Two problems on doubling measures

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Doubling measures appear in relation to quasiconformal mappings of the unit disk of the complex plane onto itself. Each such map determines a homeomorphism of the unit circle on itself, and the problem arises, which mappings $f$ can occur as boundary mappings? A famous theorem of Beurling and Ahlfors [2] states a necessary and sufficient condition: the Lebesgue measures $|f(I)|$ and $|f(J)|$ are comparable, $|f(I)| \simeq |f(J)|$, whenever $I$ and $J$ are adjacent arcs of equal length. Denoting by $\mu$ the measure on the unit circle such that $\mu(I) = |f(I)|$, this can be expressed by the inequality $\mu(2I) \leq c \mu(I)$, where $2I$ denotes an arc on the circle, concentric with $I$, of twice the length. The measure $\mu$ in the Beurling-Ahlfors theorem is the harmonic measure for a certain elliptic operator in divergence form, whence the problem of null-sets for doubling measures is closely related to that of null-sets for harmonic measure [3].

Certain estimations, such as those for singular integrals and maximal functions, which are classical in the case of Lebesgue measure, can be obtained for doubling measure in Euclidean space ([4], [5] and [6]). Doubling measures also appear in relation to inner functions in several complex variables [1].

The definition of doubling measure has meaning for any metric space $(X, \rho)$, i.e. $\mu(B(x, 2r)) \leq c \mu(B(x, r))$, and it is natural to ask which compact metric spaces $(X, \rho)$ carry non-trivial doubling measures. A necessary and sufficient condition was found by Vol'berg and Konyagin [8], and called finite uniform metric dimension: in each ball
$B(x, 2r)$ at most $N$ points can be found with mutual distances at least $r$.

In view of the original interest in singular mappings and singular measures, mutually singular doubling measures on the same metric space are of interest. We prove that such measures exist provided $(X, \rho)$ carries a doubling measure and is perfect. This answers a question stated in [8, p. 637].

A measure $\mu$ on $\mathbb{R}^1$, is called dyadic-doubling if $\mu(I) \leq c \mu(J)$ whenever $I$ and $J$ are adjacent dyadic intervals of the same length, whose union is also dyadic. These measures occur in the theory of weights and are completely characterized [7]. It is hardly surprising that the class of doubling measures and the class of dyadic doubling measures are different, but less trivial that the corresponding classes of null-sets (which we abbreviate as $\mathcal{N}$ and $\mathcal{N}_d$) are different. The class $\mathcal{N}$ is bilipschitz invariant. The class $\mathcal{N}_d$ lacks an invariance property of $\mathcal{N}$: we find a closed set $E$, not in $\mathcal{N}_d$, and a set $T$ of full measure in \(\mathbb{R}^1\), $|\mathbb{R}^1 \setminus T| = 0$, such that $t + E$ is in $\mathcal{N}_d$ for each $t$ in $T$. A previous example [9] accomplished this with a set $T$ of dimension 1. The class $\mathcal{N}_d$ is not invariant under multiplication by positive numbers $t$, but our example is not as strong as the one for addition.

1. Singular doubling measures on compact metric space.

Vol'berg and Konyagin proved [8] that a compact metric space $(X, \rho)$ carries a nontrivial doubling measure $\mu$ on $X$:

\[
\mu(B(x, 2R)) \leq \Lambda \cdot \mu(B(x, R)), \quad \text{for all } x \in X, \ R > 0,
\]

where $\Lambda \geq 1$ and $B(x, R) = \{y \in X : \rho(x, y) < R\}$, if and only if it has finite uniform metric dimension. In particular any compact set $X$ in $\mathbb{R}^n$ carries a nontrivial doubling measure.

They also raised the question: on which compact metric spaces $(X, \rho)$ are all doubling measures mutually absolutely continuous? It follows from a well-known example of Beurling and Ahlfors [2] that this is not the case even for the unit circle. We prove the following.

**Theorem 1.** Let $(X, \rho)$ be a compact metric space and $\mu$ be a doubling measure on $X$ having no atoms. Then there exists a doubling measure on $X$ singular with respect to $\mu$. 
We emphasize that a doubling measure on \( X \) satisfies the doubling condition on \( X \) only; that is, only balls with centers in \( X \) figure in the definition.

We say that \((X, \rho)\) has finite uniform metric dimension if there exists a finite \( N = N(X, \rho) \) such that for any \( x \in X \) and \( R > 0 \), there are at most \( N \) points in \( B(x, 2R) \) separated from one another by a distance at least \( R \).

**Proof of Theorem 1.** Let \((X, \rho)\) satisfy all conditions in Theorem 1, and \( \mu \) be a doubling measure on \( X \) with \( \mu(X) = 1 \). To construct a doubling measure on \( X \) singular with respect to \( \mu \), we invoke the idea of Riesz product on the measure space \((X, \mu)\). The functions \( w_k \) in the next lemma play the role of \( 1 + a_k \cos kx \) in the usual Riesz product.

**Lemma 1.** There exist measurable functions \( w_k \) on \( X \) taking values \( 1/2 \) and \( 3/2 \) only, so that

\[
\mu(w_k = 1/2) = \mu(w_k = 3/2) = 1/2, \tag{1.2}
\]

\( w_k \longrightarrow 1 \quad \text{weakly in } L^2(d\mu), \tag{1.3} \]

and

\[
w_k^{1/2} \longrightarrow \frac{1}{2}(\sqrt{1/2} + \sqrt{3/2}) \quad \text{weakly in } L^2(d\mu). \]

**Proof.** We observe that every measurable set \( E \) of measure \( \mu(E) > 0 \), can be divided into two subsets, each of measure \( \mu(E)/2 \). This is a general property of measure spaces with no atoms. Hence there is a measurable function \( w \) such that \( \mu(w < t) = t, 0 \leq t \leq 1 \). Let \( g(t) \) have period 1 on \([0, +\infty)\) with \( g = 1/2 \) on \([0, 1/2)\), \( g = 3/2 \) on \([1/2, 1]\). We set \( w_k = g(2^k w) \).

