Pluriharmonic interpolation
and hulls of $C^1$ curves
in the unit sphere

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1. Introduction.

Let $\Gamma$ be a simple closed $C^1$ curve lying in the unit sphere $\partial \mathbb{B}^n$, $\mathbb{B}^n$ being the unit ball in $\mathbb{C}^n$. By Stolzenberg's theorem [12], either $\Gamma$ is polynomially convex or $\widehat{\Gamma} \setminus \Gamma$ is a 1-dimensional analytic subvariety of $\mathbb{B}^n$. Čirka [6] and Forstnerič [7] showed that if $\Gamma$ is $C^2$ and not polynomially convex, then $\widehat{\Gamma} \setminus \Gamma$ is smooth near $\Gamma$ (i.e., singularities do not accumulate at $\Gamma$) and $\Gamma$ is transverse at each of its points in the sense that the tangents to $\Gamma$ never lie in the complex tangent space to $\partial \mathbb{B}^n$. In particular, if $\Gamma$ is $C^2$ and has at least one complex tangent, then $\Gamma$ is polynomially convex. This is no longer true when $\Gamma$ is only $C^1$: Rosay [9] constructed a $C^1$ Jordan curve in $\partial \mathbb{B}^2$, bounding an analytic disk in $\mathbb{B}^2$, and having a complex tangent at a single point. Motivated by this, the first author proved in [1] that a rectifiable curve $\Gamma$ is polynomially convex if the set of points of $\Gamma$ where its tangent (these exist almost everywhere) is complex-tangential has positive linear measure.

On the other hand, Berndtsson and Bruna [4] showed that, when $\Gamma$ is of class $C^3$ (in fact, $C^{2+\epsilon}$ was enough), the functions in $C(\Gamma)$ which can be interpolated by functions pluriharmonic on $\mathbb{B}^n$ and continuous on $\overline{\mathbb{B}^n}$ form a closed subspace of $C(\Gamma)$ of finite codimension. When $\Gamma$ is polynomially convex, this codimension is zero. When $\Gamma$ is not polynomially convex, then by Forstnerič's result, $\widehat{\Gamma} \setminus \Gamma$ is smooth near
\( \Gamma \) and the Berndtsson-Bruna theorem is related with the solvability of the Dirichlet problem in \( \tilde{\Gamma} \setminus \Gamma \). In fact, Shcherbina [11] later used this approach to characterize the codimension, for \( n = 2 \) and for \( C^\infty \) curves, in terms of the topology of \( \Gamma \).

Here we shall study the \( C^1 \) case of both types of problems-hulls and pluriharmonic interpolation. Our first result (Theorem 2.1) states that if \( \Gamma \) is not polynomially convex, then \( \tilde{\Gamma} \setminus \Gamma \) is still nice near \( \Gamma \). In fact, close to \( Q \in \Gamma, \tilde{\Gamma} \setminus \Gamma \) is the graph, over its projection on a suitable complex line, of a holomorphic function -the complex line being the normal to the sphere in the case when the tangent to \( \Gamma \) is transverse and being a complex tangent line to the sphere at \( Q \) otherwise. From this local parametrization we deduce in Section 3 our second result: If \( \Gamma \) is not polynomially convex and \( T(s) \) is the so-called index of transversality of \( \Gamma \) (i.e., \( i T(s) \) is the complex normal component of the unit tangent to \( \Gamma \)) then \( T(s) \) is greater or equal than 0 (after a possible change of orientation of \( \Gamma \)) and

\[
\int \frac{ds}{T(s)^p} < \infty,
\]

for all \( p > 0 \). This captures both Forstneric's result (because, in the \( C^2 \) case, \( T(s) = O(|s - s_0|) \) close to a complex-tangential point \( \gamma(s_0) \)) and the \( C^1 \) version of the Theorem in [1] for rectifiable curves.

Finally, in Section 4, we prove, under the hypotheses that \( T \) has constant sign and that \( \int T^{-1} ds \) converges, that the Berndtsson-Bruna result on pluriharmonic interpolation carries over to \( C^1 \) curves; in particular, this holds for all non-polynomially convex \( C^1 \) curves.

To simplify the exposition, we assume in the rest of the paper that \( n = 2 \). It is routinely checked that all proofs generalize to \( n > 2 \) with straightforward modifications.

### 2. The local structure of the hull.

Let \( \Gamma \) be a simple closed curve of class \( C^1 \) lying on the unit sphere \( S = bB^2 \), with arc-length parametrization \( \gamma(s) \). We assume that \( \Gamma \) is not polynomially convex. By Stolzenberg's theorem (see [12] or [13, Theorem 30.1], [15, Chapter 13]), \( V = \tilde{\Gamma} \setminus \Gamma \) is a one-dimensional analytic variety. We will prove here:

**Theorem 2.1.** For each point \( Q \in \Gamma \) there is a neighbourhood \( N \) and a complex line \( L \) through \( Q \) such that if \( \pi \) is the projection on \( L \) one
has:

a) \( \pi \) is one-to-one from \( \bar{V} \cap N \) onto a domain \( \bar{U} \subset L \) of class \( C^1 \), and \( \pi \) maps \( \Gamma \cap N \) onto an arc \( \tau \subset bU \).

b) There is an holomorphic function \( f \) in \( U \) of class \( C^1(\bar{U}) \) such that \( \bar{V} \cap N \) is the graph of \( f \) over \( U \).

Thus \( V \) is locally a graph in the neighbourhood of \( \Gamma \). In the proof of Theorem 2.1 we need the following "general principle" (see [12], [2], [3], [5] for the original argument; see also [15, Theorem 10.7], and [13, Lemmas 30.7 and 30.9]):

**Lemma 2.2.** Let \( X \subset \mathbb{C}^2 \) be compact and \( p \) a polynomial. Let \( \Omega_\infty \) be the unbounded component of \( \mathbb{C} \setminus p(X) \). Suppose that there is an open Jordan arc \( \sigma \), open in \( p(X) \), such that

a) \( \sigma \subset b \Omega_\infty \cap b \Omega \), where \( \Omega \) is a bounded component of \( \mathbb{C} \setminus p(X) \).

b) \( p^{-1}(\lambda) \cap X \) contains exactly one point for all \( \lambda \in \sigma \).

Then, either \( p^{-1}(\Omega) \cap \hat{X} \) is empty or \( p^{-1}(\Omega) \cap \hat{X} \) is single sheeted. In the later case, there exists \( \phi \in H^\infty(\Omega, \mathbb{C}^2) \) such that

\[
p^{-1}(\Omega) \cap \hat{X} = \{ \phi(\lambda) : \lambda \in \Omega \}.
\]

Moreover, there are no points of \( \hat{X} \setminus X \) over \( \sigma \), and \( \phi \) has a continuous extension to \( \sigma \).

