} = x^s \). Therefore \( G = C_G(x)S \).

Let \( x \in S \). Since \( S \) does not have composed order elements, \( x \) is a 2-element or a 5-element.

If \( x \) is a 2-element then \( x = y^s \) where \( y \in S \) and \( s \in S \). Hence \( C_G(x)S = C_G(y^s)S = (C_G(y))^sS = (C_G(y))^sS = G \). Thus, if \( g \in G \), it follows that \( g = ls \), where \( l \in C_G(x) \), \( s \in S \). Then \( x^g = x^{ls} = x^s \).

If \( x \) is a 5-element, then \( x \in C_5 \times C_5 \). We will see that \( G = C_G(x)S \). It is enough to prove that there exists \( g \in G \setminus S \) such that \( g \in C_G(x) \).

If \( H \leq G \) write \( H^* = H \setminus \{1\} \). Then

\[
(C_5 \times C_5)^* = \langle h_1 \rangle^* \cup \langle h_2 \rangle^* \cup \langle h_3 \rangle^* \cup \langle h_4 \rangle^* \cup \langle h_5 \rangle^* \cup \langle h_6 \rangle^*
\]

where \( h_1 = (1,0), \ h_2 = (0,1), \ h_3 = (1,1), \ h_4 = (2,1), \ h_5 = (1,2), \ h_6 = (4,1) \).

Notice that if \( h \in \langle h_i \rangle \), then \( C_G(h) = C_G(h_i) \) for \( 1 \leq i \leq 6 \) and it is enough to show that \( G = C_G(h_i)S \), \( 1 \leq i \leq 6 \).

We have,

\[
\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \in C_G(h_1) \setminus S, \quad \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \in C_G(h_2) \setminus S, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in C_G(h_3) \setminus S, \\
\begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix} \in C_G(h_4) \setminus S, \quad \begin{pmatrix} 0 & 3 \\ 0 & 2 \end{pmatrix} \in C_G(h_5) \setminus S, \quad \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix} \in C_G(h_6) \setminus S,
\]

Hence \( G = C_G(x)S \). Therefore, if \( g \in G, \ g = cs \), where \( c \in C_G(x) \) and \( s \in S \). Then \( x^g = x^{cs} = x^s \). Thus, \( S \) is c-closed in \( G \).

Now consider the chief series of \( G \):

\[
1 \trianglelefteq C_5 \times C_5 \trianglelefteq [C_5 \times C_5](\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}) \trianglelefteq [C_5 \times C_5](\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}) \trianglelefteq S \trianglelefteq G.
\]

The only 5-chief factor of this series is \( C_5 \times C_5 \).

Notice that \( G \notin \mathfrak{F} \) since \( \det \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 4 \neq 1 \).

The part of the above series from 1 to \( S \) is a chief series of \( S \) and the only 5-chief factor of this series is \( C_5 \times C_5 \).
Since \( \det \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} = 1 \) and \( \det \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = 1 \), it follows that \( S \in \mathfrak{F} \), then \( S \in \text{Iny}_\mathfrak{F}(G) \) is c-closed in \( G \), but \( G \notin \mathfrak{F} \).

3. It is said that a subgroup \( H \) in a group \( G \) has property CR (Character Restriction) if every ordinary irreducible character \( \theta \in \text{Irr}(H) \) is the restriction \( \chi_H \) of some \( \chi \in \text{Irr}(G) \). It is well known that if \( H \) satisfies CR property in \( G \) then \( H \) is c-closed in \( G \).

A number of authors have shown that property CR, together with suitable additional hypothesis on \( H \) and \( G \), does imply the existence of a normal complement for \( H \). For instance Hawkes and Humphreys ([5]) prove that CR yields a normal complement if \( G \) is solvable and \( H \) is an \( \mathfrak{F} \)-projector for \( G \), where \( \mathfrak{F} \) is any saturated formation. The last example shows that the corresponding result for Fitting classes and injectors satisfying property CR, does not work.

References


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