Calderón-Zygmund Operators on Product Spaces

Jean-Lin Journé

1. Introduction

In their well-known theory of singular integral operators, Calderón and Zygmund [3] obtained the boundedness of certain convolution operators on \( \mathbb{R}^d \) which generalize the Hilbert transform \( H \) in \( \mathbb{R}^1 \), defined for \( f \in C_0^\infty(\mathbb{R}^1) \) by

\[
Hf(x) = \lim_{\epsilon \to 0} \int_{|x-y| > \epsilon} \frac{f(y)}{x-y} \, dy.
\]

(0.1)

Typical examples of such operators are the Riesz transforms \( R_j, j \in [1, d] \), defined for \( f \in C_0^\infty(\mathbb{R}^d) \) by

\[
R_j f(x) = \lim_{\epsilon \to 0} \int_{|x-y| > \epsilon} \frac{x_j - y_j}{|x-y|^{d+1}} f(y) \, dy.
\]

(0.2)

Their program can be decomposed into two steps. In the first one they prove \( L^2 \)-boundedness using Plancherel’s theorem. In the second step they use the smoothness and size properties of the kernel and the \( L^2 \)-boundedness to prove \( L^p \)-boundedness for \( p \in [1, +\infty[ \) as well as the a.e. convergence of the r.h.s. of (0.2) for \( f \in L^p, p \in [1, +\infty[ \). Peetre [14] has shown that these operators are also bounded from \( \text{BMO}(\mathbb{R}^d) \) to \( \text{BMO}(\mathbb{R}^d) \).

The theory has been generalized in two ways.
In the first extension, one considers non-convolution operators associated to a kernel in the following sense. Let $\Delta$ be the diagonal set of $\mathbb{R}^d \times \mathbb{R}^d$ and let $K$ be a locally bounded function defined from $\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta$ to $\mathbb{C}$. Let $T: C_0^\infty(\mathbb{R}^d) \to [C_0^\infty(\mathbb{R}^d)]'$ be a linear operator defined in the weakest possible sense. Then $K$ is the kernel of $T$ if for $f, g \in C_0^\infty(\mathbb{R}^d)$ with disjoint supports, $\langle g, Tf \rangle$ is given by $\int \int g(x)K(x, y)f(y)\, dx\, dy$. Suppose moreover that $K$ satisfies some smoothness and size properties analogous to those enjoyed by the kernels of the Riesz transforms. Of course one cannot conclude that $T$ is bounded on $L^2$ and if $T$ is not a convolution operator one usually cannot use Plancherel's theorem. However it was observed that if the operator is known to be bounded on $L^2$ the second part of the program of Calderón and Zygmund can be carried out and one obtains a variety of results as in the convolution case. See [8] or [12]. In addition these operators are bounded from $L^\infty$ to BMO, the obstruction for boundedness on BMO being purely algebraic; that is, they are bounded on BMO if and only if they are well defined on BMO, as for instance, in the convolution case. The most famous non-convolution operator of this kind is the Cauchy-operator on Lipschitz curves $T_a$ defined for $a \in L^\infty(\mathbb{R})$, $|a|_\infty < 1$, $f, g \in C_0^\infty(\mathbb{R})$ by

\[(0.3) \quad \langle g, T_af \rangle = \int \int \frac{g(x)f(y)}{(x-y) + \int_a^y a(u)\, du}\, dx\, dy.\]

This example also illustrates the weakness of the theory since it leaves open the question of the $L^2$-boundedness of such operators. See however [2] and [7] for the Cauchy kernel. This gap has been recently filled, up to a certain extent, by the so-called $T_1$-theorem [9] which asserts that under a very weak regularity condition, $T$ is bounded on $L^2$ if and only if $T_1$ and $T^{*1}$, defined appropriately, both lie on BMO.

The second extension is due to R. Fefferman and E. Stein [11]. They study convolution operators which satisfy certain quantitative properties enjoyed by tensor products of operators of Calderón-Zygmund type, as for instance the double Hilbert transform defined for $f \in C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R})$ by

\[(0.4) \quad [(H_1 \otimes H_2)f](x_1, x_2) = \lim_{x_1, x_2 \to 0} \int \int \int_{(y_1, y_2) \geq (x_1, x_2)} \frac{f(y_1, y_2)}{(x_1 - y_1)(x_2 - y_2)}\, dy_1\, dy_2.\]

For such tensor products the $L^p$-boundedness for $p \in ]1, +\infty[$ is a trivial consequence of Fubini's theorem but for the more general Fefferman-Stein operators a new machinery is built in [11] which unfortunately uses at each step that the operators under consideration are convolution operators. Moreover it ignores «the BMO aspect of things» in which we shall be mostly interested, while it gives sharp results on maximal operators, which we cannot handle.
Our purpose is to unify up to a certain extent these two generalizations and to define on a product of $n$ Euclidean spaces a class of singular integral operators which coincides with the extended Calderón-Zygmund class in the case $n = 1$ and coincides in the convolution case with the Fefferman-Stein class when $n = 2$. Actually we extend the non-convolution-Calderón-Zygmund class, and then proceed by induction for $n > 2$. The basic example of an operator considered in this setting is the «$n$th-Cauchy operator» associated to the kernel $K_a$, defined for $a \in L^E_\infty(\mathbb{R}^n)$ and $|a|_\infty < 1$ by

$$K_a(x, y) = \frac{1}{\prod_{i=1}^n (x_i - y_i) + \int_{x_1}^{y_1} \ldots \int_{x_n}^{y_n} a(u_1, \ldots, u_n) \, du}.$$ 

As in the case $n = 1$, this kernel $K_a$ can be expanded in the sum $\sum_{\alpha} L_\alpha^j$ of «commutators» where

$$L_\alpha^j(x, y) = \left[ \prod_{i=1}^n (x_i - y_i) \right]^{-j-1} \left( \int_{x_1}^{y_1} \ldots \int_{x_n}^{y_n} a(u) \, du \right)^j.$$ 

Let $\tilde{L}_\alpha^j$ be the operator associated to $L_\alpha^j$. Then we show $\|\tilde{L}_\alpha^j\|_{2, 2} \leq C_\alpha \|a\|_\infty^j$. Thus we can sum the series and obtain

$$\|\tilde{K}_a\|_{2, 2} \leq \frac{1}{1 - C_n \|a\|_\infty} \quad \text{for} \quad |a|_\infty < \frac{1}{C_n}.$$ 

The general case $|a|_\infty < 1$ remains open.

The connection between $L^2$ and BMO, emphasized by the $T1$-Theorem and its proof, turns out to be extremely useful in this setting too. The BMO-space to be considered is the space of Chang-Fefferman studied in [5] which takes into account the product structure of the underlying space. As in the classical situation one makes two kinds of size and smoothness assumptions (integral or pointwise) on the kernel according to whether the associated operator is known to be bounded on $L^2$ or not. In the first case we show under rather weak assumptions on the kernel that the operator is also bounded from $L^\infty$ to BMO and therefore on all $L^p$'s for $p \in [1, \infty[$ and under somewhat stronger assumptions the boundedness on BMO, if there is no algebraic obstruction. In the second case we show a $T1$-theorem in the spirit of the classical one. In the case where $T$ is given by a kernel $K$ antisymmetric in each pair $(x_i, y_i)_{1 \leq i \leq n}$ as $K_a$ or $L_a^j$ for instance, the the $T1$-theorem reduces to: $T$ is bounded on $L^2$ if and only if $T1 \in \text{BMO}$.

In Sections 1 and 2 we recall some basic notations on singular integrals and Calderón-Zygmund operators in the classical situation and on BMO and Carleson measures on product spaces. The class of operators we wish to study is presented in Section 3, together with their more immediate properties.
In Section 4 we reduce the implication \( L^2 \)-boundedness \( \rightarrow \) \( L^\infty \)-BMO-boundedness to a geometric lemma which we prove in Section 5. This lemma may be thought of as a substitute for the Whitney decomposition in the setting of product spaces. In Section 6 we state our \( T1 \)-theorem and reduce its proof to two technical points which are studied in Section 7 and 8. Section 9 deals with a special property of antisymmetric kernels, which is new even when \( n = 1 \) and which is applied to the study of the kernel \( K_\alpha \) for \( |a| \leq \varepsilon_n \). Finally we apply in Section 10 the geometric lemma of Section 5 to extend a result of J. L. Rubio de Francia on a Littlewood-Paley inequality of arbitrary intervals of \( \mathbb{R} \) to the \( n \)-dimensional setting.

It is a real pleasure to express my gratitude to Raphy Coifman, Guy David, Yves Meyer, and Stephen Semmes for reading the manuscript and suggesting several improvements, to J. L. Rubio de Francia for bringing to my attention the conjecture solved in Section 10 and to Peter Jones for telling me it was obviously true.

1. Classical singular integral operators and Calderón-Zygmund operators on \( \mathbb{R}^d \)

The definitions we shall adopt are standard. However, the terminology will be slightly different from usual ([8], [12]).

Let \( \Omega = \mathbb{R}^d \times \mathbb{R}^d \setminus \Delta \), where \( \Delta = \{(x,y), x = y\} \), and let \( \delta \in ]0, 1[ \).

**Definition 1.** Let \( K \) be a continuous function defined on \( \Omega \) and taking its values in a Banach space \( B \). This function \( K \) is a \( B \)-\( \delta \)-standard kernel if the following are satisfied, for some constant \( C > 0 \).

For all \( (x,y) \) in \( \Omega \),

\[
|K(x,y)|_B \leq \frac{C}{|x - y|^{d + \delta}}.
\]  
(1.1)

For all \( (x,y) \) in \( \Omega \), and \( x' \) in \( \mathbb{R}^d \) such that \( |x - x'| < \frac{|x - y|}{2} \),

\[
|K(x,y) - K(x',y)|_B \leq C \frac{|x - x'|^\delta}{|x - y|^{d + \delta}} \quad \text{and}
\]

\[
|K(y,x) - K(y,x')|_B \leq C \frac{|x - x'|^\delta}{|x - y|^{d + \delta}}.
\]  
(1.2)

The smallest constant \( C \) for which (1.1) and (1.2) hold is denoted by \( |K|_{\delta,B} \). We shall omit the subscript \( B \) when it creates no ambiguity.

**Definition 2.** Let \( T : C_c^\infty (\mathbb{R}^d) \to [C_c^\infty (\mathbb{R}^d)]' \) be a continuous linear mapping. \( T \)
is a singular integral operator (SIO) if, for some $\delta \in [0, 1]$, there exists a $C^\delta$-standard kernel $K$ such that for all functions $f, g \in C^\infty_0(\mathbb{R}^d)$ having disjoint supports,

$$\langle g, Tf \rangle = \iint g(x)K(x, y)f(y)\,dx\,dy.$$ \hspace{1cm} (1.3)

Here $\langle g, Tf \rangle$ denotes the action of the distribution $Tf$ on the function $g$. We shall also say that $T$ is a $\delta$-SIO.