In case \( p \) is a coordinate function, say \( p(z) = z_1 \), then \( \phi(\lambda) = (\lambda, f(\lambda)) \), so over \( \Omega \), \( \hat{X} \setminus X \) is the graph of \( f \in H^\infty(\Omega) \).

**Proof of Theorem 2.1.** We shall distinguish two cases:

**Case A:** \( \Gamma \) is transverse at \( Q \), i.e. the tangent to \( \Gamma \) at \( Q \) has a non-zero complex normal component. We can assume without loss of generality that \( Q = (1,0) = \gamma(0) \) and transversality means that \( \gamma'(0) \) is a non-zero (pure imaginary) number.

Then \( \gamma_1(s) \) determines \( s \) for \( |s| \) small enough, say \( |s| < \varepsilon \). Since \( \Gamma \) is simple, \( \gamma_1(s) \neq 1 \) for \( |s| \geq \varepsilon \). Hence, shrinking \( \varepsilon \) if needed we see that the points of

\[
\sigma \overset{\text{def}}{=} \{ \gamma_1(s) : |s| < \varepsilon \}
\]

are
are covered only once by $z_1$ on $\Gamma$. We also assume that $\varepsilon$ is small enough so that $\sigma$ is a $C^1$-curve (because $\gamma'_1(0) \neq 0$). We apply Lemma 2.2 with $X = \Gamma$ (note that $\sigma \subset b\Omega_\infty$ because $1 \in b\Omega_\infty$), $p(z) = z_1$, and therefore over $\Omega$, the bounded component of $z_1(\Gamma)$ having $\sigma$ in the boundary, $V$ is the graph of some holomorphic function $f$. On $\sigma$

$$f(\gamma_1(s)) = \gamma_2(s), \quad |s| < \varepsilon.$$  

Thus $f$ is of class $C^1$ on $\sigma$. Now we take as $U$ a $C^1$ domain in the $z_1$-plane contained in $\Omega$ and such that $bU \cap b\Omega \overset{\text{def}}{=} \tau \subset \sigma$.

At this point we need:

**Lemma 2.3.** Let $U$ be a $C^1$-domain in the complex plane, let $f$ be holomorphic in $U$, continuous on $\overline{U}$. Let $\tau \subset bU$ be an arc on which $f|_{\partial U}$ is of class $C^1$. Then $f'$ extends continuously to the points of $\tau$.

**Proof.** Let $g: \overline{\Delta} \to \overline{U}$ be the Riemann mapping function from the unit disk $\Delta$ to $U$. Let $I \subset T = b\Delta$ the arc mapped onto $\tau$. Let $\tau' \subset \tau$ be a closed subarc of $\tau$ and $I' \subset I$ its corresponding arc in $\mathbb{T}$. It is well-known ([8, Theorem 10.1]) that $\arg g'$ has a continuous extension to $\Delta$, hence $\log g' \in \text{VMOA}$, and so $g'$ and $1/g'$ are in $L^p(T)$ for all $p > 0$. Let $h = f \circ g$, which is in the disc algebra. The hypothesis implies that $h$ is absolutely continuous in $I$ with derivative

$$h' = (f' \circ g) g'$$

at almost all points of $I$, and $h'$ is in $L^p_{\text{loc}}(I)$ because $f'$ is continuous on $\tau$. Assume without loss of generality that $I = T \cap D(1, r)$, $I' = T \cap D(1, r')$. Let $\chi$ be a $C^\infty$ function supported in $D(1, \tau)$ equal to 1 on $D(1, r')$. We consider

$$H(z) = \frac{1}{2\pi i} \int_T \frac{\chi(\zeta) h(\zeta)}{\zeta - z} \, d\zeta, \quad z \in \Delta.$$  

Note that

$$H(z) = \chi(z) h(z) - \frac{1}{2\pi i} \int_T \frac{\chi(z) - \chi(\zeta)}{\zeta - z} h(\zeta) \, d\zeta$$

and that this last integral defines a smooth function on $\overline{\Delta}$. Therefore $H$ is in the disk algebra and on $T$ it is an absolutely continuous function.
with derivative in \( L^p(\mathbb{T}) \). This implies that \( H' \) is in the Hardy class \( H^p, p > 0 \), or which is the same, the non-tangential maximal function over the Stolz angle \( S(\theta) \)

\[
(H')^*(\theta) = \sup\{|H'(z)| : z \in S(\theta)\}
\]

belongs to \( L^p(\mathbb{T}), p > 0 \). From this it follows that

\[
(h')^*(\theta) = \sup\{|h'(z)| : z \in S(\theta)\}
\]

is in \( L^p(\mathbb{T}) \) for all \( p > 0 \) and hence is in \( L^p_{\text{loc}}(I) \) for all \( p > 0 \).

Since also \((g')^{-1}\) has non-tangential maximal function in \( L^p(\mathbb{T}) \) for all \( p > 0 \), we conclude that \((f' \circ g)^* \in L^1_{\text{loc}}(I)\). We will show now that \( f' \circ g \) extends continuously to all points of \( I \), which obviously implies the lemma. Fix a closed subarc \( J \subset I \) and let \( D \) be a \( C^\infty \)-domain in \( \Delta \) such that \( J \subset bD \cap \mathbb{T} \subset I \). Then \( f' \circ g \) has non-tangential maximal function (with respect to \( D \)) in \( L^1(bD) \) and therefore belongs to \( H^1(D) \). Now, \( f' \circ g \) is continuous in \( bD \cap \mathbb{T} \) and so \( f' \circ g|_D \) extends continuously to the closure of \( D \) (here we use the fact that \( D \) being a \( C^\infty \) domain the holomorphic function theory of \( D \) is analogous to the one of \( \Delta \)). By the choice of \( D \), it then follows that \( f' \circ g \) extends continuously to all points of \( J \).

Shrinking the domain \( U \) a bit we conclude the proof of Theorem 2.1 in the case A.

\textit{Case B.} \( \Gamma \) is complex-tangential at \( Q \), i.e. the tangent to \( \Gamma \) at \( Q \) points in the complex-tangential direction. We can assume, without loss of generality, that \( Q = (1,0) = \gamma(0) \) and that \( \gamma_2'(0) = 1, \gamma_1'(0) = 0 \). It follows immediately that there is \( \varepsilon > 0 \) such that

\[
s \mapsto |\gamma_2(s)|
\]

is strictly increasing in \((0, \varepsilon)\) and strictly decreasing in \((-\varepsilon, 0)\).