To see that \( w_k \longrightarrow 1 \) weakly in \( L^2(d\mu) \), we observe that the functions \( w_k \) are independent. Therefore \( w_k \) tends weakly to its mean, as does \( w_k^{1/2} \).

For \( x \in X \) and \( r > 0 \), define

\[
h_{x,r}(y) = \begin{cases} 
1, & \text{if } \rho(x, y) \leq r, \\
0, & \text{if } \rho(x, y) \geq 3r/2, \\
3 - 2\rho(x, y)r^{-1}, & \text{if } r < \rho(x, y) < 3r/2.
\end{cases}
\]
By the doubling property of $\mu$, there exists $A > 1$ independent of $x$ and $r$, so that

$$\int_X h_{x,2r}(y) \, d\mu(y) \leq A \int_X h_{x,r}(y) \, d\mu(y),$$

for all $x \in X$ and $r > 0$.

Let $\alpha = (\sqrt{1/2} + \sqrt{3/2})/2$, $\beta = 21\alpha/20$ and $\{w_k\}$ be the functions in Lemma 1. Note that $\beta < 1$.

We shall construct continuous functions $\{u_n\}_1^\infty$ and $\{v_n\}_1^\infty$ on $X$, so that the following inequalities are true:

$$\frac{1}{2} - \frac{1}{100(n+1)} \leq u_n \leq \frac{3}{2} + \frac{1}{100(n+1)},$$

$$\int_X \prod_0^n u_i \, d\mu = 1,$$

$$\int_X \left( \prod_0^n u_i \right)^{1/2} \, d\mu \leq \beta^n,$$

$$\int_X h_{x,2r} \prod_0^n u_i \, d\mu \leq \left( 7 - \frac{1}{n+1} \right) A \int_X h_{x,r} \prod_0^n u_i \, d\mu,$$

for all $x \in X$ and $r > 0$;

$$0 \leq v_n \leq 1,$$

$$\int_X v_n \, d\mu \leq \beta^{n/2},$$

and for all $0 \leq j \leq n$,

$$\int_X (1 - v_j) \prod_0^n u_i \, d\mu \leq \left( 3 - \frac{1}{n+1} \right) \beta^j.$$

Let $u_0 \equiv 1$ and $v_0 \equiv 0$ on $X$.

Assume that $u_0, \ldots, u_n$ and $v_0, \ldots, v_n$ have been chosen so that (1.5) to (1.11) are satisfied; we shall construct $u_{n+1}$ and $v_{n+1}$.
Because of (1.2), (1.3), (1.7) and (1.11), for sufficiently large $k > k(u_0, u_1, \ldots, u_n, v_0, v_1, \ldots, v_n)$,

$$
(1.12) \quad \int_X \left( \prod_{0}^{n} u_i \right)^{1/2} w_k^{1/2} \, d\mu \leq \left( 1 + \frac{1}{100(n+1)} \right) \alpha \beta^n,
$$

and

$$
(1.13) \quad \int_X (1 - v_j) \left( \prod_{0}^{n} u_i \right) w_k \, d\mu \leq \left( 3 - \frac{1}{\sqrt{(n+1)(n+2)}} \right) \beta^j,
$$

for all $0 \leq j \leq n$.

Because $u_i, 0 \leq i \leq n$, are uniformly continuous on $X$ with values in $[1/4, 2]$, it follows from (1.4) that for all $x \in X$,

$$
\int_X h_{x,2r} \prod_{0}^{n} u_i \, d\mu \leq A \left( 1 + \frac{1}{n+1} \right) \int_X h_{x,r} \prod_{0}^{n} u_i \, d\mu,
$$

provided that $0 < r < r(u_0, u_1, \ldots, u_n)$. Since $1/2 \leq w_k \leq 3/2$,

$$
(1.14) \quad \int_X h_{x,2r} \prod_{0}^{n} u_i w_k \, d\mu \leq 3A \left( 1 + \frac{1}{n+1} \right) \int_X h_{x,r} \prod_{0}^{n} u_i w_k \, d\mu
$$

for all $x \in X$ and $0 < r < r(u_0, u_1, \ldots, u_n)$.

Now we can see by (1.2), the compactness of $X$ and the continuity of $h_{x,r}(y)$ with respect to the variables $x, y$ and $r$ that

$$
\lim_{k \to \infty} \int_X h_{x,r}(y) \prod_{0}^{n} u_i(y) w_k(y) \, d\mu(y) = \int_X h_{x,r}(y) \prod_{0}^{n} u_i(y) \, d\mu(y)
$$

uniformly for $x \in X$, $r \geq r(u_0, \ldots, u_n)$. Moreover the integrals on the right have a positive lower bound for all $x$ and $r \geq r(u_0, \ldots, u_n)$. We deduce from (1.8) and (1.14) that for sufficiently large $k$,

$$
\int_X h_{x,2r} \prod_{0}^{n} u_i w_k \, d\mu \leq \left( 7 - \frac{1}{\sqrt{(n+1)(n+2)}} \right) A \int_X h_{x,r} \prod_{0}^{n} u_i w_k \, d\mu,
$$

(1.15)
for all \( x \in X \) and \( r > 0 \).

Now choose and fix one \( w_{k(n)} \), so that (1.12), (1.13) and (1.15) are satisfied.