**Definition 3.** Let $T$ be a $\delta$-SIO. It is a $\delta$-Calderón-Zygmund operator ($\delta$-CZO) if it extends boundedly from $L^2$ to itself.

The following theorem gives necessary and sufficient conditions for a $\delta$-SIO to be a $\delta$-CZO. The statement of these conditions is explained afterwards.

**Theorem 1** [9]. Let $T$ be a $\delta$-SIO. It is a $\delta$-CZO if and only if

$$T1 \in \text{BMO}$$ \hspace{1cm} (1.4)

$$T^*1 \in \text{BMO}$$ \hspace{1cm} (1.5)

$T$ has the weak-boundedness property \hspace{1cm} (1.6)

In order to give a meaning to (1.4) we must show how $T$ acts on bounded $C^\infty$ functions. The meaning of (1.5) will then be clear since $T^*$, defined by

$$\langle g, T^*f \rangle = \langle f, Tg \rangle$$

for all $f, g \in C^\infty_0(\mathbb{R}^d)$, is also a $\delta$-SIO if $T$ is.

The action of an SIO, $T$ on $C^\infty_0(\mathbb{R}^d)$, the set of bounded $C^\infty$ functions, is described the following way ([8], [9]). For $f \in C^\infty_0(\mathbb{R}^d), Tf$ will be a distribution acting on $C^\infty_0(\mathbb{R}^d)$, the subspace of $C^\infty_0(\mathbb{R}^d)$ of functions $g$ such that $\int g\,dx = 0$.

Let $g$ be such a function and let $h \in C^\infty_0(\mathbb{R}^d)$ be equal to $f$ on a neighborhood of supp $g$, so that $g$ and $f - h$ have disjoint supports.

If $f$ has compact support,

$$\langle g, Tf \rangle = \langle g, Th \rangle + \langle g, T(f - h) \rangle,$$

where, by (1.3),

$$\langle g, T(f - h) \rangle = \iint g(x)K(x, y)[f(y) - h(y)]\,dx\,dy.$$ 

Since $g$ has mean value 0, this is also equal to

$$\iint g(x)[K(x, y) - K(x_0, y)]f(y)\,dx\,dy,$$

where $x_0$ is any point of supp $g$. Notice that by (1.2), this integral is absolutely convergent even if $(f - h)$ has non-compact support, and is independent of $x_0$. This integral can therefore serve as a definition of $\langle g, T(f - h) \rangle$. Obviously $\langle g, Th \rangle + \langle g, T(f - h) \rangle$ does not depend on the choice of $h$. Hence we can set

$$\langle g, Tf \rangle = \langle g, T(f - h) \rangle + \langle g, Th \rangle,$$

and this defines the desired extension.
This description yields immediately an effective method for computing $Tf$ when $f \in C^\infty_0(R^d)$.

**Lemma 1.** Let $0 \in C^\infty_0(R^d)$ and equal to 1 on $|x| < 1$. Let $\theta_q$ be defined for $q \in \mathbb{N}$ by $\theta_q(x) = \theta(\frac{x}{q})$, and for $f$ on $C^\infty_0(R^d)$ let $f_q = f \theta_q$. Then for all $g$ in $C^\infty_0(R^d)$,

$$\langle g, Tf \rangle = \lim_{q \to +\infty} \langle g, Tf_q \rangle. \quad (1.7)$$

We shall now give the meaning of (1.6). See [9].

**Definition 4.** Let $T$ be a $\delta$-SIO. It has the weak boundedness property if for any bounded subset $B$ of $C^\infty_0(R^d)$ there exists $C_B > 0$ such that for any pair $(\eta, \xi)$ of elements of $B$ and any $(x, t)$ in $\mathbb{R}^d_{x+1}$,

$$|\langle \eta^*_t, T^*\xi \rangle| \leq C_B t^{-d}, \quad (1.8)$$

where $\xi^*_t$ is defined by $\xi^*_t(y) = \int_0^t \xi(y - \frac{x - s}{t})$ and $\eta^*_t$ similarly.

We shall also write that $T$ has the WBP.

Note that any operator $T$ bounded on $L^2$ has the WBP since there exists a constant $C'$ such that $\|\xi^*_t\|_2 \leq C' t^{-d/2}$ for all $(x, t)$ in $\mathbb{R}^d_{x+1}$ and $\xi$ in $B$.

It is easy to show that $T$ has the WBP if there exists a constant $C$ and an integer $N$ such that for all cubes $Q$ of length $\delta(Q)$ and all functions $f$ and $g$ supported in $Q$, $|\langle g, Tf \rangle| \leq C|Q|P(N, g, Q)P(N, f, Q)$, where

$$P(N, g, Q) = \sum_{|\alpha| \leq N} \left| \frac{\partial^\alpha}{\partial x^\alpha} g \right|_\infty. \quad (1.9)$$

It is well known that CZO's are bounded from $L^\infty$ to BMO. However, there exist conditions much weaker than (1.2) that will ensure that an operator $T$, bounded on $L^2$, associated in the sense of (1.3) to a kernel $K$, is bounded from $L^\infty$ to BMO. The weakest of the known conditions is

$$\int_{|x-y| > 2|x-x'|} |K(x, y) - K(x', y)| \, dy < \infty \quad (1.10)$$

and is due to Calderón and Zygmund. For our purposes it will be best to assume something slightly stronger

$$\int_{|x-y| > 2|x-x'|} |K(x, y) - K(x', y)| \, dy \leq C 2^{-k\epsilon} \quad (1.10)$$

for some $\epsilon > 0$ and all $k \in \mathbb{N}$.

**Definition 5.** A locally integrable function $K$ satisfying to (1.10) is an $\epsilon$-kernel. An operator $T$ bounded on $L^2$ and associated to an $\epsilon$-kernel is a
Calderón-Zygmund operator of type \( (CZ) \). If \( K \) takes its values in a normed space \( V \), then it is a \( V \)-e-kernel.

We denote by \( |K|_{\nu, \nu} \) the smallest \( C \) for which (1.10) holds.

This distinction between pointwise conditions like (1.2) and integral conditions like (1.10) becomes crucial when the operator \( T \) maps functions of \( C_0^\infty(\mathbb{R}) \) into Hilbert-space valued distributions, that is, distributions acting on functions taking their values in a Hilbert space \( H \). In this case the kernel \( K \) takes its values in \( H \) and there are two possible ways to extend (1.10) in this setting, namely

\[
\int_{|x - y| > 2^{k} |x - x'|} |K(x, y) - K(x', y)|_H dy < C 2^{-k s}
\]

or, for all \( \lambda \in H \) such that \( \|\lambda\|_H = 1 \),

\[
\int_{|x - y| > 2^{k} |x - x'|} \langle \lambda, K(x, y) - K(x', y) \rangle_H dy < C 2^{-k s}.
\]  

(1.11)

Observe that an operator \( T \) bounded from \( L^2 \) to \( L^2_H \) associated to a kernel \( K \) satisfying (1.11) is bounded from \( L^p \) to \( BMO_H \) and therefore from \( L^p \) to \( L^p_H \) for all \( p \in [2, +\infty[ \), [15].

A slightly stronger version of (1.11) appears in the proof of the following theorem of J. L. Rubio de Francia.

**Theorem 2** [15]. Let \( \{I_k\}_{k=1}^\infty \) be a collection of disjoint intervals of \( \mathbb{R} \) and let \( S_{I_k} \) be the Fourier multiplier with symbol \( \chi_{I_k} \). Finally let \( \Delta \) be defined on \( L^2 \) by \( \Delta f = \Sigma (S_{I_k} f)^2 \)^{1/2}. Then \( \Delta \) is bounded on \( L^p \) for all \( p \in [2, +\infty[ \).

We shall conclude this section with a lemma of Coifman and Meyer, some notations and a remark.

The letter \( \phi \) will always denote a \( C_0^\infty \) radial function supported in the unit ball and such that \( \int \phi \, dx = 1 \). Let us define \( \phi_t \) by \( \phi_t(y) = \frac{1}{t^d} \phi\left(\frac{y}{t}\right) \). Then \( P_t \) is the convolution with \( \phi_t \).

The letter \( \psi \) will denote a radial \( C_0^\infty \) function supported in the unit ball and such that, for all \( \xi \in \mathbb{R}^d \), (1.12) \( \int_0^{\infty} \left| \hat{\psi}(t\xi) \right|^2 t^{-1} \, dt = 1 \). We define \( \psi \) and \( Q_t \) like \( \psi \) and \( P_t \).

**Lemma 2.** Let \( T \) be a \( \delta \)-SIO having the WBP. For all bounded subsets \( B \) of \( C_0^\infty(\mathbb{R}^d) \) and \( \eta, \xi \in B \) such that \( \int \eta \, dx = 0 \) or \( \int \xi \, dx = 0 \),

\[
\left| \langle \eta\gamma, T\xi \rangle \right| \leq C_B \omega_{\delta, r}(x - y),
\]

(1.13)

where \( \omega_{\delta, r}(x - y) = \frac{t^b}{t^{d + b} + |x - y|^{d + b}} \).

Conversely every continuous operator \( T: C_0^\infty(\mathbb{R}^d) \rightarrow [C_0^\infty(\mathbb{R}^d)]' \) having the WBP and satisfying (1.13) is a \( \delta' \)-SIO for all \( \delta' < \delta \).
We omit the proof of this lemma, which is elementary. For the converse part one uses the decomposition of \( T \) as 
\[- \int_0^t \frac{\partial}{\partial t} (P_t TP_t) \, dt.\]

This lemma suggests the following convention. In order to unify (1.1), (1.2) and (1.8) in an inequality analogous to (1.13) we shall remove the assumption \( \int \eta \, dx = 0 \) or \( \int \xi \, dx = 0 \) when \( x = y \). In the rest of the paper and without explicit mention we shall assume that if two functions \( \eta \) and \( \xi \) appear in an inequality of type (1.13) and \( x \neq y \), then \( \int \eta \, dx = 0 \) or \( \int \xi \, dx = 0 \).

Finally let us observe that if a function \( f \) is, say, in \( L^2 \) and \( T \) is a \( CZ \), then \( Tf \) is a \( L^2 \)-function and \( Q_t \, Tf \) is a \( C^\infty \) function. Let \( x \in \mathbb{R}^d \) and suppose \( f(z) = 0 \) when \( |x - z| \leq 2t \). Then we can write

\[
Q_t \, Tf(x) = \int_{|x-y| > 2t} (Q_t \, T)_{xy} f(z) \, dz \tag{1.14}
\]

where \( (Q_t \, T)_{xy} = \int \psi_{2t}(x - y)[K(y, z) - K(x, z)] \, dy \).