Since \( |\gamma_1(s)|^2 + |\gamma_2(s)|^2 = 1 \),

\[
s \mapsto |\gamma_1(s)|
\]

is strictly decreasing in \((0, \varepsilon)\) and strictly increasing in \((-\varepsilon, 0)\). Let us define

\[
\sigma_+ = \{\gamma_1(s) : 0 < s < \varepsilon\},
\]
\[ \sigma_+ = \{ \gamma_1(s) : -\varepsilon < s < 0 \}, \]
\[ \sigma = \{ \gamma_1(s) : -\varepsilon < s < \varepsilon \}. \]

As before, since $\Gamma$ is simple, and shrinking $\varepsilon$ if needed we may assume that $\gamma_1(s) \not\in \sigma$ for $|s| \geq \varepsilon$.

We know that both $\sigma_+, \sigma_-$ meet each circle $|\lambda| = \rho$ at most once. We claim that $\sigma_+, \sigma_-$ do not intersect. This is seen as in [9]: suppose that $\sigma_+$ and $\sigma_-$ meet at $\lambda_0 = \gamma_1(a) = \gamma_1(b)$, with $0 < a < \varepsilon$, $-\varepsilon < b < 0$. Let $\nu = \gamma_1(b, a)$. Let $R$ be a smooth simply connected domain in the $z_1$ plane separating $\nu$ from $z_1(\Gamma) \setminus \nu$ and containing $z_1(\Gamma) \setminus \nu$. The domain $R$ admits a peaking function $H(\lambda)$ for the point $\lambda_0$. The function equal to $H$ on $\overline{R}$ and to 1 in the domain bounded by $\nu$ can be uniformly approximated, by Mergelyan's theorem, by a sequence of polynomials $p_n(\lambda)$. This shows that the arc $\Gamma_1 = \gamma((b, a)) \subset \Gamma$ is a peak set for the algebra $P(\Gamma)$. Analogously, $\Gamma_2 = \Gamma \setminus \Gamma_1$ is also a peak set for $P(\Gamma)$. But $\Gamma_1, \Gamma_2$ are smooth arcs and so $P(\Gamma_1) = C(\Gamma_1)$, $P(\Gamma_2) = C(\Gamma_2)$. By general theory of uniform algebras (in fact an easy duality argument works), it follows that $P(\Gamma) = C(\Gamma)$ and $\Gamma$ would be polynomially convex.

Therefore, $\sigma_+$ and $\sigma_-$ do not meet, which means that $\gamma_1$ is one to one in $(-\varepsilon, \varepsilon)$ and Lemma 2.2 applies again as before. The main difference here with respect the situation in case A is that here the curve $\sigma$ is in general not smooth at 1. This is why the $z_1$ projection does not work in this case and we shall look now to the $z_2$ projection.

Let $g$ be the holomorphic function on $\Omega$ given by Lemma 2.2 so that $\lambda \mapsto (\lambda, g(\lambda))$ parametrizes $V$ over $\Omega$. Note that on $\sigma$
\[ g(\gamma_1(s)) = \gamma_2(s), \quad |s| < \varepsilon, \]
defines a curve $\tau$ in the $z_2$-plane which we can assume smooth because $\gamma_2(0) = 1$.

We denote, for small $\delta$,
\[ \Omega_\delta = \{ \lambda \in \Omega : |\lambda| > 1 - \delta \}. \]
The function $g$ extends continuously to $\overline{\Omega_\delta}$, and
\[ V_\delta = \{ (\lambda, g(\lambda)) : \lambda \in \overline{\Omega_\delta} \}\]
is a neighbourhood of $Q = (1, 0)$ in $\hat{\Gamma}$. The boundary $b\Omega_\delta$ consists of
\[ C_\delta = \{ \lambda \in \Omega : |\lambda| = 1 - \delta \}\]
and two arcs $\sigma_+^\delta$, $\sigma_-^\delta$ included respectively in $\sigma_+, \sigma_-$. We denote $\sigma^\delta = \sigma_+^\delta \cup \sigma_-^\delta \cup \{1\} = \gamma_1(I_\delta)$ and $\tau^\delta = \gamma_2(I_\delta) = g(\sigma^\delta)$, a smooth subarc of $\tau$.

We claim that for small enough $\delta$, $z_2$ does not vanish at $V_\delta$ except at $Q$, i.e. $g$ does not have zeros in $\Omega_\delta$. To see this, choose first $\delta$ such that $g$ does not vanish on $\overline{C_\delta}$. $\Omega_\delta$ is a simply-connected rectifiable domain and $g(b \Omega_\delta)$ is a closed piecewise smooth curve containing the arc $\tau^\delta = g(\sigma^\delta)$. Since $g(C_\delta)$ does not pass through 0 in the neighbourhood of 0 there are two components of $\mathbb{C} \setminus g(b \Omega_\delta)$, which we call $R_+^\delta$ and $R_-^\delta$. Let $m_+, m_-$ be the number of preimages (counting multiplicities) in $\Omega_\delta$ of points of $R_+^\delta$, $R_-^\delta$, respectively. Let $N = \max\{m_+, m_\}$.

If $\lambda$ is a zero of $g$ in $\Omega_\delta$, as $g$ is an open mapping, the image of a neighbourhood of $\lambda$ is a neighbourhood of 0 and hence meets both $R_+^\delta$ and $R_-^\delta$. Therefore there are at most $N$ zeros of $g$ in $\Omega_\delta$, and $g$ has no zeros in $\Omega_\delta$ for small enough $\delta$.

We will show next that $m_+ - m_-$ is either $+1$ or $-1$. We have

$$m_+ = \frac{1}{2\pi} \Delta_{\Omega_\delta} \arg(g(\lambda) - a), \quad a \in R_+^\delta,$$

$$m_- = \frac{1}{2\pi} \Delta_{\Omega_\delta} \arg(g(\lambda) - b), \quad b \in R_-^\delta,$$

or

$$2\pi m_+ = \Delta_{g(\Omega_\delta)} \arg(\zeta - a)$$
$$= \Delta_{g(C_\delta)} \arg(\zeta - a) + \Delta_{\tau^\delta} \arg(\zeta - a),$$

$$2\pi m_- = \Delta_{g(\Omega_\delta)} \arg(\zeta - b)$$
$$= \Delta_{g(C_\delta)} \arg(\zeta - b) + \Delta_{\tau^\delta} \arg(\zeta - b).$$