Denote by \( d\nu_n = \prod_0^n u_i \, d\mu \). It follows from Lusin’s theorem that there exists a continuous \( \tilde{w}_{k(n)} \) on \( X \) taking values in \([1/2, 3/2]\) that agrees with \( w_{k(n)} \) on \( X \) outside a set \( E_n \) of small \( \nu_n \) measure. And let

\[
u_{n+1} = \left( \int_X \tilde{w}_{k(n)} \, d\nu_n \right)^{-1} \tilde{w}_{k(n)}.
\]

Clearly

\[
\int \prod_0^{n+1} u_i \, d\mu = \int \nu_{n+1} \, d\nu_n = 1,
\]

and

\[
\frac{1}{2} - \frac{1}{100(n + 2)} \leq u_{n+1} \leq \frac{3}{2} + \frac{1}{100(n + 2)}
\]

if \( \nu_n(E_n) \) is sufficiently small. Moreover, \( \nu_n(E_n) \) can be chosen small enough, so that (1.12), (1.13) and (1.15) remain true for slightly bigger constants when \( w_k \) is replaced by \( \nu_{n+1} \):

\[
\left( \prod_0^{n+1} u_i \right)^{1/2} d\mu \leq \beta^{n+1},
\]

\[
\int_X (1 - v_j) \prod_0^{n+1} u_i \, d\mu \leq \left( 3 - \frac{1}{n + 2} \right) \beta^j, \quad 0 \leq j \leq n,
\]

(the case \( j = n + 1 \) shall be provided later) and

\[
\int_X h_{x,2r} \prod_0^{n+1} u_i \, d\mu \leq \left( 7 - \frac{1}{n + 2} \right) A \int_X h_{x,r} \prod_0^{n+1} u_i \, d\mu.
\]

Finally, choose \( v_{n+1} \) continuous on \( X, 0 \leq v_{n+1} \leq 1 \), and

\[
v_{n+1} = \begin{cases} 
0, & \text{on the set where } \prod_0^{n+1} u_i \leq \beta^{n+1}, \\
1, & \text{on the set where } \prod_0^{n+1} u_i \geq 2 \beta^{n+1}.
\end{cases}
\]
It follows from (1.16) that
\[
\int_X v_{n+1} \, d\mu \leq \mu\left(\prod_{i=0}^{n+1} u_i \geq \beta^{n+1}\right) \leq \beta^{(n+1)/2},
\]

and
\[
\int_X (1 - v_{n+1}) \prod_{i=0}^{n+1} u_i \, d\mu \leq \int \prod_{i=0}^{n+1} u_i \, d\mu \leq 2 \beta^{n+1}.
\]

Hence \(u_{n+1}\) and \(v_{n+1}\) satisfy all properties (1.5) to (1.11).

Finally let \(\nu\) be a \(w^*\) limit of \(\prod u_i \, d\mu\), or of some subsequence.

In view of (1.10) and (1.11), \(\int v_j \, d\mu \leq \beta^{j/2}\) and \(\int (1 - v_j) \, d\nu \leq 3 \beta^j\) for all \(j\). Thus \(v_j \to 0\) almost everywhere with respect to \(d\mu\) and \(v_j \to 1\) almost everywhere with respect to \(d\nu\). Therefore \(\mu\) and \(\nu\) are mutually singular.

From (1.8), it follows that for all \(x \in X\) and \(r > 0\),
\[
\nu(B(x, 2r)) \leq \int h_{x, 2r} \, d\nu \leq 7A \int h_{x, r} \, d\nu \leq 7A \nu\left(B\left(x, \frac{3}{2}r\right)\right).
\]

Therefore \(\nu\) is a doubling measure on \(X\). This completes the proof of Theorem 1.

Let \((X, \rho)\) be a compact metric space of finite uniform metric dimension. Let \(E_X\) be the set of accumulation points in \(X\) and \(F_X\) be the set of isolated points in \(X\). Then \(X = E_X \cup F_X\).

**Lemma 2.** Let \(\mu\) be any doubling measure on \(X\). Then every point in \(F_X\) has positive \(\mu\)-measure, and every point in \(E_X\) has zero \(\mu\)-measure.

**Proof.** It is clear that every isolated point has positive \(\mu\)-measure by the doubling condition. Let \(x \in E_X\), and pick \(\{x_n\} \subseteq X\) so that \(0 < \rho(x, x_n) < \rho(x, x_{n-1})/10\). Then \(\{B(x_n, 2 \rho(x, x_n)/3)\}\) are mutually disjoint, and \(x \in B(x_n, 4 \rho(x, x_n)/3)\). Therefore
\[
\mu(X) \geq \sum_n \mu\left(B\left(x, \frac{2}{3} \rho(x, x_n)\right)\right)
\geq c \sum_n \mu\left(B\left(x, \frac{4}{3} \rho(x, x_n)\right)\right) \geq c \sum_n \mu(\{x\}).
\]
Since \( \mu(X) < \infty \), \( \mu(\{x\}) \) must be zero.

Therefore, for a doubling measure on \( (X, \rho) \) we may call \( \mu|_{E_X} \) the continuous part of \( \mu \) and \( \mu|_{F_X} \) the atomic part of \( \mu \).

**Corollary 1.** If \( X \) is a perfect set, then with respect to each doubling measure on \( X \) there exists a singular one.

The following statement, which follows easily from the proof of Theorem 1, answers the question of Vol'berg and Konyagin.

**Corollary 2.** All doubling measures on \( (X, \rho) \) are mutually absolutely continuous if and only if every doubling measure on \( X \) is purely atomic. A necessary topological condition is that \( F_X = X \).

Although \( F_X = X \) is a necessary condition, it is far from being a sufficient condition for the mutual absolute continuity of all doubling measures on \( X \); see the example below.

**Example.** There are compact subsets \( X, Y \) and \( Z \) of \( \mathbb{R}^1 \), so that the sets of accumulation points \( E_X, E_Y \) and \( E_Z \) are all perfect sets, and the closures of isolated points \( F_X, F_Y \) and \( F_Z \) equal to \( X, Y \) and \( Z \) respectively. However, all doubling measures on \( X \) are purely atomic; every doubling measure on \( Y \) contains a nontrivial continuous part; some doubling measures on \( Z \) are purely atomic and others have a nontrivial continuous part.