By (1.10), we have

\[
\int_{|x - z| > 2t} \left| (Q_t \, T)_{xy} \right| \, dz \leq C \, 2^{-k_x}. \tag{1.15}
\]

As a consequence, the following inequality holds for all \( u \in \mathbb{R}_+ \)

\[
\int_{|x - z| \leq 2t \cap u \leq |x - z|} |(Q_t \, T)_{xy}| \, dz \, dt \leq C_{x, t}. \tag{1.16}
\]

2. Carleson measures and BMO on product spaces

Let \( \Omega \) be an open subset in \( \mathbb{R} \times \mathbb{R} \). \( S(\Omega) \) is the subset of \( \mathbb{R}_+^2 \times \mathbb{R}_+^2 \) of \( (\theta, t_1, x_1, t_2, x_2) \)'s such that \( |x_1 - t_1, x_1 + t_1 \times x_2 - t_2, x_2 + t_2| \subseteq \Omega \).

**Definition 6** [4]. A Carleson measure on \( \mathbb{R}_+^2 \times \mathbb{R}_+^2 \) is a measure \( d\mu(x_1, t_1, x_2, t_2) = d\mu(x, t) \) such that for all \( \Omega \)

\[
\int_{S(\Omega)} d\mu(x, t) \leq C_{\Omega} |\Omega|.
\]

**Definition 7.** A function \( b \) is in \( BMO(\mathbb{R} \times \mathbb{R}) \) if it can be written as \( a_0 + H_j a_1 + H_j a_2 + H_j a_3 \) with \( \sum_{j=0}^3 |a_i|_\infty < +\infty \) and where the \( H_j \)'s, \( j \in \{1, 2, \ldots, 4\} \) are the partial Hilbert transforms. Moreover, \( \inf \sum_{j=0}^3 |a_i|_\infty \), where the inf is taken over all possible decompositions of \( b \), is a norm that makes \( BMO(\mathbb{R} \times \mathbb{R}) \) a Banach space.

Let \( Q_\theta \) be defined on \( C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R}) \) by \( Q_\theta[f \otimes g] = [Q_\theta, f] \otimes g \) and similarly for \( Q_{\theta_2} \). Clearly \( Q_{\theta_1} \) and \( Q_{\theta_2} \) extend by linearity to \( L^{1}_{loc}(\mathbb{R}^2) \). A. Chang and R. Fefferman have proved the following.
**Theorem A** [5]. A function $b$ in $L^1_{\text{loc}}$ is in $\text{BMO}$ if and only if 
$((Q_1, Q_2, b)(x_1, x_2))^i dx_1 dx_2 (t_1, t_2)^{-1} dt_1 dt_2$. Is a Carleson measure on $\mathbb{R}^2_+ \times \mathbb{R}^2_+$. 

**Theorem B** [6]. A linear operator $T$ bounded from $L^2$ to $L^2$ and from $L^\infty(\mathbb{R}^2)$ to $\text{BMO}(\mathbb{R} \times \mathbb{R})$ is bounded on all $L^p$'s for $p \in [2, +\infty[$.

It is a routine exercise to rewrite these definitions and theorems when $\mathbb{R} \times \mathbb{R}$ is replaced by $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \times \ldots \times \mathbb{R}^{d_n}$ and $\mathbb{R}^n_+ \times \mathbb{R}^2_+$ is replaced by $\mathbb{R}^{d_1 + 1} \times \ldots \times \mathbb{R}^{d_n + 1}$. Moreover, Theorems A and B remain valid if the functions under consideration are Hilbert-space valued. This will be used without mention in Section 10. In order to avoid minor technical complications we shall suppose from now on that all the $d_i$'s are equal to 1.

3. **Extension of the definitions of Section 1 in the setting of product spaces**

Let $T_1$ and $T_2$ be two classical $\delta$-SIO's on $\mathbb{R}$ and let $T = T_1 \otimes T_2$. This operator $T$ is a priori defined from $\mathcal{C}_0(\mathbb{R}) \otimes \mathcal{C}_0(\mathbb{R})$ to its algebraic dual by the formula

$$\langle g_1 \otimes g_2, T f_1 \otimes f_2 \rangle = \langle g_1, T_1 f_1 \rangle \langle g_2, T_2 f_2 \rangle.$$ 

Let $L_1$ and $L_2$ be the kernels of $T_1$ and $T_2$. If $g_1$ and $f_1$ have disjoint supports, we can write

$$\langle g_1 \otimes g_2, T f_1 \otimes f_2 \rangle = \int g_1(x) L_1(x, y) f_1(y) g_2(y) \text{ } dx \text{ } dy.$$ 

Let us put on the set of $\delta$-CZO's the norm $\| \cdot \|_{\text{CZO}}$ defined by $\|S\|_{\text{CZO}} = \|S\|_{L^2} + |K|$ where $K$ is the kernel of $S$. This makes the set of $\delta$-CZO's a Banach space which we denote by $\delta$CZO. Let $K_1(x, y) = L_1(x, y)T_2$. Then $K_1$ is a $\delta$CZO-valued function and is actually a $\delta$CZO-$\delta$-standard kernel and one has

$$\langle g_1 \otimes g_2, T f_1 \otimes f_2 \rangle = \int \int g_1(x) g_2(y) K_1(x, y) f_1(y) \text{ } dx \text{ } dy.$$ 

We can define $K_2(x, y)$ in a similar fashion. Now we forget that $T$ is a tensor product and set the following definition.

**Definition 8.** Let $T: \mathcal{C}_0(\mathbb{R}) \otimes \mathcal{C}_0(\mathbb{R}) \to [\mathcal{C}_0(\mathbb{R}) \otimes \mathcal{C}_0(\mathbb{R})]'$ be a continuous linear mapping. It is a $\delta$-SIO on $\mathbb{R} \times \mathbb{R}$ if there exists a pair $(K_1, K_2)$ of $\delta$CZO-$\delta$-standard kernels so that, for all $f, g, h, k \in \mathcal{C}_0(\mathbb{R})$, with $\text{supp } f \cap \text{supp } g = \emptyset$,

$$\langle g \otimes k, T f \otimes h \rangle = \int \int g(x) k(x, K_1(x, y) h(f(y)) \text{ } dx \text{ } dy, \quad (3.1)$$

$$\langle k \otimes g, Th \otimes f \rangle = \int \int g(x) k(x, K_2(x, y) h(f(y)) \text{ } dx \text{ } dy. \quad (3.2)$$
Let $\hat{T}$ be defined by

$$\langle g \otimes k, \hat{T}f \otimes h \rangle = \langle f \otimes k, Tg \otimes h \rangle.$$  

It is readily seen that $\hat{T}$ is a $\delta$-SIO if $T$ is. Its kernels $\hat{K}_1$ and $\hat{K}_2$ will be given by $\hat{K}_1(x, y) = K_1(y, x)$ and $\hat{K}_2(x, y) = [K_2(x, y)]^*$.  

**Definition 9.** A $\delta$-SIO $T$ on $\mathbb{R} \times \mathbb{R}$ is a $\delta$-CZO if $T$ and $\hat{T}$ are bounded on $L^2$. The role of $\hat{T}$ becomes clear in Section 6.

We can again put a norm on the set of $\delta$-CZO’s on $\mathbb{R} \times \mathbb{R}$ by setting

$$|T|_{\delta\text{CZO}(\mathbb{R} \times \mathbb{R})} = \|T\|_{2,2} + \|\hat{T}\|_{2,2} + \sum_{i=1}^{2} |K_i|_{\delta\text{CZO}(\mathbb{R})}.$$  

Using this remark one can easily define $\delta$-CZO’s on a product space with an arbitrary number of factors, by induction on this number.

We can repeat the same procedure to define $\text{CZO}$’s on product spaces. However for $\text{CZO}$’s there is no need to consider the partial adjoints as for $\delta$-CZO’s.

Let $T$ be a $\text{CZO}$ on $\mathbb{R}$ and $K$ its kernel. We define $|T|_{\text{CZO}}$ as $|T|_{2,2} + |K|_{\text{CZO}}$. A $\text{CZO}$ $T$ on $\mathbb{R} \times \mathbb{R}$ will be a bounded operator on $L^2$ associated in the sense of Definition 8 to a pair of $\text{CZO}$ kernels and we shall put $|T|_{\text{CZO}} = |T|_{2,2} + \sum_{i=1}^{2} |K_i|_{\text{CZO}}$.

In order to state an analogue of Theorem 1 in the product setting we need to observe that a $\delta$-SIO on $\mathbb{R} \times \mathbb{R}$ has a natural extension from $C_0^\infty \otimes C_0^\infty$ to $[C_0^\infty \otimes C_0^\infty]'$. This can be shown by an iteration of the argument sketched in Section 1. It also follows that Lemma 1 can be extended, using the same notations.

**Lemma 3.** For all $g_1, g_2 \in C_0^\infty(\mathbb{R})$ and $f_1, f_2 \in C_0^\infty(\mathbb{R})$,

$$\lim_{q \to +\infty, q' \to +\infty, q'' \to +\infty} \langle g_1 \otimes g_2, T[(f_1)_{q'} \otimes (f_2)_{q''}] \rangle = \lim_{q \to +\infty} \langle g_1 \otimes g_2, T[(f_1)_q \otimes f_2] \rangle = \langle g_1 \otimes g_2, T(\mathcal{F}_1 \otimes f_2) \rangle.$$  

In order to extend the definition of the WBP in the product setting it is convenient to introduce the following notations.

Let $T$ be a $\delta$-SIO on $\mathbb{R} \times \mathbb{R}$ and $f, g \in C_0^\infty(\mathbb{R})$. The operator $\langle g, T^\dagger f \rangle : C_0^\infty(\mathbb{R}) \to C_0^\infty(\mathbb{R})'$ is defined by

$$\langle h, \langle g, T^\dagger f \rangle k \rangle = \langle g \otimes h, T f \otimes k \rangle.$$  

It is easy to see that $\langle g, T^\dagger f \rangle$ is a $\delta$-SIO on $\mathbb{R}$ with kernel $\langle g, T^\dagger f \rangle (x, y) = \langle g, K_2(x, y) f \rangle$. One defines $\langle g, T^2 f \rangle$ similarly. The notation $T^\dagger f = 0$ simply means $\langle g, T^\dagger f \rangle = 0$ for all $g$. Notice that all this makes sense if $f \in C_0^\infty(\mathbb{R})$ and
\( g \in C_c^0(\mathbb{R}) \). In particular \( T^1 1 = 0 \) is equivalent to \( \langle k, T^2 h \rangle 1 = 0 \) for all \( k, h \in C_c^0(\mathbb{R}) \). Similarly \( T^* 1 = 0 \) means \( \langle k, T^2 h \rangle^* 1 = 0 \) in the same conditions. Exchanging the role of indices we obtain the meaning of \( T^2 1 = 0 \) or \( T^* 1 = 0 \).

In the following, the notations are those of Definition 4.