Recall that $\tau^\delta$ is a smooth curve. Now substract both equations and make $a, b \to 0$ to get

$$2\pi (m_+ - m_-) = \lim_{a \to 0} \Delta_{\tau^\delta} \arg(\zeta - a) - \lim_{b \to 0} \Delta_{\tau^\delta} \arg(\zeta - b) = \pm 2\pi.$$

If instead we add the equations we get

$$2\pi (m_+ + m_-) = 2 \Delta_{g(C_\delta)} \arg \zeta + \lim_{a \to 0} \Delta_{\tau^\delta} \arg(\zeta - a) + \lim_{b \to 0} \Delta_{\tau^\delta} \arg(\zeta - b).$$

Since $\tau^\delta$ is smooth the limits

$$\lim_{\delta \to 0} \lim_{a \to 0} \Delta_{\tau^\delta} \arg(\zeta - a), \quad \lim_{\delta \to 0} \lim_{b \to 0} \Delta_{\tau^\delta} \arg(\zeta - b),$$
are $\pi$, $-\pi$ or $-\pi, \pi$ respectively. Hence we obtain

$$\lim_{\delta \to 0} \Delta C_\delta \arg g(\lambda) = \lim_{\delta \to 0} \Delta g(C_\delta) \arg \zeta = \pi (m_+ + m_-).$$

Let $m = m_+ + m_-$, an odd integer; $m$ is positive, because $g$ is not constant. Since $g$ does not vanish in $\Omega_\delta$ we can consider $h = g^{1/m}$. Then

$$\lim_{\delta \to 0} \Delta C_\delta \arg h(\lambda) = \pi.$$

As the argument shows, this holds for all arcs in $\Omega_\delta$ joining $\sigma_+^\delta$ and $\sigma_-^\delta$.

Next we will see that for small enough $\delta$, $h$ is a one-to-one map from $\Omega_\delta$ to a domain $R_\delta$, which is smooth in the neighbourhood of $0 \in bR_\delta$. Recall that $g(\gamma_1(s)) = \gamma_2(s)$, $\sigma_\delta = \gamma_1(I_\delta)$ and $\gamma_2(0) = 1$. Without loss of generality we can assume that $h(\gamma_1(s))$, $s > 0$, is the principal determination of $\gamma_2(s)^{1/m}$, so that $h(\sigma_+^\delta)$ is a $C^1$ arc having limiting tangent $(1,0)$ at 0, as it easily seen using polar coordinates. In the same way, $h(\sigma_-^\delta)$ is a $C^1$ arc having as tangent at 0 the opposite of some $m$-root of $(-1)$. Since $\pi$ is the variation of the argument, and $m$ is odd, this root must be of course $-1$ and hence $h(\sigma_\delta)$ is a smooth arc. The fact that $h$ is one-to-one follows then from the argument principle.

Let $f : R_\delta \to \Omega_\delta$ be the inverse map of $h$, $h(\lambda) = \zeta$, $g(\lambda) = \zeta^m$. We thus get the parametrization

$$R_\delta \longrightarrow V_\delta,$$
$$\zeta \mapsto (f(\zeta), \zeta^m),$$

$f(0) = 1$, and $R_\delta$ is smooth near 0. Also, $f \in C(\overline{R_\delta})$.

The final step is to show that $m$ must, in fact, be 1. Suppose that $m \geq 3$. Let $F : \Delta \to R_\delta$ be the Riemann mapping function, $F(1) = 0$ from the unit disk to $R_\delta$. Then we have a parametrization

$$\overline{\Delta} \longrightarrow V_\delta,$$
$$z \mapsto (f(F(z)), F(z)^m) = (G(z), F(z)^m).$$

Shrinking $R_\delta$ we may suppose that $R_\delta$ is a $C^1$-domain so the mapping $F$ satisfies, as said in the proof of Lemma 2.3, that $F' \in H^p$ for all $p$. In particular, $F$ satisfies a Lipschitz condition of order $\beta$ for all $\beta < 1$ and $G$ is then in the disk algebra. Let $\alpha \subset b\Delta$ be the arc parametrizing $V_\delta \cap \Gamma$, $1 \in \alpha$. 

We will show that $G(e^{it})$ has a non-zero derivative $\mu$ at $t = 0$. Once this is seen, since

$$|F(e^{it})| = O(|t|^\beta)$$

for all $\beta < 1$, taking $\beta > 1/m$ we see that $F^m(e^{it})$ has zero-derivative and then $(\mu, 0)$ is tangent to $\Gamma$ at $Q = (1, 0)$, in contradiction with the assumption that $\Gamma$ is complex-tangential at $Q$.

Let $G = IK'$ be the inner-outer factorization of $G$,

$$K'(z) = \exp \left( \frac{1}{2\pi} \int_{-\pi}^{+\pi} \frac{e^{it} + z}{e^{it} - z} \log |G(e^{it})| dt \right).$$

Since $G(1) = 1$, the inner part $I$ has an analytic continuation near 1; using that $|I(re^{it})|^2 \leq 1$ and $|I(e^{it})|^2 = 1$ it is immediate to obtain that $I'(1) I(1) \geq 0$. We want to show now that the formal rule

$$K''(1) = -K(1) \frac{1}{4\pi} \int_{-\pi}^{+\pi} \frac{1}{\sin^2 \frac{t}{2}} \frac{1}{\log |G(e^{it})|} dt$$

(which makes sense because $\log |G(e^{it})| \simeq 1 - |G(e^{it})| \simeq |F(e^{it})|^{2m} = O(t^2)$ near $t = 0$) obtained by differentiating $K$ under the integral sign is fully justified. This will give

$$K''(1) = k K(1)$$

with $k > 0$, because $\log |G| \leq 0$ and then

$$\mu = G'(1) = I'(1) K(1) + I(1) K'(1) = I'(1) I(1) + K'(1) (K(1)) = k + I'(1) I(1) > 0.$$

It remains thus to show that the formal rule above holds true. For this it is enough to show that the part of the integral over $\alpha$ satisfies the rule. Now on $\alpha$, $|G(e^{it})|^2 = 1 - |F(e^{it})|^{2m}$. Writing

$$u(t) = \frac{1}{2} \log(1 - |F(e^{it})|^{2m})$$

and

$$Hu(z) = \frac{1}{2\pi} \int_{\alpha} \frac{e^{it} + z}{e^{it} - z} u(t) dt$$
we shall show that $Hu(z)$ has an unrestricted derivative at 1, i.e.

$$\frac{Hu(z) - Hu(1)}{z - 1} - \frac{1}{2\pi} \int_\alpha \frac{2e^{it}}{(e^{it} - 1)^2} u(t) \, dt \to 0.$$  

The last expression can be written

$$\frac{1}{\pi} \int_\alpha \frac{u(t)}{e^{it} - 1} \left( \frac{e^{it}}{e^{it} - z} - \frac{e^{it}}{e^{it} - 1} \right) \, dt = C(v)(z) - C(v)(1),$$

where $v(t) = u(t)/(e^{it} - 1)$ and $C(v)$ denotes the Cauchy integral of $v$ over the arc $\alpha$. Now

$$v'(t) = \frac{u'(t)}{e^{it} - 1} - \frac{i e^{it} u(t)}{(e^{it} - 1)^2}.$$  

But $u' = m |F^{2(m-1)}(F)| (\Re F') (1 - |F|^2)^{-1}$, so that $v' \in L^p$ for all $p$. Thus $v$ satisfies a Lipschitz condition of order $\beta$ for all $\beta$, hence so does $C(v)$ and we are done.