**Construction of \( X \).** Let \( E \) be the Cantor ternary set on \([0, 1], F\) be the centers of all maximal intervals in \([0, 1]\setminus E, \) and \( X = E \cup F \). Let \( \{a_{n,j}\}_{j=1}^{2^n-1} \) be all points in \( F \) of distance \( 3^{-n}/2 \) to \( E \) and \( I_{n,j} = [a_{n,j} - 3^{-n+1}/2, a_{n,j} + 3^{-n+1}/2], \) thus \( \{I_{n,j}\}_{j=1}^{2^n-1} \) forms a covering of \( E \). Let \( \mu \) be a doubling measure on \( X \). Then there exists \( c > 0 \), so that

\[
\sum_j \mu(\{a_{n,j}\}) \geq c \sum_j \mu(I_{n,j} \cap X) \geq c \mu(E),
\]

for each \( n \geq 1. \) Since \( \mu(X) < \infty \), \( \mu(E) \) must be zero.

**Construction of \( Y \).** Let \( E \) be the Cantor ternary set on \([0, 1], \) and \([0, 1]\setminus E = \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{2^n-1} I_{n,j}, \) where \( \{I_{n,j}\}_{j=1}^{2^n-1} \) are the maximal intervals.
in \([0, 1)\setminus E\) of length exactly \(3^{-n}\), arranged in ascending order with respect to \(j\). Given \(0 < \beta_n < 1/4\), let \(a_{n,j}\) and \(b_{n,j}\) \((a_{n,j} > b_{n,j})\) be the two points in \(I_{n,j}\) of distance \(\beta_n 3^{-n}\) to \(E\),

\[
F = \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{2^{n-1}} \{a_{n,j}, b_{n,j}\}
\]

and \(Y = E \cup F\).

Suppose that

\[
(1.17) \quad \sum \left(\log \frac{1}{\beta_n}\right)^{-1} < \infty.
\]

Then every doubling measure \(\mu\) on \(Y\) has a nontrivial continuous part.

Denote by \(\{E_{n,j}\}_{j=1}^{2^n}\) the intervals in \([0, 1)\setminus \bigcup_{k=1}^{n} \bigcup_{j=1}^{2^{k-1}} I_{k,j}\) in ascending order with respect to \(j\). Note that \(|E_{n,j}| = 3^{-n}\), and that

\[
(1.18) \quad \bigcup_{j=1}^{2^n} E_{n,j} \cap F = \bigcup_{k=n+1}^{\infty} \bigcup_{j=1}^{2^{k-1}} \{a_{k,j}, b_{k,j}\}.
\]

Moreover for each \((n, j)\),

\[
\text{dist} (a_{n,j}, E) = \text{dist} (a_{n,j}, E_{n,2j}) = \text{dist} (b_{n,j}, E) = \text{dist} (b_{n,j}, E_{n,2j-1}) = \beta_n 3^{-n}.
\]

Suppose that \(\mu\) is a doubling measure on \(Y\) which is purely atomic, i.e. \(\mu(E) = 0\). Then there exists \(c > 0\) depending only on \(\mu\), so that

\[
(1.19) \quad \mu(E_{n,2j} \cap F) \geq c \left(\log \frac{1}{\beta_n}\right) \mu(\{a_{n,j}\}),
\]

and

\[
(1.20) \quad \mu(E_{n,2j-1} \cap F) \geq c \left(\log \frac{1}{\beta_n}\right) \mu(\{b_{n,j}\}).
\]

In fact, fix \((n, j)\) and write \(E_{n,2j}\) as \([p/3^n, (p + 1)/3^n]\) for some integer \(p\) and \(a_{n,j}\) as \(p/3^n - \beta_n/3^n\). Let \(x_q = p/3^n + 1/3^n + 2q \in E\), and \(B_q = B(x_q, 2 \cdot 3^{-n-2q-1})\). Note that \(\{B_q\}_q\) are mutually disjoint and
\( B_q \subseteq E_{n,2j} \); moreover if \( \beta_n 3^{-n} < 3^{-n-2q-1} \), then \( a_{n,j} \in 2B_q \) and \( 2B_q \cap F = \{a_{n,j}\} \cup (2B_q \cap E_{n,2j}) \), where \( 2B_q \) is the interval \( B(x_q, 4 \cdot 3^{-n-2q-1}) \). Therefore there exist \( c' > 1 \), so that for \( 1 \leq q \leq (1/3) \log 1/\beta_n \),

\[
\mu(\{a_{n,j}\}) \leq \mu(2B_q) \leq c' \mu(2B_q) = c' \mu(B_q \cap F).
\]

Summing over \( q, 1 \leq q \leq (1/3) \log 1/\beta_n \), we obtain

\[
\mu(\{a_{n,j}\}) \log 1/\beta_n \leq 3c' \mu(E_{n,2j} \cap F).
\]

The proof of (1.20) is similar.

Denote by \( m_n = \mu(\bigcup_{j=1}^{2n-1} \{a_{n,j}, b_{n,j}\}) \) and recall that \( \sum_{1}^{\infty} m_n = \mu(Y) < \infty \). Summing over \( j \)'s in (1.19) and (1.20), we deduce from (1.18) that \( \sum_{n=1}^{\infty} m_k \geq c (\log 1/\beta_n) m_n \), for each \( n \geq 1 \). Denote \( \sum_{n=1}^{\infty} m_k \) by \( r_n \) and \( \log 1/\beta_n \) by \( N_n \), we have \( r_{n+1} \geq r_n (cN_n)/(1 + cN_n) \) for \( n \geq 1 \). Thus

\[
r_{n+1} \geq \prod_{k=1}^{n} \frac{cN_k}{1 + cN_k} \mu(Y).
\]

As \( n \to \infty \), the left hand side approaches 0, and the right hand side has a positive limit under the hypothesis (1.17), which is impossible.

Therefore every doubling measure on \( Y \) must have a nontrivial continuous part.