**Definition 10.** Let \( T \) be a \( \delta \)-SIO on \( \mathbb{R} \times \mathbb{R} \). \( T \) has the WBP if for \( i \in \{1, 2\} \)

\[
\| \langle \eta_i, T^i \xi_i^* \rangle \|_{C^{\alpha}_t} \leq C_{\delta} t^{-1}. \tag{3.3}
\]

It is easy to see that a \( \delta \)-CZO on \( \mathbb{R} \times \mathbb{R} \) has the WBP.

Next we indicate the extension of Lemma 2 in the product setting.

**Lemma 4.** Let \( T \) be a \( \delta \)-SIO with the WBP. Then for all \( B, (\eta, \xi) \in B \times B, \)

\( (x, y) \in \mathbb{R} \times \mathbb{R}, t > 0 \) and \( i \in \{1, 2\}, \)

\[
\| \langle \eta_i, T^i \xi_i \rangle \|_{C^{\alpha}_t} \leq C_{\delta} (x - y). \tag{3.4}
\]

Conversely every bounded operator \( T \) defined from \( C^0_0 \otimes C^0_0 \) to its dual satisfying to (3.4) is a \( \beta \)-SIO having the WBP for all \( \beta < \delta \).

In this statement we made use of the convention of Section 1. The proof of this lemma is routine and we omit the details. Of course lemmas 3 and 4 extend in the setting of an arbitrary product of copies of \( \mathbb{R} \).

To conclude this section, we shall give the analogue of (1.14), (1.15) and (1.16) in the product setting.

Suppose first that there are only two factors in the product. Let \( T \) be a CZ\( \epsilon \) on \( \mathbb{R} \times \mathbb{R} \) and \( f \in L^2(\mathbb{R}^2) \). Then \( (Q_1 Q_2 T f)(x_1, x_2) \) is a CZ\( \epsilon \) function of \( (x_1, x_2) \).

If \( x_1, t_1 \) are fixed and \( f(z_1, z_2) = 0 \) for \( |x_1 - z_1| \leq 2t_1 \), then we can write

\[
\int (Q_1 Q_2 T f)(x_1, x_2) = \int (Q_1 T)_{z_1}(f(z_1, z_2))(x_2) \, dz_1, \tag{3.5}
\]

where \( (Q_1 T)_{z_1} \) is a CZ\( \epsilon \) acting on functions of \( z_2 \), and given by

\[
(Q_1 T)_{z_1} = \int \psi_1(x_1 - y_1) [K_1(y_1, z_1) - K_1(x_1, z_1)] \, dy_1.
\]

Here \( K_1 \) is the first kernel of \( T \) and the symbol \( \langle \cdot \rangle \) over \( z_1 \) simply means that \( z_1 \) has become a parameter in (3.5). It is not clear that the integral in (3.5) converges absolutely. However by (3.6) below, that will be the case if \( f(z_1, z_2) \) is uniformly in \( L^2(dz_2) \), in particular if \( f \) is bounded with compact support.

The definition of a CZ\( \epsilon \) on \( \mathbb{R} \times \mathbb{R} \) immediately yields the following generalization of (1.15),

\[
\int |x_1 - z_1| \geq 2^{k_1} |(Q_1 T)_{z_1}| \, dz_1 \leq C 2^{-kt}. \tag{3.6}
\]

The case of a product of three spaces or more is very similar.
For all $I \subseteq [1, n]$, $(x_i, i \in I) \in \mathbb{R}^I$ and $(t_i, i \in I) \in (\mathbb{R}_+)^I$ and $(z_i, i \in I) \in \mathbb{R}^I$ such that for all $i \in I$, $|z_i - x_i| \geq 2t_i$ (we write also $|z_i - x_i| \geq 2t_i$) the symbol $[Q_{i_j}T]|_{x_jz_j}$ denotes a CZe acting on $L^2(\mathbb{R}^I)$, where $J = \{1, n\} \setminus I$. This CZe is defined by induction on $|I|$. If $I = \{i\}$ and $K_i$ is the kernel of $T$ in the variable $i$, then

$$[Q_{i_j}T]|_{x_jz_j} = \int \psi_t(x_i - y_i)[K_i(y_i, z_i) - K_i(x_i, z_i)] dy_i.$$ 

Now if $[Q_{i_j}T]|_{x_jz_j}$ is defined and $I' = I \cup \{i\}$ we define $[Q_{i_j}T]|_{x_jz_j} = [Q_i[T]|_{x_jz_j} = z_j|x_jz_j|$. This makes sense since $[Q_{i_j}T]|_{x_jz_j}$ is itself a CZe and has a kernel in the $i$-variable. On the other hand it is readily seen that $[Q_{i_j}T]|_{x_jz_j}$ depends only on $t_j$, $x_j$ and $z_j$ and not on the decomposition of $I'$ as $I \cup \{i\}$. So the notation is consistent.

Let $I \subseteq [1, n]$ and $J = [1, n] \setminus I$ and let $f \in L^\infty(\mathbb{R}^n)$ have compact support and suppose $f(x) = 0$ if $|x_i - z_i| \geq 2t_i$ for some $i \in I$. Then with obvious notations we write

$$[Q_{i_j}T](x) = \int [Q_{i_j}[Q_{i_j}T]|_{x_jz_j}]f(z_i, z_j) dz_i. \quad (3.7)$$

From (3.10) below it follows that this integral is absolutely convergent. Indeed (1.15) and (1.16) extend easily to the following, where $i \not\in I$ and $I' = I \cup \{i\}$:

$$\int_{|x_j - z_j| \geq 2t_i} [Q_{i_j}T]|_{x_jz_j}|cz_i dz_i \leq C 2^{-k_i} \|[Q_{i_j}T]|_{x_jz_j}\|_{cz_i} \tag{3.8}$$

and for $u \in \mathbb{R}_+$

$$\int_{t_i \leq u} \|[Q_{i_j}T]|_{x_jz_j}|cz_i dz_i \frac{dt_i}{t_i} \leq C \|[Q_{i_j}T]|_{x_jz_j}\|_{cz_i}. \quad \tag{3.9}$$

Moreover it follows from (3.8),

$$\int_{|x_j - z_j| \geq 2t_i} [Q_{i_j}T]|_{x_jz_j}|cz_i dz_i \leq C \|T\|_{cz_i} \times 2^{-k_i}. \tag{3.10}$$

4. $L^\infty$-BMO boundedness of CZe\'s on $\mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R} = \mathbb{R}^n$

We wish to show the following.

**Theorem 3.** Let $T$ be a CZe on $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}$. Then $T$ admits a bounded extension from $L^\infty(\mathbb{R}^n)$ to $\text{BMO}(\mathbb{R} \times \ldots \times \mathbb{R})$.

By interpolation it follows that $T$ is bounded on all $L^p$'s for $p \in [2, +\infty[$ and if $T^*$ is also a CZe, then $T$ is bounded on all $L^p$'s for $p \in [1, +\infty[$. This situation occurs automatically in the convolution case where we can conclude the following.
Corollary. Let $T$ be a CZè on $\mathbb{R}^n = \mathbb{R} \times \ldots \times \mathbb{R}$ and a convolution operator. Then $T$ admits a bounded extension from BMO($\mathbb{R}^n$) to itself.

To prove the corollary we use the $H^1$-BMO duality [5] and an argument of [10], p. 150. Since $L^2$ is dense in $H^1$ (this is a trivial consequence of the atomic decomposition for $H^1$ [6]) it is enough to show that for all $f \in L^2 \cap H^1$, $\| T^* f \|_{H^1} \leq C \| f \|_{H^1}$, or equivalently that $T^* f, H_1 T^* f, H_2 T^* f$ and $H_1 H_2 T^* f$ are all in $L^1$ with a norm less than $C \| f \|_{H^1}$. But as functions of $L^2$ these four functions are equal to $T^* f, T^* H_1 f, T^* H_2 f$ and $T^* H_1 H_2 f$ which are in $L^1$ since by Theorem 3 $T^*$ maps $H^1$ in $L^1$ and $f, H_1 f, H_2 f$, and $H_1 H_2 f$ are all in $H^1$. The corollary is proved.

There are other CZè's which are candidates for being bounded on BMO, namely those defined on BMO. In the case $n = 2$ to be defined on BMO is equivalent to the conditions $T^1 1 = 0 = T^2 1$. It turns out that one can still prove that $T$ is then bounded on BMO but the assumptions on the kernels of the CZè's have to be strengthened in order to know that $TH_1, TH_2$ and $TH_1 H_2$ are also CZè's if $T$ is and satisfies $T^1 1 = 0 = T^2 1$. We omit the details.

We now turn to the proof of Theorem 3. In order to use the induction hypothesis it is convenient to have the following formulation of Theorem 3, which is clearly equivalent by Theorem A.

Theorem 3'. There exists $C_{n, e} > 0$ such that for all bounded open subsets $\Omega$ of $\mathbb{R}^n$, all $b \in L^n(\mathbb{R}^n)$ with compact support and all $T \in CZè(\mathbb{R} \times \ldots \times \mathbb{R})$,

$$
\int_{\Omega} |Q_t T b(x)|^2 \, dx = \frac{dt}{t} \leq C_{n, e} \| T \|_{CZè} \| b \|_{L^n}^2 |\Omega|.
$$

We shall need the following lemma.

Lemma 5. There exists a constant $C_{n, e}$ such that for all bounded open subsets $\Omega$ of $\mathbb{R}^n$ there exists $n$ functions $T_1, \ldots, T_n$ defined from $S(\Omega)$ to $\mathbb{R}_+$ such that $T_i(x, t) \geq 2t_i$ and with the following properties:

$$
\text{If } \Omega_n = \bigcup_{(t, x) \in S(\Omega)} \prod_{1 \leq i \leq n} |x_i - T_i, x_i + T_i|,
$$

then $|\Omega_n| \leq C_n |\Omega|$.

For all $T \in CZè(\mathbb{R} \times \ldots \times \mathbb{R})$, all $I \subseteq [1, n]$, $I \neq \phi$, let

$$
E_{x_i, t_i, z_j} = \bigcup_{T_j(x, t) < |x_j - z_j|} \prod_{j \notin I} |x_j - t_j, x_j + t_j|.
$$

Then

$$
\int |E_{x_i, t_i, z_j}| \cdot |(Q_t T)_{x_i, z_j}| \, dt \frac{dt}{t} \leq C_{n, e} |\Omega| \| T \|_{CZè}.
$$

(4.3)
Of course when \( I = [1, n] \) (4.3) has to be interpreted the following way: 
\([Q_I T]_{x_I t_I} \) is a real number and \(|E_{x_I t_I z_I}| = 1\) if \( T_i(x, t) \leq |x_i - z_i| \) for all \( i \in [1, n] \) and \(|E_{x_I t_I z_I}| = 0\) otherwise.

We postpone the proof of Lemma 5 to the next section.