In conclusion we have proved that $m = 1$. This means, with the notations used before, that for small $\delta$, $g$ is a one-to-one map from $\Omega_\delta$ to $R_\delta = g(\Omega_\delta)$ with inverse $f$. We now take as $U$ a $C^1$ domain included in $R_\delta$ so that $bU \cap bR_\delta = \tau_\delta$. On $\tau_\delta$

$$f(\gamma_2(s)) = \gamma_1(s)$$

and hence $f$ is $C^1$ on $\tau_\delta$. With Lemma 2.3 we conclude as before.

This completes the proof of Theorem 2.1.

3. **Analytical properties of the curve $\Gamma$.**

Theorem 2.1 has several consequences regarding the curve $\Gamma$ itself. Here we will draw one of them, to be used in the next section. If $\gamma(s)$ is the arc-length parametrization of $\Gamma$, as mentioned in the introduction

$$\gamma(s) \gamma'(s) = iT(s)$$

with $T$ a real-valued continuous function (we use the notation $\overline{a \overline{b}}$ for $\sum_j a_j \overline{b_j}$ if $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n)$). $T$ can be said to measure the transversality of $\Gamma$. We prove:
Theorem 3.1. If \( \Gamma \) is a \( C^1 \) simple closed curve which is not polynomially convex, then \( T \) has constant sign and
\[
\int \frac{ds}{|T(s)|^p} < +\infty, \quad \text{for all } p.
\]

Corollary 3.2. If \( \int \frac{ds}{T(s)} = +\infty \), \( \Gamma \) is polynomially convex.

The Corollary implies Forstneric’ result [7] according to which a \( C^2 \) curve which is complex-tangential at one point is polynomially convex and should be compared as well with Alexander’s in [1], stating that a rectifiable curve whose set of complex tangencies has positive length is also polynomially convex.

Proof of Theorem 3.1. Of course it is enough to prove the result locally around a complex-tangent point \( Q \in \Gamma \). Let’s consider the parametrization described in Theorem 2.1. Without loss of generality we assume \( \gamma = (1,0) \), that \( U \) is a \( C^1 \)-domain in the \( z_2 \)-plane, \( f \) is holomorphic in \( U \), of class \( C^1 \) up to \( \overline{U} \) and
\[
\overline{U} \rightarrow \overline{V} \cap N \\
\lambda \mapsto (f(\lambda), \lambda)
\]
is the parametrization. Let \( \tau \subset bU \) be the arc parametrizing \( \Gamma \cap N \).

As before, let \( F : \overline{\Delta} \rightarrow \overline{U} \) be the Riemann mapping function from the unit disk to \( U \), \( F(1) = 0 \), and the parametrization
\[
\overline{\Delta} \phi \rightarrow \overline{V} \cap N \\
z \mapsto (G(z), F(z)).
\]

Let \( \alpha \subset \mathbb{T} \) be the arc mapped onto \( \tau \) by \( F \). We know that \( F' \in H^p \) for all \( p \). In particular \( F(e^{it}) \) is absolutely continuous and \( G \) is in the disk algebra. If \( e^{it_0} \in \alpha \) is a point where \( F(e^{it}) \) is differentiable, since \( f \) is \( C^1 \) on \( \tau \), \( G(e^{it}) \) is also differentiable at \( t_0 \). Hence \( G(e^{it}) \) is differentiable almost everywhere on \( \alpha \) and moreover \( d(G(e^{it}))/dt \) is in \( L^p(\alpha) \). By [14, Theorem IV.5], the non-tangential limit at such point
\[
\lim_{z \rightarrow e^{it_0}} G'(z) \overset{d}{=} G'(e^{it_0})
\]
exists and equals \(-i e^{-i\theta} d(G(e^{it})) dt |_{t=t_0}\). It obviously follows that this limit also equals the non-tangential limit

\[
\lim_{z \to e^{it_0}} \frac{G(z) - G(e^{it_0})}{z - e^{it_0}}.
\]

Therefore at almost all points of \(\alpha\) the tangential and radial derivatives of \(F\) and \(G\) exist and belong to \(L^p(\alpha)\) for all \(p\). In particular,

\[
ds = (|G'|^2 + |F'|^2)^{1/2} dt
\]

and \(T\) is given almost everywhere on \(\phi(\alpha)\) by

\[
i T = \frac{1}{(|G'|^2 + |F'|^2)^{1/2}} \frac{i e^{it} G' + i e^{it} F'}{(G' e^{it}) + (F' e^{it})}.
\]

We claim that there is a constant \(c > 0\) such that

\[
e^{it}(G' \overline{G} + F' \overline{F}) \geq c, \quad \text{almost everywhere on } \alpha.
\]

**Lemma 3.3.** Let \(\phi : \overline{\mathbb{D}} \to \mathbb{B}^2\), \(\phi = (G, F)\) be an analytic disk such that the tangential and radial derivatives of \(F\), \(G\) exist almost everywhere on an arc \(\alpha \subset \mathbb{T}\), with \(\phi(\alpha) \subset S\). Let \(a = \phi(0) \in \mathbb{B}^2\). Then

\[
e^{it}(G' \overline{G} + F' \overline{F}) \geq \frac{1 - |a|}{1 + |a|}, \quad \text{almost everywhere on } \alpha.
\]

**Proof.** Since \(|F|^2 + |G|^2 = 1\) on \(\alpha\) and \(|F(r e^{i\theta})|^2 + |G(r e^{i\theta})|^2 \leq 1\), at one point where everything makes sense, one has

\[
0 = \frac{d}{dt}(|F(e^{it})|^2 + |G(e^{it})|^2)
= 2 \text{Re} \, i e^{it} \left( F'(e^{it}) \overline{F(e^{it})} + G'(e^{it}) \overline{G(e^{it})} \right),
\]

\[
0 \leq \frac{d}{dr} \left|_{r=1} \right. \left( |F(r e^{i\theta})|^2 + |G(r e^{i\theta})|^2 \right) = 2 \text{Re} \, e^{it} (F' \overline{F} + G' \overline{G}).
\]
Therefore \( e^{it} \left( G' \overline{F} + G' \overline{F} \right) \) is real and non-negative.