The construction of \( Z \) uses Whitney modication of measures. Let \( E \) be a closed set on \( \mathbb{R}^1 \) and \( \mu \) be a measure on \( \mathbb{R}^1 \). We call \( \mu^E \), a measure on \( \mathbb{R}^1 \), a Whitney modication of \( \mu \) if \( \mu^E \equiv \mu \) on \( E \), and for some Whitney decomposition \( W = \{I\} \) of \( \mathbb{R}^1 \setminus E \), \( \mu^E(\{x_I\}) = \mu^E(I) = \mu(I) \) for every \( I \in W \) and \( x_I \) the center of \( I \).

Recall that intervals in \( W \) have mutually disjoint interiors, \( \bigcup_{W} I = \mathbb{R}^1 \setminus E \) and \( \text{dist}(I, E)/4 \leq |I| \leq 4 \text{dist}(I, E) \) for each \( I \in W \). A measure \( \mu \) is said to have the doubling property on a closed set \( S \), if (1.1) is satisfied for all \( x \in S \) and \( R > 0 \).

**Lemma 3.** If \( \mu \) is a doubling measure on \( \mathbb{R}^1 \), then any Whitney modification \( \mu^E \) of \( \mu \) has the doubling property on \( E \cup F \), where \( F \) consists of the centers of intervals in \( W \), and \( W \) is the Whitney decomposition associated with \( \mu^E \).
Proof. For $x \in E$, let $I_x = \{x\}$, and for $x \in F$, let $I_x$ be the interval in $W$ centered at $x$. For any $x \in E \cup F$ and $R > 0$, we claim that

\begin{equation}
\mu^E(B(x, R)) \cong \mu(B(x, \mathrm{dist}(x, E)),
\end{equation}

if $B(x, R) \cap (E \cup F) = \{x\}$, and

\begin{equation}
\mu^E(B(x, R)) \cong \mu(B(x, R)),
\end{equation}

if $B(x, R) \cap (E \cup F)$ has at least two points.

By $c \cong d$ we mean $c/d$ is bounded above and below by positive numbers depending only on the constant $\Lambda$ in the doubling property of $\mu$.

Let $a = \inf\{y \in E \cup F : y > x - R\}$ and $b = \sup\{y \in E \cup F : y < x + R\}$. Note that $a = x - R$ if $x - R \in E$, and $a \in F$ otherwise; and that $b = x + R$ if $x + R \in E$, and $b \in F$ otherwise.

If $a = b$, then $a = b = x$. In this case,

$$\mu^E(B(x, R)) = \mu^E(\{x\}) = \mu(I_x) \cong \mu(B(x, \mathrm{dist}(x, E)).$$

If $a \neq b$, then $b - a \geq \max\{\mathrm{dist}(a, E), \mathrm{dist}(b, E)\}/64$. Note that $x + R - b \leq 64 \mathrm{dist}(b, E) + a - (x - R) \leq 64 \mathrm{dist}(a, E)$. Therefore $2R \geq b - a \geq 2^{-12}R$. In this case

$$\mu^E(B(x, R)) = \mu^E([a, b]) = \mu([a - |I_a|/2, b + |I_b|/2]) \cong \mu(B(x, R)).$$

Doubling property of $\mu^E$ on $E \cup F$ follows immediately from (1.21) and (1.22).

This property of the Whitney modification has a natural generalization to $\mathbb{R}^n$. For the converse, we raise the following question.

Question. For which $(X, \mu)$, $X$ perfect in $\mathbb{R}^n$ and $\mu$ doubling on $X$, is $\mu$ the restriction of a doubling measure in $\mathbb{R}^n$?

Construction of $Z$. It follows from Lusin’s theorem and an example of Beurling and Ahlfors [1] that there exist a nontrivial doubling measure $\mu$ on $\mathbb{R}^1$ and a perfect set $E \subset [0, 1]$ of positive length so that $\mu(E) = 0$. Let $W$ be any Whitney decomposition of $\mathbb{R}^1 \setminus E$, $F$ be the centers of intervals in $W$ and $Z = E \cup (F \cap [-100, 100])$. Then the Whitney modification $\mu^E$ has the doubling property on $Z$ (a modification of Lemma 3) and is purely atomic.
Let \( \sigma \) be the Lebesgue measure on \( \mathbb{R}^1 \) and \( \sigma^E \) be a Whitney modification. Then \( \sigma^E(E) = \sigma(E) > 0 \), and \( \sigma^E \) has the doubling property on \( Z \).

**Question.** Do there exist \( X \) compact in \( \mathbb{R}^1 \), a doubling measure \( \mu \) on \( X \), such that \( F_X = X \) and that \( \mu|_{E_X} \) is also a nontrivial doubling measure on \( E_X \)? (Recall that \( E_X \) is the set of accumulation points and \( F_X \) is the set of isolated points.)

**Question.** Given a compact set \( X \) on \( \mathbb{R}^n \), and \( \alpha > 0 \), does there exist a measure \( \mu \) doubling on \( X \) such that \( \mu \) has full measure on a Borel set of Hausdorff dimensions less or equal than \( \alpha \)?

We believe that the answer is positive when \( n = 1 \).

2. Null sets for dyadic doubling measures.

A measure \( \mu \) on \( \mathbb{R}^1 \) is called a **doubling measure** if (1.1) holds for all \( x \in \mathbb{R}^1 \) and \( R > 0 \), equivalently, there exists \( \lambda \geq 1 \) so that \( \mu(I) \leq \lambda \mu(J) \) for all neighboring intervals \( I \) and \( J \) of the same length. A measure \( \mu \) on \( \mathbb{R}^1 \) is called a **dyadic doubling measure** if there exists \( \lambda \geq 1 \) so that \( \mu(I) \leq \lambda \mu(J) \) whenever \( I \) and \( J \) are two dyadic neighboring intervals of same length and \( I \cup J \) is also a dyadic interval. We shall refer to the constant \( \lambda \) above as \( \lambda(\mu) \).