Let \( \Omega \) and \( \Omega_n \) be as in Lemma 5, \( b \in L^\infty(\mathbb{R}^n) \) with compact support and \( \|b\|_\infty \leq 1 \) and let \( T \in CZC(\mathbb{R}^n) \) with \( \|T\|_{CZC} \leq 1 \). We want to prove

\[
\int_{S(\Omega)} |Q_I T b(x)|^2 \frac{dx dt}{t} \leq C_{n, \epsilon} |\Omega|.
\]  

Using (4.2) we immediately reduce to the case where \( b \) is supported out of \( \Omega_n \).

Just write \( b = b_{x \Omega_n} + b_{x \Omega_n} \) and observe that

\[
\int_{S(\Omega)} |(Q_I T b_{x \Omega_n})(x)|^2 \frac{dx dt}{t} \leq C_n \|b_{x \Omega_n}\|_2^2 \leq C_n |\Omega_n| \leq C_n |\Omega|.
\]

Suppose from now on that \( b \) is supported out of \( \Omega_n \). Then, for each \( (x, t) \in S(\Omega) \) and \( z \in \text{supp } b \), \( |z - x| \geq T_i \) for at least one index \( i \). This yields the following decomposition for \( b \):

\[
b = \sum_{I \subseteq [1, n], I \neq \emptyset} (-1)^{|I|} b_{x, t, I},
\]

where

\[
b_{x, t, I}(z) = b(z) \prod_{i \in I} x_i, |x_i - z_i| \geq T_i(x, t).
\]

Thus

\[
(Q_I T b)(x) = \sum_{I \subseteq [1, n], I \neq \emptyset} (-1)^{|I|} b_{x, t, I} (x).
\]

Therefore, to prove (4.4) it is enough to prove for all \( I \subseteq [1, n], \ I \neq \emptyset \),

\[
\int_{S(\Omega)} |Q_I T b_{x, t, I}(x)|^2 \frac{dx dt}{t} \leq C_{n, \epsilon} |\Omega|.
\]  

Since \( T_i(x, t) \geq 2t_i \), we can use (3.7), which reads

\[
(Q_I T b_{x, t, I})(x) = \int_{t_I} [Q_I, [Q_I, T]_{x_I t_I} b_{x, z_I}](x) x_i, z_i, t_i \leq T_i, \ t_j \geq T_j.
\]

For \( x_j, t_j \) fixed, let \( E_{x_j t_j} = \bigcup \prod_{j \neq I} [x_j - t_j, x_j + t_j] \), the union being over the \( t_i \)'s such that \( (x_i, x_j, t_i, t_j) \in S(\Omega) \). Minkowski's inequality and (4.6) yield
\[
\int_{S(E_{x,t_j})} |Q \cdot T b_{x,t_j}(x)|^2 \ dx_j \frac{dt_j}{t_j} \leq \left( \int_{S(E_{x,t_j})} \left( \int_{S(E_{x,t_j})} [Q_jT_j]_{z_j} b(z_j) (x_j) \right)^2 \times x_j |x_j - z_j| t_j \frac{dx_j \ dt_j}{t_j} \right)^{1/2} \ dz_j \right)^2.
\]

Now let \(x_j, t_j, z_j\) be fixed and \(E_{x,t_j,z_j}\) as defined in Lemma 5. If \((x_j, t_j) \in S(E_{x,t_j})\) and \(T_j(x, t) \leq |x_j - z_j|\), then \((x_j, t_j) \in S(E_{x,t_j,z_j})\). Therefore we need only to dominate
\[
\left( \int_{|z_j - x_j| \geq 2 t_j} \left( \int_{S(E_{x,t_j})} \left( \int_{S(E_{x,t_j})} [Q_jT_j]_{z_j} b(z_j) \right)^2 |x_j| \ dx_j \frac{dx_j \ dt_j}{t_j} \right)^{1/2} \ dz_j \right)^2.
\]

The induction hypothesis under the form (4.1) yields the following majorant
\[
\left( \int_{|z_j - x_j| \geq 2 t_j} \left( \int_{S(E_{x,t_j})} [Q_jT_j]_{z_j} \right)^2 C_{z_j} \ dz_j \right)^2.
\]

By (3.10) and Cauchy-Schwarz, this is less than
\[
C_{n,t} \left( \int_{|z_j - x_j| \geq 2 t_j} \left[ \int_{S(E_{x,t_j})} [Q_jT_j]_{z_j} \right] C_{z_j} \ dz_j \right).
\]

It remains to integrate against \(dx_j \ dt_j/t_j\) and use (4.3). In the case where \(I = [1, n]\) some minor modifications in notations are needed. They are left to the reader. The proof is therefore reduced to showing Lemma 5.

5. Proof of Lemma 5

When \(n = 1\) this lemma is trivial. Let \(\Omega\) be a bounded open subset of \(\mathbb{R}\) and for \(x \in \Omega\), let \(I(x)\) be the connected component of \(x\) in \(\Omega\). Then simply set \(T(x, t) = |I(x)|\) for \((x, t) \in S(\Omega)\). Clearly \(T(x, t) \geq 2t\) since \(|x - t, x + t| \subseteq I(x)\). Moreover \(|x - I(x), x + I(x)| \subseteq 3I(x)\) which implies (4.2) with \(C_1 = 3\). Finally (4.3) reduces to
\[
\int_{S(\Omega)} \int_{x \in I(x)} \left( |Q \cdot T|_{x,z} \right) dx \ dz \ dz \ dx \ dt \leq C_1 |\Omega| \ T_{CZ},
\]

which follows trivially from (3.9) with \(u = |I(x)|/2\). This observation will permit us to illustrate in a simple case one point of the strategy of the proof.
Lemma 6. Suppose we have built $T_1 \ldots T_n$ such that (4.3) holds for $I = [2, n]$. Then if $T_1 \geq |I_{x_1}(x_1)|$, (4.3) holds for $I = [1, n]$.

Here $I_{x_1}(x_1)$ denotes the connected component of $x_1$ in $E_{x_1}$, as defined in Section 4.

Let $x_i, t_i, z_i$ be fixed. To deduce (4.3) for $I$ form (4.3) for $I$, it is enough to show that

$$\int_{E_{x_1}, z_i} |[Q_{x_1} T]_{x_1, z_i}| \frac{dx_1}{t_1} \leq C [Q_{x_1} T]_{x_1, z_i} \leq C [Q_{x_1} T]_{x_1, z_i},$$

and then integrate against $dx_i \frac{dt_i}{t_i} dz_i$.

This inequality actually means

$$\int_{E_{x_1}, z_i} |[Q_{x_1} T]_{x_1, z_i}| \frac{dx_1}{t_1} \leq C [E_{x_1}, z_i] \leq C [Q_{x_1} T]_{x_1, z_i}.$$

Thanks to the formula

$$[Q_{x_1} T]_{x_1, z_i} = [Q_{x_1} [Q_{x_1} T]_{x_1, z_i}],$$

we are almost in position to use (5.1). We only need that the conditions on $(x_1, z_i, t_i)$ imply

i) $(x_1, t_i) \in S(E_{x_1}, z_i)$

ii) $|x_1 - z_i| \geq |I_{x_1, t_i}(x_1)|$, where $I_{x_1, t_i}(x_1)$ is the component of $x_1$ in $E_{x_1}, z_i$.

i) follows from the definition of $E_{x_1, t_i}(x_1)$ and from the condition $|x_1 - z_i| \geq T_i$. ii) follows from the fact that $E_{x_1, t_i} \subseteq E_{x_1, t_i}$ for all $z_i$. Therefore $|x_1 - z_i| \geq T_i \geq |I_{x_1, t_i}(x_1)| \geq |I_{x_1, t_i}(x_1)|$. This implies ii) and lemma 6 is proved.

In general one point in the strategy will be to define the $T_i$'s by induction on $i$ in such a way that if $I$ is a set of indices and $t_0 < \inf I$, and if the $T_i$'s are such that (4.3) holds for $I$, then it holds for $\{t_0\} \cup I$, almost independently of the choice of the $T_i$'s for $i > t_0$.

Lemma 7. Let $\Omega \subseteq \mathbb{R}^n$, $(x_1, t_1) \in \mathbb{R}^2_+$ and for $(x_2, \ldots, x_n) \in E_{x_1, t_1}$, let $\tau(x_1, t_1, x_2, x_3, \ldots, x_n) = 2t_1 \vee \inf \alpha, (x_1, \alpha_{x_1, t_1}(x_n)) \geq \frac{1}{2}$. For $(x, t) \in S(\Omega)$, let $\tau(x, t) = \sup \tau(x_1, t_1, x_2, \ldots, x_n)$, the sup being over those $(y_n)_{2 \leq i \leq n}$ such that $|x_i - y_i| \leq t_i$ for all $i \in [2, n]$. For $z_i \in \mathbb{R}$ such that $|z_i - x_i| \geq 2t_1$ let

$$E_{x_1, t_1, z_1} = \bigcup_{\tau(x, t) < |x_1 - z_1|} \prod_{i=2}^n |x_i - t_i, x_i + t_i|. $$
Then, for \( T \in CZ_{\mathbb{R} \times \ldots \times \mathbb{R}} \) with \( \| T \|_{CZ} \leq 1 \)

\[
(5.2) \quad \int_{|x_1 - z_1| \geq 2t_1} |\mathcal{E}_{x_1, z_1}| \| (Q(T)_{x_1, z_1})_{CZ} \| dx_1 \frac{dt_1}{t_1} \, dz_1 \leq C_n |\Omega|.
\]

Moreover, if

\[
\Omega' = \bigcup_{(x, t) \in S(\Omega)} |x_1 - \tau, x_1 + \tau| \times \prod_{i \geq 2} |x_i - t_i, x_i + t_i|,
\]

then

\[
(5.3) \quad \chi_{\Omega'} \leq \frac{1}{2} (\chi_{\Omega})^*,
\]

where * is the strong Hardy-Littlewood maximal operator.

In order to prove (5.3), it is enough to prove that, for all \((x, t) \in S(\Omega)\),

\[
\frac{\left( \prod_{i \geq 2} |x_i - t_i, x_i + t_i| \cap E_{x_1, t_1} \setminus E_{x_1, \beta} \right)}{2^n \prod_{i \geq 2} t_i} \geq \frac{1}{2}.
\]

If \( \tau = 2t_1 \) this is obvious since \((x, t) \in S(\Omega)\). If \( \tau > 2t_1 \) and \((y_i)_{i \leq n} \) such that \( |x_i - y_i| \leq t_i \) for \( i \in [2, n] \), and \( \tau_1(x_1, t_1, y_2, \ldots, y_n) > \beta \). Therefore \((x_{E_{x_1, t_1}} \setminus E_{x_1, \beta})^*(y_2, \ldots, y_n) < \frac{1}{2} \) and in particular

\[
\prod_{i \geq 2} |x_i - t_i, x_i + t_i| \cap E_{x_1, t_1} \setminus E_{x_1, \beta} \leq \frac{1}{2}.
\]

Since \( |x_i - t_i, x_i + t_i| \leq E_{x_1, t_1} \), this is equivalent to

\[
\frac{\left( \prod_{i \geq 2} |x_i - t_i, x_i + t_i| \cap E_{x_1, \beta} \right)}{2^n \prod_{i \geq 2} t_i} > \frac{1}{2}.
\]

Since \( |x_1 - \beta, x_1 + \beta| \times E_{x_1, \beta} \leq \Omega \), this implies

\[
\frac{\left| x_1 - \beta, x_1 + \beta| \times \prod_{i \geq 2} |x_i - t_i, x_i + t_i| \cap E_{x_1, \beta} \right|}{2^n \prod_{i \geq 2} t_i} > \frac{1}{2}.
\]

Letting \( \beta \) tend to \( \tau \), we obtain the desired inequality and (5.3).