Assume now that \( a = 0 \). Then we can write \( F(z) = z F_0(z) \), \( G(z) = z G_0(z) \) and apply the above to \((G_0, F_0)\) because \(|G_0|^2 + |F_0|^2 = 1\) on \( \alpha \). Then

\[
e^{it} \left( G' \overline{G} + F' \overline{F} \right) = 1 + e^{it} \left( G'_0 \overline{G}_0 + F'_0 \overline{F}_0 \right) \geq 1
\]

and the result is proved when \( a = 0 \).

Assume now \( a \neq 0 \). We can choose complex orthonormal coordinates such that \( a = (\lambda, 0) \). Let \( \varphi_a = (\varphi_1, \varphi_2) \) be the automorphism of \( B \) with

\[
\varphi_1 = \frac{(s-1)z_1 + \lambda - sz_1}{1 - \lambda z_1}, \quad \varphi_2 = \frac{sz_2}{1 - \lambda z_1}
\]

where \( s^2 = 1 - |\lambda|^2 \), so that \( \varphi_0(a) = 0 \), \( \varphi_a^{-1} = \varphi_a \) (see [10, Chapter 2]).

If \( \varphi_a \circ \phi = \phi_0 = (G_0, F_0) \), then \( G = \varphi_1 (G_0, F_0) \), \( F = \varphi_2 (G_0, F_0) \). Therefore, with \( D_i = \partial / \partial z_i \)

\[
\begin{align*}
G' &= (D_1 \varphi_1) G_0' + (D_2 \varphi_1) F_0', \\
F' &= (D_1 \varphi_2) G_0' + (D_2 \varphi_2) F_0',
\end{align*}
\]

and

\[
G' \overline{G} + F' \overline{F} = ((D_1 \varphi_1) \overline{\varphi_1} + (D_2 \varphi_2) \overline{\varphi_2}) G_0' \\
+ ((D_2 \varphi_1) \overline{\varphi_1} + (D_2 \varphi_2) \overline{\varphi_2}) F_0'.
\]

A computation shows that the brackets at \((z_1, z_2)\) equal

\[
\frac{s^2 z_1}{|1 - \lambda z_1|^2}, \quad \frac{s^2 z_2}{|1 - \lambda z_1|^2},
\]

respectively. Hence,

\[
e^{it} \left( G' \overline{F} + F' \overline{F} \right) = \frac{s^2}{|1 - \alpha \phi_0|^2} e^{it} \left( G'_0 \overline{G}_0 + F'_0 \overline{F}_0 \right).
\]

Since \( \phi_0(0) = 0 \), \( e^{it} (G'_0 \overline{G}_0 + F'_0 \overline{F}_0) \geq 1 \) almost everywhere on \( \alpha \) and the lemma is proved.
Note that the lemma also gives a proof of the fact that an analytic disk \( \phi : \overline{\Delta} \to \overline{\mathbb{B}^2} \) with \( \phi(T) \subset S \) passing through \( a \in \mathbb{B}^2 \) must have a boundary of length \( \geq 2\pi(1 - |a|)/(1 + |a|) \).

This already shows that \( T \) has constant sign. Finally
\[
\int_{\phi(a)} \frac{ds}{|T(s)|^p} = \int_{\alpha} \frac{(|G'|^2 + |F'|^2)^{(p+1)/2}}{|G' G + F' F|^p} \, dt \\
\leq C \int_{\alpha} \frac{(|G'|^2 + |F'|^2)^{(p+1)/2}}{dt} < +\infty,
\]
which ends the proof of Theorem 3.1.

4. Pluriharmonic interpolation from \( \Gamma \).

In this section we assume that \( \Gamma \) is a simple closed \( C^1 \)-curve on \( S = b \mathbb{B}^2 \), with arc-length parametrization \( \gamma(s) \), such that its index of transversality defined by
\[
i T(s) = \overline{\gamma'(s)} \gamma(s)
\]
satisfies
\[
T(s) \geq 0 \quad \text{and} \quad \int \frac{ds}{T(s)} < +\infty.
\]
We may say that \( \Gamma \) is close to transverse. As seen in the previous section, this is the case if \( \Gamma \) is not polynomially convex, but we don’t assume this here. Our purpose is to prove

**Theorem 4.1.** With the assumptions above, the space PHC of pluriharmonic functions in \( \mathbb{B}^2 \), continuous up to \( b \mathbb{B}^2 \) has a closed trace of finite codimension in \( C(\Gamma) \). In particular, if \( \Gamma \) is polynomially convex, any continuous function on \( \Gamma \) can be interpolated by a pluriharmonic function in PHC.

This was proved in [4] for \( C^3 \) curves without any other assumption.

**Proof.** The scheme of the proof is the same as that in [4], but each of the steps needs substantial modifications due to the lack of extra smoothness. Let \( E \subset \Gamma \) be the set of complex-tangential points of \( \Gamma \)
and let \( C_0(\Gamma) \) be the space of continuous functions in \( \Gamma \) vanishing on \( E \). The first step is to construct an operator

\[
K : C_0(\Gamma) \rightarrow \text{PHC}
\]

such that

\[
(K\varphi)(\gamma(s)) = \int L(t, s) \varphi(t) \, dt + \varphi(s),
\]

where the integral operator of the right-hand side is compact. The second step consists in showing that \( E \) is an interpolation set for the ball algebra, so that by a general result in [13, Theorem 22.2], there is a linear continuous operator

\[
I : C(E) \rightarrow A(B)
\]

such that \( I\psi|_E = \psi \). Then the operator

\[
P : C(\Gamma) \rightarrow C(\Gamma)
\]

defined by

\[
P\varphi = K(\varphi - I\varphi) + I\varphi
\]

satisfies

\[
(P\varphi)(\gamma(s)) = \varphi(s) + \int L(t, s) (\varphi(t) - I\varphi(t)) \, dt.
\]

Now, Range \( P \) consists of boundary values of pluriharmonic functions in \( \text{PHC} \). Moreover, by Fredholm theory, Range \( P \) is closed and of finite codimension. Then, a functional analysis argument ends the proof of the theorem (see [4, Section 6]).