Denote by \( \mathcal{D} \) the collection of all doubling measures on \( \mathbb{R}^1 \) and by \( \mathcal{D}_d \) the collection of all dyadic doubling measures on \( \mathbb{R}^1 \). Denote by \( \mathcal{N} \) the collection of null sets for \( \mathcal{D} \), i.e., \( \mathcal{N} = \{ E \subseteq \mathbb{R}^1 : \mu(E) = 0 \text{ for all } \mu \in \mathcal{D} \} \), and \( \mathcal{N}_d \) its dyadic counterpart \( \{ E \subseteq \mathbb{R}^1 : \mu(E) = 0 \text{ for all } \mu \in \mathcal{D}_d \} \). Clearly \( \mathcal{N}_d \subseteq \mathcal{N} \), and \( \mathcal{N} \) is invariant under any bilipschitz mapping on \( \mathbb{R}^1 \). However \( \mathcal{N}_d \) is not invariant under translation, or under multiplication.

**Theorem 2.** There exist a perfect set \( S \subseteq [0,1] \) which is in \( \mathcal{N} \setminus \mathcal{N}_d \), and a set \( T \subseteq \mathbb{R}^1 \) of full measure (i.e., \( \mathbb{R}^1 \setminus T \) has zero length) such that \( t + S \in \mathcal{N}_d \) for each \( t \in T \).

A weaker version of Theorem 2 was proved in [9] with \( \dim T = 1 \). The present proof has the same structure, but uses more refined estimations.
The analogue of Theorem 2 under multiplication is more difficult. We are only able to find perfect sets $S$ and $T$ with $\dim T = 0$, so that $S \in \mathcal{N} \setminus \mathcal{N}_d$ but $tS \in \mathcal{N}_d$ for each $t \in T$. We shall report this elsewhere.

The following lemmas from [9] are needed in our proof.

Lemma 4. Let $\mu$ be a dyadic doubling measure on $\mathbb{R}^1$. Then there exists $c > 1$ depending on $\lambda(\mu)$ only, so that for any dyadic interval $S$ and any subinterval $T$ of $S$,

$$\frac{1}{4} \left( \frac{|T|}{|S|} \right)^{\frac{c}{1+c}} \mu(S) \leq \mu(T) \leq 4 \left( \frac{|T|}{|S|} \right)^{1/c} \mu(S).$$

Lemma 5. Let $S$ be any dyadic interval and $\mu$ and $\nu$ be two dyadic doubling measures on $\mathbb{R}^1$ satisfying $\mu(S) = \nu(S)$. Then the new measure $\omega \equiv \nu$ on $S$, with $\lambda(\omega) \leq \mu(S)$, is a dyadic doubling measure on $\mathbb{R}^1$ with $\lambda(\omega) \leq \max \{ \lambda(\mu), \lambda(\nu) \}$.

Lemma 6. Given $a, \varepsilon, \delta \in (0, 1)$ with $\varepsilon + \delta^a < 1/16$, then there exists a measure $\tau \in D_d$, with $\lambda(\tau) \leq 10^{1/8}$, which satisfies $\tau([0, 1]) = 1$, $\tau([0, \varepsilon]) = c$ and $\tau([1 - \delta, 1]) = \delta^a$.

We shall use $\tau_{a, \varepsilon, \delta}$ to denote the restriction of this $\tau$ measure to the interval $[0, 1]$.

Proof of Theorem 2. Let $\alpha > 1$ and choose $\beta > \alpha$, $0 < \alpha < \min \{ 1/5, \alpha \beta^{-1} \}$, $0 < c_m < 1/4$ and positive integers $L_m$ ($m \geq 1$) so that the following are true:

\begin{align}
(2.1) \quad \frac{1}{c_m} \alpha^{-\beta} &= o(1), \quad \text{as } m \to \infty, \\
(2.2) \quad \sum m^{2a-a} &< \infty, \\
(2.3) \quad \sum (1 - m^{-\beta})^{L_m} &< \infty, \\
\text{and} \\
(2.4) \quad \sum (1 - 4c_m)^{L_m} &= \infty.
\end{align}

For example, choose $\beta = 4\alpha$, $a = (\alpha - 1)/5\alpha$, $c_m = (4m^{2\alpha})^{-1}$ and $L_m = [m^{2\alpha}]$, with $[\cdot]$ the greatest integer function.
Let $K_1 = 0$, and $K_{m+1} = K_m + L_m$ for $m \geq 1$. Define $n_k$ inductively by letting $n_0 = 10$ and

$$n_{k+1} = n_k + 10 + \lceil \beta \log_2 m - \log_2 c_m \rceil$$

when $K_m \leq k < K_{m+1}$.

Given $m \geq 1$, $1 + K_m \leq k \leq K_{m+1}$ and integer $j$, denote by $L'$s, $I'$s and $J'$s the dyadic intervals:

$$L_{k,j} = \left[ \frac{j}{2^{n_k}}, \frac{j+1}{2^{n_k}} \right],$$

$$I_{k,j} = \left[ \frac{j}{2^{n_k}}, \frac{j+1}{2^{n_k}} + \frac{1}{2^{n_k+5 \hat{m}^\alpha}} \right]$$

and

$$J_{k,j} = \left[ \frac{j+1}{2^{n_k}} - \frac{1}{2^{n_k+5 \hat{m}^\beta}}, \frac{j+1}{2^{n_k}} \right],$$

where $\hat{m}^\alpha = 2^{(\alpha \log_2 m)}$, $\hat{m}^\beta = 2^{(\beta \log_2 m)}$ and $\lceil \cdot \rceil$ is the greatest integer function. Note that for $1 + K_m \leq k \leq K_{m+1}$,

$$|J_{k,j}|/|I_{k,j'}| = O(m^{\alpha-\beta}) \to 0, \quad \text{as } m \to \infty$$

and

$$|J_{k,j}|/|L_{k+1,j'}| = 2^{n_{k+1}-n_k-\lceil \beta \log_2 m \rceil - 5} \geq m^{2\alpha}.$$

To construct $S$, first we permanently remove from $S_1 = [0, 1]$ a group of mutually disjoint intervals $I_{k,j}$ of different sizes, with $k$ ranging from $1 + K_1$ to $K_2$, and collect some of the $J$ intervals from the remaining part of $S_1$; call the union $S_2$. Next we permanently remove from $S_2$ a group of intervals $I_{k,j}$ with $k$ ranging from $1 + K_2$ to $K_3$, and collect some of the $J$ intervals from the remaining part of $S_2$; call the union $S_3$, etc. Finally let $S = \cap S_m$.