To prove (5.2) observe that

\[
\mathcal{E}_{x_1, z_1} \subseteq \{(y_2, \ldots, y_n) \in \mathbb{R}^{[2, n]}, \tau_1(x_1, t_1, y_2, \ldots, y_n) < |x_1 - z_1| \} \subseteq \{(y_2, \ldots, y_n), (x_{E_{x_1, t_1}} \setminus E_{x_1, \beta})^*(y_2 \ldots y_n) \geq \frac{1}{2} \}.
\]
This latter inclusion follows from the trivial fact that \((x_{E_{x_1t_1}} \setminus E_{x_1^0})^\times (y_2, \ldots, y_n)\)

is an increasing function of \(\alpha\). These inclusions imply \(|E_{x_1t_1, z_1}| <

\leq C_n|E_{x_1t_1} \setminus E_{x_1|x_1 - z_1}|\). At this point we need the following.

**Lemma 8.** Let \(x_1 \in \mathbb{R}, T \in C\mathcal{Z}(\mathbb{R} \times \ldots \times \mathbb{R})\) with \(||T||_{C\mathcal{Z}} \leq 1\), and let \(F: \mathbb{R}_+ \rightarrow \mathbb{R}_+\) be a decreasing function vanishing for \(t\) large. Then

\[
(5.4) \quad \int_{|x_1 - z_1| \geq 2t_1} \left[ F(t_1) - F(|x_1 - z_1|) \right] \left| (Q_{t_1}T)_{x_1, z_1} \right|_{C\mathcal{Z}} \frac{dt_1}{t_1} dz_1 \leq C_{n} F(0^+) \cdot
\]

It is easy to reduce to the case where \(F\) is \(C^1\). In this case write \(F(t_1) - F(|x_1 - z_1|) = -\int_{t_1}^{x_1 - z_1} F'(u) du\). Using (3.9) with \(I = \phi\) and \(I' = \{1\}\), we obtain, since \(-F' > 0\), a domination of the L.H.S. of (5.4) by \(-||T||_{C\mathcal{Z}} \times \int_0^{\infty} F'(u) du\), which proves Lemma 8.

To prove (5.3) we apply (5.4) with \(F(t) = E_{x_1t_1}\). The restriction \(|x_1 - z_1| \geq 2t_1\) is irrelevant since otherwise \(E_{x_1t_1, z_1} = \phi\). An application of (5.4) and the inequality \(|E_{x_1t_1, z_1}| \leq C(|E_{x_1t_1}| - |E_{x_1|x_1 - z_1}|)\) yield

\[
\int |E_{x_1t_1, z_1}| \left| (Q_{t_1}T)_{x_1, z_1} \right|_{C\mathcal{Z}} \frac{dt_1}{t_1} dz_1 \leq C_{n} |E_{x_1t_1}|, \quad \text{where} \quad E_{x_1t_1} = \bigcup_{t_1 > 0} E_{x_1t_1}.
\]

An integration in \(x_1\) yields \(C_{n} \int |E_{x_1t_1}| dx_1\) as a majorant of the l.h.s. of (5.3).

But this is exactly \(C_{n} |\Omega|\), and Lemma 7 is proved.

We shall use Lemma 7 with many indices playing the role of index 1 and with many sets instead of \(\Omega\); we shall specify which index and which set are considered, e.g. \(\tau_{i(x_1, t_1, x_2, t_2, \ldots, x_n, t_n, \Omega)}\).

A direct consequence of Lemma 7 is the following. If \(T_i(x, t) \geq \tau_{i(x, t, \Omega)}\), then \(E_{x_1t_1, z_1} \subseteq E_{x_1t_1, z_1}\), and (5.2) implies (4.3) for \(I = \{1\}\). Now we define the \(T_i\)’s by induction on \(i\). The letter \(\omega\) will denote an open subset of \(\mathbb{R}^k\) for some \(k \in [1, n]\) which will be specified by the context. We shall use the notation \(E_{x_1t_1, \Omega}\) as in Section 4 but we shall specify the set under consideration, e.g. \(E_{x_1t_1, \Omega}\). Finally \(I_{x_1} = I_{x_1t_1}(x_1)\) with the notations of Lemma 6.

We set

\[
T_1 = |I_{x_1}| \vee \sup_{I \subseteq [2, n]} \tau_{i(x_1, t_1, x_j, t_j, \omega),} \quad J \subseteq [2, n] \setminus I \quad \omega \subseteq E_{x_1t_1, \Omega}, 
\]

\(x_1, t_1, x_j, t_j \in S(\Omega)\)

\(\Omega_1 = \bigcup |x_1 - T_1, x_1 + T_1| \times \prod_{i \geq 2} |x_i - t_i, x_i + t_i|,\)

\(\Omega_1 = \bigcup |x_1 - T_1, x_1 + T_1| \times \prod_{i \geq 2} |x_i - t_i, x_i + t_i|,\)
the union being taken over \((x_i, t_i) \in S(\Omega)\),

\[
T_2 = \sup_{I \subseteq [3, n]} \tau_2(x_1, T_1, x_2, t_2, x_j, t_j, I, \omega),
\]

\[
I = [3, n] \setminus J
\]

\[
J = [3, n] \setminus I
\]

\[
\omega \subseteq E_{x_j, t_j}(\Omega_i)
\]

\[
(x_1, T_1, x_2, t_2, x_j, t_j) \in S(\omega)
\]

and \(\Omega_2 = \bigcup I x_1 - T_1, x_1 + T_1[x]x_2 - T_2, x_2 + T_2[\times \prod_{j \neq 1}] x_1 - t_1, x_1 + t_1|.

Suppose \(T_1, \ldots, T_{i-1}, T_i\) are already defined and let

\[
\Omega_i = \bigcup_j \prod_{j \neq 1} [x_j - T_j, x_j + T_j] \times \prod_{k \neq 1} [x_k - t_k, x_k + t_k].
\]

We define \(T_{i+1}\) as follows

\[
T_{i+1} = \sup_{I \subseteq [i + 2, n]} \tau_{i+1}(x_1, T_1, \ldots, x_i, T_i, x_{i+1}, t_{i+1}, x_j, t_j, \omega).
\]

\[
I = [i + 2, n] \setminus J
\]

\[
J = [i + 2, n] \setminus I
\]

\[
\omega \subseteq E_{x_j, t_j}(\Omega_i)
\]

\[
(x_1, T_1, \ldots, x_i, T_i, x_{i+1}, t_{i+1}, x_j, t_j) \in S(\omega)
\]

Finally let

\[
\Omega_{n-1} = \bigcup \left( \prod_{i \leq n-1} [x_i - T_i, x_i + T_i] \times [x_n - t_n, x_n + t_n] \right)
\]

and let \(T_n = \tau_n(x_1, T_1, \ldots, x_{n-1}, T_{n-1}, x_n, t_n, \Omega_{n-1})\).

The property (4.2) will be a trivial consequence of the following.

**Lemma 9.** For all \(i \in [1, n-1]\), \((x_0) = \frac{1}{2} x_{n-i}^*\).

If \(i = n - 1\), this is an immediate consequence of (5.3) applied with index \(n\) and set \(\Omega_{n-1}\).

If \(i < n - 1\), let \((x, t) \in S(\Omega), \alpha > 0\) be such that \(t_{i+1} < \alpha < T_{i+1}\). There exists \(I \subseteq [i + 2, n]\) and \(\omega \subseteq E_{x_j, t_j}(\Omega_i)\) such that \((x_1, T_1, \ldots, x_i, T_i, x_{i+1}, t_{i+1}, x_j, t_j) \in S(\omega)\) and \(\tau_{i+1}(x_1, T_1, \ldots, x_i, T_i, x_{i+1}, t_{i+1}, x_j, t_j, \omega) > \alpha\). The proof of Lemma 7 shows that

\[
\prod_{j \leq i} [x_j - T_j, x_j + T_j] \times [x_{i+1} - t_{i+1}, x_{i+1} + t_{i+1}] \times [x_j - t_j, x_j + t_j]
\]

has at least half of its volume in \(\omega\). Since \(\omega \subseteq E_{x_j, t_j}(\Omega_i), \omega \times [x_j - t_j, x_j + t_j] \subseteq \Omega_i\). Hence
\[
\prod_{j \leq i} |x_j - T_j, x_j + T_j| \times |x_{i + 1} - \tau_{i + 1}, x_{i + 1} + \tau_{i + 1}| \times \prod_{j > i + 1} |x_j - t_j, x_j + t_j|
\]

has half of its volume in \(\Omega_i\). Let \(\alpha\) tend to \(T_{i+1}\) and the same is proved for \(T_{i+1}\) instead of \(\tau_{i+1}\). Finally we have proved that \(\Omega_{i+1}\) is the union of rectangles that have at least half of their volume in \(\Omega_i\). This implies the lemma. Actually we have skipped the case where \(i = 1\) and \(T_i = L_{\alpha}\), but then the argument is trivial.

We are left with proving (4.3). To do so we replace \(E_{i'j'z'}\) by a larger set \(F_{x'y'z'}\) defined as follows. Let \(i_0 = \inf I\). Then

\[
F_{x'y'z'} = \bigcup_{T_{i'<j}<x_j-z_j} \prod_{j<i_0} |x_j - T_j, x_j + T_j| \times \prod_{j > i_0, j \notin I} |x_j - t_j, x_j + t_j|.
\]

Now we shall prove by induction on \(|I|\) that

(5.5) \[
\int |F_{x'y'z'}| \cdot |[Q_{i'T}T]_{y'z'}|_{CZ} \frac{dt}{t_j} \cdot d\nu_j \cdot dx_j \leq C_{n, \epsilon} \|T\|_{CZ}.
\]

This will be sufficient since \(t_j < T_j\) for all \(j\) and \(E_{i'j'z'} \subseteq F_{x'y'z'}\). Also, by Lemma 6 it is enough to consider the case \(|I| < n\).

If \(I\) has a single element \(i\), then (5.5) is a direct consequence of (5.2) applied with the set \(\Omega_i\) and the index \(i\), since \(T_i(x, \epsilon) \ni \tau_i(x_1, T_1, \ldots, x_{i-1}, T_{i-1}, x_i, t_i, \ldots, x_n, t_n, \Omega_i)\) and \(|\Omega_i| \leq C_i \|\Omega\|\) by Lemma 9.