To start with, let

\[
K(t, z) = \frac{1}{\pi} \text{Im} \frac{\gamma(t)\gamma'(t)}{1 - z\gamma(t)}, \quad K\varphi(z) = \int K(t, z) \varphi(t) \, dt,
\]

\[
L(t, x) = K(t, \gamma(x)) = \frac{1}{\pi} T(t) \text{Re} \frac{1}{1 - \gamma(x)\gamma(t)}.
\]

Note that \( K(t, z) \) is positive and that

\[
\text{Re}(1 - \gamma(x)\overline{\gamma(t)}) = \frac{1}{2} |\gamma(x) - \gamma(t)|^2 \simeq |t - z|^2,
\]

\[
\text{Im}(1 - \gamma(x)\overline{\gamma(t)}) = \text{Im} \int_x^t \gamma'(s)\overline{\gamma(t)} \, ds = \int_x^t T(s) \, ds + O(|x - t|^2).
\]
Hence
\[
|1 - \gamma(x)\gamma(t)| \approx (t - x)^2 + \int_t^x T(s)ds.
\]

Lemma 4.2. With the assumption of Theorem 4.1,

a) \(L(t, x)\) satisfies
\[
\sup_{|x - t| \leq \delta} \int_{|x - t| \leq \delta} L(t, x) dt \to 0, \quad \text{as} \quad \delta \to 0.
\]

b) \(\int K(t, z) dt \leq C, \quad \text{for all} \quad z \in \mathbb{B}^2.\)

Proof.
\[
|L(t, x)| \lesssim T(t) \frac{|t - x|^2}{\left( |t - x|^2 + \int_t^x T(\xi) d\xi \right)^2} \leq T(t) \frac{|t - x|^2}{\int_t^x T(\xi) d\xi}. \]

Let \(\phi(t) = \int_0^t T(\xi) d\xi\), \(\phi\) is strictly increasing; let \(\psi = \phi^{-1}\), we make the change of variables \(u = \phi(t)\). If \(v = \phi(x)\), since \(T(t) dt = du\).
\[
\int_{|t - x| \leq \delta} T(t) \frac{|t - x|^2}{\int_t^x T(\xi) d\xi} dt \leq \int_{|u - v| \leq \varepsilon(\delta)} \frac{|\psi(u) - \psi(v)|^2}{|u - v|^2} du
\]
with \(\varepsilon(\delta) \to 0\) as \(\delta \to 0\). Now we apply Hardy’s inequality, or rather its proof:
\[
|\psi(u) - \psi(v)|^2 = \left( \int_v^u \psi'(\xi) d\xi \right)^2
\]
\[
\leq \left( \int_v^u |\psi'(\xi)| |\xi - v|^{1/2} |\xi - v|^{-1/2} d\xi \right)^2
\]
\[
\leq 2 |u - v|^{1/2} \int_v^u |\psi'(\xi)|^2 |\xi - v|^{1/2} d\xi
\]
by Holder’s inequality with the measure $|\xi - v|^{-1/2} \, d\xi$. Then,

$$
\int_{|u-v| \leq \varepsilon(t)} \frac{\|\psi(u) - \psi(v)\|^2}{|u-v|^2} \, du \\
\leq 2 \int_{|u-v| \leq \varepsilon(t)} |u-v|^{-3/2} \left[ \int_v^u |\psi'(\xi)|^2 \, |\xi - v|^{1/2} \, d\xi \right] \, du \\
\leq 2 \int_{|\xi - v| \leq \varepsilon(t)} |\psi'(\xi)|^2 \, |\xi - v|^{1/2} \left( \int_{|u-v| \geq |\xi - v|} |u-v|^{-3/2} \, du \right) \, d\xi \\
\leq K \int_{|\xi - v| \leq \varepsilon(t)} |\psi'(\xi)|^2 \, d\xi.
$$

Hence for $a$) it is enough to have $\int |\psi'(\xi)|^2 \, d\xi < +\infty$. But changing variables again, $u = \psi(\xi)$, $du = \psi'(\xi) \, d\xi$, $\psi'(\xi) = 1/T(u)$ this is

$$
\int \frac{du}{T(u)} < +\infty.
$$

This proves part $a$) of Lemma 4.2. For $b$), let for fixed $z$, $s = s(z)$ be such that

$$
|1 - \gamma(s) \, z| = \min\{|1 - \gamma(t) \, z| : \text{all } t\}.
$$

Then

$$
|1 - \gamma(t) \, z| \simeq |1 - \gamma(s) \, z| + |1 - \gamma(t) \, \gamma(s)|.
$$

The inequality $\lesssim$ is immediate because $|1 - \overline{ab}|^{1/2}$ satisfies a triangle inequality. On the other hand,

$$
|1 - \gamma(t) \, \gamma(s)| + |1 - \gamma(s) \, z| \lesssim |1 - \gamma(t) \, z| + 2|1 - \gamma(s) \, z| \\
\lesssim 3|1 - \gamma(t) \, z|
$$

by the choice of $s$. Hence

$$
|1 - \gamma(t) \, z| \simeq |1 - \gamma(s) \, z| + |t - s|^2 + \left| \int_s^t T(\xi) \, d\xi \right|.
$$

$K(t, z) = \frac{1}{\pi} \frac{T(t) \text{ Re} \frac{1}{1 - \gamma(t) \, z}}{1 - \gamma(t) \, z}$
\[
\begin{align*}
&= \frac{1}{\pi} T(t) \frac{\text{Re} (1 - \gamma(t) \bar{z})}{|1 - \gamma(t) z|^2} \\
&\simeq T(t) \frac{\text{Re} (1 - \gamma(t) \bar{z})}{\left( |1 - \gamma(s) z| + |t - s|^2 + \left| \int_s^t T(\xi) \, d\xi \right|^2 \right)^{1/2}}.
\end{align*}
\]

Next,
\[
2 \text{Re} (1 - \gamma(t) \bar{z}) = 1 - \gamma(t) \bar{z} + 1 - \gamma(t) \bar{z}
\]
\[
= |\gamma(t) - z|^2 + 1 - |z|^2
\]
\[
\lesssim |\gamma(t) - \gamma(s)|^2 + |z - \gamma(s)|^2 + 1 - |z|^2
\]
\[
\lesssim |t - s|^2 + |z - \gamma(s)|^2 + 1 - |z|^2.
\]

Write \( r = r(z) = |z - \gamma(s)|^2 + 1 - |z|^2, \) \( R = |1 - \gamma(s) z| \). Then
\[
K(t, z) \lesssim T(t) \frac{|t - s|^2}{\left( |t - s|^2 + \int_0^t T(\xi) \, d\xi \right)^2} + T(t) \frac{r(z)}{\left( R + \int_s^t T(\xi) \, d\xi \right)^2}
\]
\[
= K_1(t, s) + K_2(t, z).
\]

In proving a) we have already seen that \( \int K_1(t, s) \, dt = O(1) \).