Let $S_1 = [0, 1]$,

$$C_1' = \{I_{1,j} : J_{1,j-1} \cup I_{1,j} \subseteq S_1\},$$

$$C_1'' = \{J_{1,j} : J_{1,j} \cup I_{1,j+1} \subseteq S_1\}.$$

For $1 + K_1 \leq k < K_2$, let
Two problems on doubling measures

\[ C_{k+1}^I = C_k^I \cup \{I_{k+1,j} : J_{k+1,j-1} \cup I_{k+1,j} \text{ is contained in } S_1, \]
\[ \text{but neither } J_{k+1,j-1} \text{ nor } I_{k+1,j} \]
\[ \text{is contained in any interval in } C_k^I \cup C_k^I \}, \]

\[ C_{k+1}^I = C_k^I \cup \{J_{k+1,j} : J_{k+1,j-1} \cup I_{k+1,j+1} \text{ is contained in } S_1, \]
\[ \text{but neither } J_{k+1,j} \text{ nor } I_{k+1,j+1} \]
\[ \text{is contained in any interval in } C_k^I \cup C_k^I \} \].

Note that all intervals in \( C_{K_3}^I \cup C_{K_3}^I \) have mutually disjoint interiors, and that intervals in \( C_{K_2}^I \) and those in \( C_{K_2}^I \) appear in pairs sharing common end points.

Let
\[ S_2^I = \text{union of all intervals in } C_{K_2}^I, \]
\[ S_2 = \text{union of all intervals in } C_{K_2}^I. \]

We keep the interior of \( S_2^I \) in the complement of \( S \) permanently, and construct \( S_2^I \) and \( S_2 \) as subsets of \( S_2 \). Let
\[ C^I_{1+K_2} = \{I_{1+K_2,j} : J_{1+K_2,j-1} \cup I_{1+K_2,j} \subseteq S_2\}, \]
\[ C^I_{1+K_2} = \{J_{1+K_2,j} : J_{1+K_2,j-1} \cup I_{1+K_2,j+1} \subseteq S_2\}. \]

And define for \( 1 + K_2 \leq k < K_3 \),
\[ C^I_{k+1} = C_k^I \cup \{I_{k+1,j} : J_{k+1,j-1} \cup I_{k+1,j} \text{ is contained in } S_2, \]
\[ \text{but neither } J_{k+1,j-1} \text{ nor } I_{k+1,j} \]
\[ \text{is contained in any interval in } C_k^I \cup C_k^I \}, \]
\[ C^I_{k+1} = C_k^I \cup \{J_{k+1,j} : J_{k+1,j-1} \cup I_{k+1,j+1} \text{ is contained in } S_2, \]
\[ \text{but neither } J_{k+1,j} \text{ nor } I_{k+1,j+1} \]
\[ \text{is contained in any interval in } C_k^I \cup C_k^I \} ; \]

and let
\[ S_3^I = \text{union of all intervals in } C_{K_3}^I, \]
\[ S_3 = \text{union of all intervals in } C_{K_3}^I. \]
We keep the interior of $S^I_0$ in the complement of $S$ permanently, and construct $S^I_0$ and $S_4$ from $S_3$ as above. Continue this process to obtain $C^I_{K_m}, C^I_{K_{m+1}}, S^I_m$ and $S_m$ for all $m \geq 5$. Let

$$S = \bigcap_{m=1}^{\infty} S_m.$$  

The rest of the proof is based on the simple fact that $J_{k,j}$ and $I_{k,j+1}$ are two adjacent intervals of very uneven sizes (2.6), whose common boundary point $(j + 1)/2^n$ is a dyadic number.

To prove $S \in \mathcal{N}$, we note from (2.6) that for any $\nu \in \mathcal{D}$, and $1 + K_m \leq k \leq K_{m+1},$

$$\nu(J_{k,j}) \leq m^{c(\alpha - \beta)} \nu(J_{k,j} \cup I_{k,j+1}),$$

for some $c > 0$ depending only on the doubling constant of $\nu$. Summing over all $J_{k,j}$ in $C^I_{K_{m+1}}$ we have $\nu(S) \leq \nu(S_{m+1}) \leq m^{c(\alpha - \beta)} \nu([0,1])$. Thus $\nu(S) = 0$. Alternatively, $S$ is a porous set with large holes, therefore it is in $\mathcal{N}$, see [9].

To show $S \notin \mathcal{N}_d$, we apply scaled versions of Lemmas 5 and 6 repeatedly, to obtain a measure $\mu \in \mathcal{D}_d$ on $\mathbb{R}^1$, periodic with period 1, such that for $1 + K_m \leq k \leq K_{m+1}$ and all integers $j$  

(2.7)  
$$\mu(I_{k,j}) = (32 \tilde{m}^a)^{-1} \mu(L_{k,j})$$

and

(2.8)  
$$\mu(J_{k,j}) = (32 \tilde{m}^\beta)^{-a} \mu(L_{k,j}).$$

More precisely, $\mu$ is the weak limit of a subsequence of measures $\{\mu_{k_m}\}$ to be constructed as follows. Let $\mu_0$ be the Lebesgue measure on $\mathbb{R}^1$. Assume that $\mu_{k_m} \in \mathcal{D}_d$, has been constructed with period 1. Then inductively for $1 + K_m \leq k \leq K_{m+1}$, let $f_{k,j}$ be the linear map that maps $L_{k,j}$ onto $[0,1]$, and define for $E \subseteq L_{k,j}$,

$$\mu_{k_{m+1}}(E) = \mu_{k_m}(L_{k,j}) \tau_{(32 \tilde{m}^a)^{-1}}(f_{k,j}(E)),$$

where $\tau$ is the measure in Lemma 6. In view of Lemma 5, the measure $\mu_{k_{m+1}}$ is in $\mathcal{D}_d$ and satisfies (2.7) and (2.8) with $\mu$ replaced by $\mu_{k_{m+1}}$.