If \(I\) has more than a single element let \(K = I \setminus \{i_0\}\), and let \(G_{x'z'}_{k'z'}^{i_0}\) be defined as

\[
\bigcup_{T_k \ni x_k - z_k} \prod_{j < i_0} |x_j - T_j, x_j + T_j| \times |x_{i_0} - l_{i_0}, x_{i_0} + l_{i_0}| \times \prod_{j > i_0, j \notin I} |x_j - t_j, x_j + t_j|.
\]

Clearly \(G_{x'z'}_{k'z'}^{i_0} \subseteq F_{x'z'}_{k'z'}\). Moreover we have the following.

**Lemma 10.**

\[
\int |F_{x'y'z'}| \cdot |[Q_{i'T}T]_{y'z'}|_{CZ} \frac{dt}{t_{i_0}} \cdot d\nu_{i_0} \cdot dx_{i_0} \leq C_{n, \epsilon} \|Q_{i'T}T\|_{CZ} |G_{x'z'}^{i_0}_{k'z'}|.
\]

With Lemma 10, one deduces immediately (5.5) for \(I\) from (5.5) for \(K\). Therefore the induction, and the proof of Lemma 5, reduce to Lemma 10 which we now prove. To do so we shall apply (5.2) with the set \(G_{x'z'}^{i_0}_{k'z'}\), the index \(i_0\) and the operator \((Q_{k'T}T)_{x'z'}\). Let \(F_{x'y'z'} = \bigcup_{j \notin I} |y_j - s_j, y_j + s_j|\), where the union is taken over those \((y_j, s_j) \notin I\) such that

\[
\tau_{i_0}(x_{i_0}, l_{i_0}, y, s, G_{x'z'}^{i_0}_{k'z'}) \leq |x_{i_0} - z_{i_0}| \quad \text{and} \quad (x_{i_0}, l_{i_0}, y, s) \in S(G_{x'z'}^{i_0}_{k'z'}).
\]
Then (5.2) reads as follows:

\[ \int \left| \int_{x_{i_0}}^{x_{i_0} + t} \mathcal{F}_{x_{i_0}} \frac{d}{dt} \right| c_{x_{i_0}} \, dx_{i_0} \, dz_{i_0} \leq C_n \left\| \mathcal{F}_{x_{i_0}} \mathcal{T}_{x_{i_0}} \right\| _{G_{x_{i_0} + t}^{0} \cdot \mathcal{T}_{x_{i_0}}^{0}} \left| c_{x_{i_0}} \cdot G_{x_{i_0} + t}^{0} \cdot \mathcal{T}_{x_{i_0}}^{0} \right| . \]

Therefore we need only to prove \( F_{x_{i_0}} \subseteq \mathcal{F}_{x_{i_0}} \cdot \mathcal{T}_{x_{i_0}} \). In other words we must show that if \( T_1(x, t) < |x_1 - z_1| \), that is \( T_0(x, t) < |x_0 - z_0| \) and \( T_0(x, t) < |x_0 - z_0| \), then \( (x_{i_0}, T_0) < i_{i_0}, (x_{i_0}, t_{i_0}), (x_{i_0}, t_{i_0}) > i_{i_0}, e \) \( \in S(G_{x_{i_0} + t}^{0} \cdot \mathcal{T}_{x_{i_0}}^{0}) \) and the associated \( \tau_{i_0} \cdot G_{x_{i_0} + t}^{0} \cdot \mathcal{T}_{x_{i_0}}^{0} \leq |x_{i_0} - z_{i_0}| \). The first assertion follows from the definition of \( T_0 \). Indeed \( G_{x_{i_0} + t}^{0} \cdot \mathcal{T}_{x_{i_0}}^{0} \subseteq E_{x_{i_0} + t}^{0} \cdot \Omega_0 \) (with \( \Omega_0 = \Omega \)), and therefore \( T_0 \geq \tau_{i_0} \cdot G_{x_{i_0} + t}^{0} \cdot \mathcal{T}_{x_{i_0}}^{0} \) where \( \cdot \) means \( (x_{i_0}, T_0) < i_{i_0}, (x_{i_0}, t_{i_0}), (x_{i_0}, t_{i_0}) > i_{i_0}, e \). Now \( \tau_{i_0} \cdot G_{x_{i_0} + t}^{0} \cdot \mathcal{T}_{x_{i_0}}^{0} \leq T_0(x, t) \leq |x_0 - z_0| \) and the lemma is proved.

6. A «T1-theorem» in the product setting

If \( T \) is a \( \delta \)-SIO on \( \mathbb{R} \times \mathbb{R} \) and has the WBP, the conditions \( T_1 = 0 \) and \( T^*1 = 0 \) do not imply that \( T \) is bounded on \( L^2 \). This is why we introduced in Section 3 the partial adjoint \( \tilde{T} \). Now if \( T_1 = T^*1 = \tilde{T}1 = \tilde{T}^*1 = 0 \), then \( T \) is bounded on \( L^2 \). Moreover the following is true.

**Theorem 4.** Let \( T \) be a \( \delta \)-SIO on \( \mathbb{R} \times \mathbb{R} \) having the WBP and such that \( T_1 \), \( T^*1 \), \( \tilde{T}1 \) and \( \tilde{T}^*1 \) lie in \( \text{BMO} (\mathbb{R} \times \mathbb{R}) \). Then \( T \) extends boundedly from \( L^2 \) to \( L^2 \).

Let us consider an example. Let \( (a_{k_1, k_2})_{k_1, k_2} \) be a bounded real-valued sequence on \( \mathbb{Z} \times \mathbb{Z} \) and let \( \tilde{a} = \sum a_{k_1, k_2} \tilde{e}^{2 \pi i x \cdot \xi} \) be the tempered distribution such that \( \langle \tilde{a}, \psi \rangle = \sum a_{k_1, k_2} \langle \psi, \tilde{e}^{2 \pi i x \cdot \xi} \rangle \) for all \( \psi \in S(\mathbb{R}^2) \). Let \( \tilde{\varphi}_0 \in S(\mathbb{R}^2) \) be such that \( \tilde{\varphi}_0(0) = 0 \) and \( \sum_{k_1, k_2} \tilde{\varphi}_0(2^{-k_1} \xi) = 1 \) for \( \xi = 0 \) and let \( B_k \) be the Fourier multiplier of symbol \( \tilde{\varphi}_0(2^{-k} \xi) \). Let \( \mathbb{T}2: C^0(\mathbb{R}) \otimes C^0(\mathbb{R}) \to C^0(\mathbb{R}) \otimes C^0(\mathbb{R}) \) be defined by \( \langle g_1 \otimes g_2, \mathbb{T}2 f_2 \rangle = \sum k_1, k_2 a_{k_1, k_2} \langle \Delta_k, g_1, e^{2 \pi i x \cdot \xi} \rangle \langle g_2, e^{2 \pi i x \cdot \xi} \rangle \). It is easy to show that this series is absolutely convergent. Moreover, if the rows and columns of the matrix \( (a_{k_1, k_2}) \) are uniformly bounded, then \( \mathbb{T}2 \) is a 1-SIO, satisfies \( \mathbb{T}2 1 = \mathbb{T}2 1 = \mathbb{T}2 1 = 0 \) and \( \mathbb{T}2 1 = \tilde{a} \), and \( \mathbb{T}2 \) has the WBP. Finally \( \mathbb{T}2 \) is bounded if and only if \( (a_{k_1, k_2}) \) is bounded on \( \mathbb{T}2 (\mathbb{Z}) \) and \( \mathbb{T}2 1 \) is in \( \text{BMO} \) only if \( (a_{k_1, k_2}) \) is Hilbert-Schmidt.

This example shows that \( \tilde{T}1 \) and \( \tilde{T}^*1 \) have to be taken into account in order to obtain \( L^2 \)-boundedness but the conditions \( \tilde{T}1 \) and \( \tilde{T}^*1 \in \text{BMO} \) are not necessary.

From Theorems 3 and 4 applied to \( T \) and \( \tilde{T} \) we obtain the following.
Corollary. Let \( T \) be a \( \delta \)-SIO on \( \mathbb{R} \times \mathbb{R} \). It is a \( \delta \)-CZO if and only if \( T \), \( \bar{T} \), \( T^* \), and \( \bar{T}^* \) lie in BMO and it has the WBP.

To avert the suspicion of vain aestheticism, we shall now explain why we require \( \bar{T} \) to be bounded on \( L^2 \) in the definition of a CZO. This is not merely to have a nice characterization of CZO’s but to have statements which extend in the setting of an arbitrary finite product of copies of \( \mathbb{R} \). It is a very good exercise to extend the proof of Theorem 4 that we shall give below and an opportunity to see why one needs to take into account \( \bar{T} \) in the definition of \( \| \mathcal{C}_0 \|_{\mathbb{R} \times \mathbb{R}} \). We shall leave it to the reader and stick from now on to the case \( n = 2 \) (except in Section 10).

The proof of Theorem 4 can be decomposed in three steps.

In the first step, one simply observes that if \( T \) satisfies \( T^1 = T^1 = 0 \) and has the WBP, then if can be viewed as a classical vector valued SIO, \( \bar{T} \) acting on \( C_0^\infty(\mathbb{R}) \otimes H \), where \( H = L^2(\mathbb{R}, dx) \), and for which \( \bar{T} = \bar{T}^* = 0 \). The proof of the \( L^2 \)-boundedness of such an operator is the hilbertian version of the proof of [9] based on the Cotlar-Knapp-Stein lemma.

The second step is the decomposition of an operator \( T \) having the WBP, such that \( T = T^1 = \bar{T} = \bar{T}^* = 0 \) as the sum of two operators \( S \) and \( T - S \) having the WBP and such that \( S^2 = S^* S \) and \( (T - S)^1 = (T - S)^* = 0 \). The \( L^2 \)-boundedness of \( T \) is then a consequence of the first step. The construction of \( S \) is given in Section 7.

The last step is, as in the classical situation, to construct for all functions \( b \in \text{BMO} \), a CZO \( V_b \) such that \( V_b 1 = b \) and \( V_b^* 1 = \bar{V}_b 1 = \bar{V}_b^* = 0 \). Now if \( T \) satisfies the assumptions of the theorem and \( b_1, b_2, b_3 \) and \( b_4 \) are \( T \), \( T^1 \), \( \bar{T} \), and \( \bar{T}^* \) respectively, the operator \( T - V_{b_1} - V_{b_2} - V_{b_3} - V_{b_4} \) is of the type studied in the second step, so that \( T \) is bounded on \( L^2 \). The operator \( V_b \) is described in Section 8.