Next, with \( l \) equal to the length of \( \gamma \), assuming \( s = 0 \), and with the change of variables \( u = \int_0^t T(\xi) \, d\xi \),
\[
\int_0^t \frac{T(t) \, dt}{\left( R + \int_0^t T(\xi) \, d\xi \right)^2} = \int_0^M \frac{du}{(R + u)^2} \leq \int_0^\infty \frac{du}{(R + u)^2} = \frac{1}{R}.
\]

But \( r \lesssim R \). Hence \( \int K_2(t, z) \, dt \leq C \), for all \( z \in \mathbb{B}^2 \), and part b) is also proved.

**Lemma 4.3.** If \( \varphi \) vanishes whenever \( T \) vanishes, then
\[
\lim_{z \to \gamma(x)} K \varphi(z) = \varphi(x) + \int \varphi(t) L(t, x) \, dt.
\]
PROOF.

\[
\left| \int K(t, z) \varphi(t) \, dt - \varphi(x) - \int \varphi(t) L(t, x) \, dt \right|
\leq \int_{|x-t| \leq \delta} K(t, z) |\varphi(t) - \varphi(x)| \, dt + |\varphi(x)| \int K(t, z) \, dt - 1
\]

\[
+ \int_{|x-t| \geq \delta} |\varphi(t)| |K(t, z) - L(t, x)| \, dt + \int_{|x-t| \leq \delta} |\varphi(t)| |L(t, x)| \, dt
\]

\[
def T_1 + T_2 + T_3 + T_4.
\]

Let \( w(\delta) \) be the modulus of continuity of \( \varphi \). Then,

\[
T_1 \leq w(\delta) \int K(t, z) \, dt \leq C w(\delta),
\]

\[
T_4 \leq \|\varphi\|_\infty \int_{|x-t| \leq \delta} L(t, x) \, dt.
\]

We break the integral in \( T_2 \), accordingly to

\[
K(t, z) = \frac{1}{\pi} \Im \frac{z \gamma'(t)}{1 - z \gamma(t)} + O(|z - \gamma(t)|) \frac{|1 - z \gamma(t)|}{|1 - z \gamma(t)|}
\]

\[
= -\frac{1}{\pi} \frac{d}{dt} \Im \log(1 - z \gamma(t)) + O(|1 - z \gamma(t)|^{-1/2}).
\]

Note that

\[
|1 - z \gamma(t)| \approx \int s T(\xi) \, d\xi.
\]

If \( z \) is close to \( \gamma(x) \), with \( T(x) \neq 0 \), and \( \delta \) is small, so that \( T \) is bounded below by some constant \( C(x) \) between \( s \) and \( t \), one has \( |1 - z \gamma(t)| \geq C(x) |s - t| \). Hence

\[
\int_{|x-t| \leq \delta} |1 - z \gamma(t)|^{-1/2} \leq C(x) \delta^{1/2},
\]

for \( \delta \) small and \( z \) close to \( \gamma(x) \). All these gives, taking into account that \( \lim_{z \to \gamma(x)} T_3 = 0 \) by dominated convergence (\( K(t, z) \) is singular at
\[ z = \gamma(t) \text{ only}, \] and Lemma 4.2,

\[
\lim_{z \to \gamma(t)} \sup K\varphi(z) - (\varphi(x) + \int \varphi(t) L(t, x) \, dt) \leq C w(\delta) + C \delta^{1/2} + |\varphi(x)| \left( \frac{1}{\pi} \left( \arg (1 - \gamma(x) \gamma(x - \delta)) - \arg (1 - \gamma(x) \gamma(x + \delta)) \right) - 1 \right) + \|\varphi\|_{\infty} \int_{|x - t| \leq \delta} L(t, x) \, dt.
\]

Since

\[
1 - \gamma(x) \gamma(x + \delta) = -i T(x) \delta + o(\delta),
\]

\[
1 - \gamma(x) \gamma(x - \delta) = i T(x) \delta + o(\delta),
\]

we obtain Lemma 4.3 by making \( \delta \to 0 \). When \( T(x) = 0 \) this argument does not control the term \( T_2 \), but it vanishes because \( \varphi(x) = 0 \).

Lemmas 4.2 and 4.3 complete the first step of the proof. Indeed, obviously \( K\varphi \) is pluriharmonic and \( K\varphi \) has a continuous extension to \( \mathbb{B}^2 \setminus \Gamma \). Lemma 4.3 shows that \( K\varphi \in \text{PHC} \) if \( \varphi \in C_0(\Gamma) \). Finally, the operator \( \varphi \mapsto \psi \) where

\[
\psi(x) = \int \varphi(t) L(t, x) \, dt
\]

is compact in \( C(\Gamma) \). For this, we must show, to prove equicontinuity, that

\[
\int |L(t, x) - L(t, y)| \, dt
\]

is small for \( |x - y| \) small, and this follows from part a) of Lemma 4.2 and the continuity of \( L \) off the diagonal.

It only remains to prove that \( E \) is an interpolation set for the ball algebra. This is a well-known result that can be proved for instance applying the Davie-Øksendal theorem ([10, Theorem. 10.4.3]): it is enough to see that for each \( \varepsilon \) there are Koranyi balls \( V(\xi_1, \delta_1), \ldots, V(\xi_m, \delta_m) \) where

\[
V(\xi, \delta) = \{ z \in S : |1 - \xi \bar{z}| < \delta \}
\]
such that $\sum_{i=1}^{m} \delta_i < \varepsilon$ and $E \subset V(\xi_1, \delta_1) \cup \cdots \cup V(\xi_m, \delta_m)$.

Given $\varepsilon$, let $\gamma(s_1), \ldots, \gamma(s_m) \in E$ be such that

$$E \subset \gamma\left(\bigcup_{i=1}^{m}(s_i - \varepsilon, s_i + \varepsilon)\right)$$

with $m = O(1/\varepsilon)$. By the mean value theorem

$$\gamma(s_i - \varepsilon, s_i + \varepsilon) \subset V(\gamma(s_i), \delta_i)$$

with

$$\delta_i \approx \varepsilon w(\varepsilon),$$

where $w$ is the modulus of continuity of $\gamma'$. Hence

$$\sum_{i=1}^{m} \delta_i \leq m \varepsilon w(\varepsilon) = O(w(\varepsilon))$$

can be made arbitrarily small.

This ends the proof of Theorem 4.1.

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