We note from the construction that

$$\mu_{k_2}([0,1] \setminus (S^I_0 \cup S_2)) \leq \left(1 - \frac{1}{2} (32^{-1} + 32^{-a})\right)^{K_2 - K_1}. $$
The occurrence of 1/2 above is due to the fact that each $J \in \mathcal{C}_d^I$ and its companion $I$ interval are not contained in the same $L$ interval, but rather in two adjacent $L$ intervals. Therefore
\[
\mu_k(S_2) \geq \mu_k\left(S_2^I \cup S_2\right) (1 - 32^{a-1})
\geq \left(1 - \left(1 - \frac{1}{2} \left(32^{-1} + 32^{-a}\right)\right)^{K_2 - K_1}\right) (1 - 32^{a-1})
\geq \left(1 - \left(1 - \frac{1}{2} 32^{-a}\right)^{K_2 - K_1}\right) (1 - 32^{a-1}) .
\]
From the construction of $S$,
\[
\mu(S) \geq \prod_{m=1}^{\infty} \left(1 - \left(1 - \frac{1}{2} 32^{-a} m^{-\beta a}\right)^{K_{m+1} - K_m}\right) (1 - 32^{a-1} m^{\beta a - \alpha}) ,
\]
which is positive in view of (2.2) and (2.3). Therefore $S \notin \mathcal{N}_d$.
For $x \in \mathbb{R}^1$, denote by $\|x\|$ the distance from $x$ to the nearest integer. Let $T$ be the set of $t$'s such that there are infinitely many $m$'s so that
\[
(2.9) \quad \| t 2^{5+n_k} \hat{m} \|^a > c_m , \quad \text{for every } k, 1 + K_m \leq k \leq K_{m+1} .
\]
Denote points $t$ in $[0,1]$ by their binary expansion $\sum_{n=1}^{\infty} t_n 2^{-n}$ with $t_n = 1$ or 0. Then $\| t 2^{5+n_k} \hat{m} \|^a > c_m$ provided that not all $t_n$ equal 0 for those $n$ in the interval $(5 + n_k + [\alpha \log_2 m], 7 + n_k + [\alpha \log_2 m] - \log_2 c_m)$, and not all $t_n$ equal 1 for the same range of $n$'s. In view of (2.5),
\[
n_{k+1} > n_k + [\alpha \log_2 m] - \log_2 c_m + 7 ; \text{ thus for } m \geq 1,
\]
\[
| \{ t \in [0,1] : \text{ (2.9) holds} \} | \geq (1 - 4 c_m)^{K_{m+1} - K_m} .
\]
Since $[0,1] \setminus T = \bigcup_{M \geq 10} (2.9)$ fails for every $m \geq M$,
\[
|[0,1] \setminus T| \leq \sum_{M=10}^{\infty} \prod_{m=M}^{\infty} \left(1 - (1 - 4 c_m)^{K_{m+1} - K_m}\right) = 0 ,
\]
because of (2.4). Similar argument show that $|\mathbb{R}^1 \setminus T| = 0$.
Given $t \in T$, assume that for a certain $m$,
\[
\| t 2^{n_k} \hat{m} \|^a > c_m , \quad \text{for every } k, 1 + K_m \leq k \leq K_{m+1} ;
\]
then for each integer $j$,

$$(2.10) \quad \frac{p}{2^{n_k+5 r_1^*}} + \frac{cm}{2^{n_k+5 r_1^*}} \leq t + \frac{j + 1}{2^{n_k}} \leq \frac{p + 1}{2^{n_k+5 r_1^*}} - \frac{cm}{2^{n_k+5 r_1^*}}$$

for some integer $p$. Note that $t + (j + 1)/2^{n_k}$ is the common boundary point for intervals $t + J_{k,j}$ and $t + I_{k,j+1}$, and that in view of (2.10),

$$t + J_{k,j} \subseteq \left[ \frac{p}{2^{n_k+5 r_1^*}}, \frac{j + 1}{2^{n_k}} \right],$$

$$t + I_{k,j+1} \supseteq \left[ t + \frac{j + 1}{2^{n_k}}, \frac{p + 1}{2^{n_k+5 r_1^*}} \right] \equiv I_{k,j+1}^*.$$

Suppose $\nu$ is in $D_d$. Because

$$\left[ \frac{p}{2^{n_k+5 r_1^*}}, \frac{p + 1}{2^{n_k+5 r_1^*}} \right]$$

is a dyadic interval, it follows from Lemma 4 that

$$\frac{\nu(t + J_{k,j})}{\nu(t + I_{k,j+1})} \leq \left( \frac{|J_{k,j}|}{|I_{k,j+1}|} \right)^c \leq \left( \frac{m^{a - \alpha}}{cm} \right)^c$$

for some $c > 0$ depending on $\nu$ only. Summing over all $J_{k,j}$ in $C_{K_m}^L$, we have

$$\nu(t + S) \leq (m^{a - \alpha} c^{-1})^c \nu([0,1]).$$

Because $t$ is in $T$, $m$ can be made arbitrarily large. Therefore $\nu(t + S) = 0$ by (2.1). This proves that $t + S \in \mathcal{N}_d$ for every $t \in T$.

References.


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