### 7. Decomposition of \( T \) when \( T^1 = T^* = \bar{T} = \bar{T}^* = 0 \)

Let \( b \in \text{BMO}(\mathbb{R}) \) and let \( U_b : C_0^\infty(\mathbb{R}) \to [C_0^\infty(\mathbb{R})]' \) be defined by \( \langle g, U_b f \rangle = \int_0^\infty \langle (Q_\beta g), (Q_\beta (P_t f)) \rangle \, dt / t \). It is classical that this integral is absolutely convergent and that \( U_b \) is a 1-CZO with \( \| U_b \|_{1-CZO} \leq C \| b \|_{\text{BMO}} \). Moreover \( U_0 = \beta \) and \( U_{\delta} 1 = 0 \) (9).

Now let \( T \) be a \( \delta \)-SIO on \( \mathbb{R} \times \mathbb{R} \) such that \( T^1 = T^* = \bar{T} = \bar{T}^* = 0 \). We define the operator \( N \) as follows.

For all \( f_1, f_2, g_1, g_2 \in C_0^\infty(\mathbb{R}) \)

\[
\langle g_1 \otimes g_2, N f_1 \otimes f_2 \rangle = \langle g_1, U_{(t_1, t_2)} f_1 \rangle.
\]

(7.1)
Lemma 11. The operator $N$ is a $\delta'$-SIO having the WBP for all $\delta' < \delta$. Moreover $N^21 = (N^2)'1 = (N^1)'1 = 0$ and $(T - N)'1 = 0$.

In order to prove Lemma 11 we shall need the following.

Lemma 12. Let $T : C_0^0(\mathbb{R}) \otimes C_0^0(\mathbb{R}) \to [C_0^0(\mathbb{R}) \otimes C_0^0(\mathbb{R})]'$ be a continuous linear mapping. Suppose that for every bounded subset $B$ of $C_0^0(\mathbb{R})$, there exists a constant $C_B$ such that:

i) for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$, all $t_1, t_2 \in \mathbb{R}^+$, and all $\eta_1, \eta_2, \xi_1, \xi_2 \in B$

$$|\langle \eta_1^{t_1} \otimes \eta_2^{t_1}, T \xi_1^{t_2} \otimes \xi_2^{t_2} \rangle| \leq C_B \omega_{t_1}(x_1 - y_1) \omega_{t_2}(x_2 - y_2);$$

ii) for all $(x, y) \in \mathbb{R}$, $t > 0$, $i \in \{1, 2\}$ and $\eta, \xi \in B$,

$$|\langle \eta^{t_i}, T^{t_i} \xi^{t_i} \rangle|_{\text{BMO}} \leq C_B \omega_{t_i}(x - y) \text{ and } |\langle \eta^{t}, T^{t} \xi^{t} \rangle|_{\text{BMO}} \leq C_B \omega_{t}(x - y).$$

Then $T$ is a $\delta'$-SIO on $\mathbb{R} \times \mathbb{R}$ and has the WBP for all $\delta' < \delta$. Conversely any $\delta$-SIO having the WBP satisfies i) and ii).

This lemma is an immediate consequence of Lemma 2 and of Theorem 1.

Let us apply it to $N$. Applying the converse part of Lemma 12 to $T$, using (7.1) and the properties of the operators $U_f$ for $\beta \in \text{BMO}(\mathbb{R})$, one obtains easily the property i) for $N$ as well as the property ii) for $i = 2$. From the formula $\langle f, N^2g \rangle = U_{\langle f, T^2g \rangle 1}$ we also conclude $(T - N)'1 = 0$ and $(N^1)'1 = 0$. We are left with showing that $N$ satisfies ii) with $i = 1$. In fact we shall prove $N^21 = (N^2)'1 = 0$, or, in other words $\langle g, N^1f \rangle 1 = (g, N^1f)'1 = 0$ for all $f, g \in C_0^0(\mathbb{R})$. For this we shall use the assumptions $T1 = \bar{T}1 = 0$.

To show $\langle g, N^1f \rangle 1 = 0$, it is enough by Lemma 1 to show that for all $h \in C_0^0(\mathbb{R})$, $\lim_{q \to +\infty} \langle h, \langle g, N^1f \rangle_{\theta_q} \rangle = 0$, which means $\lim_{q \to +\infty} \langle g, U_{\langle h, T^2g \rangle 1}f \rangle = 0$, where $\theta_q$ is defined in Lemma 1. This is immediate from the two following lemmas.

Lemma 13. Let $(\beta_q)_{q \in \mathbb{N}}$ be a bounded sequence taking its values in $\text{BMO}(\mathbb{R})$. If $\lim_{q \to +\infty} \beta_q = 0$ for $a^*(H^1, \text{BMO})$, then for all $f, g \in C_0^0(\mathbb{R})$, $\lim_{q \to +\infty} \langle g, U_{\langle h, T^2g \rangle 1}f \rangle = 0$.

Lemma 14. Let $T$ be a $\delta$-SIO on $\mathbb{R} \times \mathbb{R}$ such that $T1 = 0$. Then for all $h \in C_0^0(\mathbb{R})$ and for $i \in \{1, 2\}$, the sequence $(\langle h, T^i \theta_q \rangle 1)_{q \to q_0}$ satisfies the hypothesis of Lemma 13 for $q_0$ big enough.

To prove Lemma 13, observe that the integrals $\int_0^t (Q_i g)(\xi) (Q_i \beta_q)(\xi) \cdot (P_i f)(\alpha) t^{-1} \, dx dt$ are uniformly absolutely convergent since $|Q_i \beta_q| \leq C$, $\int_0^t |Q_i g| t^{-1} \, dt < +\infty$ and $\sup_{t > 0} |P_i f|_2 < +\infty$. Therefore we can take the limit under the integral sign. Since by assumption $\lim_{q \to +\infty} \langle Q_i \beta_q \rangle (x) = 0$ for all $(x, t) \in \mathbb{R}^2$, Lemma 13 is proved.
To prove Lemma 14 we pick a function \( k \in C_0^\infty(\mathbb{R}) \) and we want to prove that \( |\langle k, \langle h, T'_{\theta_q} 1 \rangle \rangle| \) is less than \( C|k|_{H^1} \) for \( q > q_0 \), \( q_0 \) and \( C \) being independent of \( k \), and that \( \lim_{q' \to +\infty} |\langle k, \langle h, T'_{\theta_q} 1 \rangle \rangle| = 0 \). This latter fact follows from \( T1 = 0 \) and from Lemma 3 in Section 3. To prove the first fact it is enough to prove that for \( q > q_0 \) and \( q' > q_0 \)

\[
|\langle k, \langle h, T'_{\theta_q} 1 \rangle \rangle| \leq C|k|_{H^1}, 
\]

(7.2)

and then take the limit when \( q' \to +\infty \). Notice now that if \( \text{supp } h \cap \text{supp } (\theta_q - \theta_q) = \phi \), then \( \langle h, T'_{\theta_q - \theta_q} \rangle = \int \int h(x) K(x, y)(\theta_q - \theta_q)(y) \, dx \, dy \). This will be true if \( q_0 \) is chosen large enough and in this case a straightforward computation (using \( \int h \, dx = 0 \)) yields \( \langle h, T'_{\theta_q - \theta_q} \rangle \leq C \), which implies (7.2).

We have proved \( N^{2*1} = 0 \). The proof of \( N^{2*1} = 0 \) is identical. One just has to use \( T1 = 0 \) instead of \( T1 = 0 \). This proves Lemma 11.

We also need another operator \( M \), similar to \( N \), defined by

\[
\langle g_1 \otimes g_2, M f_1 \otimes f_2 \rangle = \langle g_1, U_{(f_1^*, T(f_2^*, *)^*)} f_1 \rangle.
\]

This operator \( M \) is also an SIO and has the WBP. Moreover, \( M^{21} = M^{2*1} = M^{1*1} = 0 \) and \( (T - M)^{1*1} = 0 \). This can be shown using the same arguments as for \( N \).

We now set \( S = M + N \) so that \( S \) has the WBP, \( S^{2*1} = 0 \) and \( (T - S)^{1*1} = 0 \).

8. Construction of the operator \( V_b \)

The construction of \( V_b \) is inspired by the construction of the operators \( U_{\beta} \), \( \beta \in \mathbb{B} \), \( \beta \in \mathbb{B} \) of Section 7; see [9].

The family of operators \( (P_t)_{t > 0} \) is defined as in Section 2, but now \( Q_t \) denotes \( -t^\beta \partial \), so that \( Q_{t}^{dt} = I \) and \( Q_t^{dt} = C_0 I \), where \( C_0 \) is not necessarily 1.

Let \( b \in \mathbb{B} \) and let \( W_b: C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R}) \to [C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R})]^* \) be defined by

\[
\langle f_1 \otimes f_2, W_b g_1 \otimes g_2 \rangle = \int \int \langle Q_t f_1 \otimes Q_{t'} f_2, (Q_{t'} Q_t b) P_{t_1} g_1 \otimes P_{t_2} g_2 \rangle \, \frac{dt_1}{t_1} \frac{dt_2}{t_2}.
\]

The \( L^2 \)-boundedness of \( W_b \) is, as in the classical situation [9], a consequence of the fact that \( (Q_t Q_{t'} b)^2 \, dx_1 dx_2 (t_1 \cdot t_2)^{-1} \, dt_1 dt_2 \) is a Carleson measure [5] and of the properties of such measures on product spaces [4]. On the other hand one sees easily that \( W_b \) is a 1-SIO whose kernels take their values in
\{ U_b, \beta \in \text{BMO}(\mathbb{R}) \}.  
Moreover, an application of Lemma 3 shows that 
\( W_b1 = Cb1, \tilde{W}_b^*1 = 0, \tilde{W}_b1 = 0 \) and \( \tilde{W}_b^*1 = 0 \).  
It remains to show that \( \tilde{W}_b \) is bounded on \( L^2 \).  
In order to do this, it is enough to show that \( \tilde{W}_b \) maps \( L^\infty \) into BMO.  
Similarly \( \tilde{W}_b^* \) will map \( L^\infty \) into BMO so that the \( L^2 \)-boundedness of \( W_b \) will follow by interpolation between \( H^1 \to L^1 \) and \( L^\infty \to \text{BMO} \) [13].

We want the estimate \( \| \tilde{W}_b f \|_{\text{BMO}} \leq C b \| f \|_\infty \).  
Consider the operator \( T_f: b \rightarrow \tilde{W}_b f \).  
We need to show that it maps BMO to itself.  
Observe that \( T_f \), which is given by

\[
\langle h_1 \otimes h_2, T_f b_1 \otimes b_2 \rangle = \iint \langle P_i h_1 \otimes Q_i h_2, (Q_i P_i f) Q_i b_1 \otimes Q_i b_2 \rangle \frac{dt_1}{t_1} \frac{dt_2}{t_2},
\]

is itself a 1-SIO, and satisfies \( T_f^*1 = T_f 1 = 0 \).  
Therefore we already know that \( T_f \) maps \( L^2 \) to \( L^2 \).  
From Theorem 3 it follows that \( T_f \) maps \( L^\infty \) to BMO.  
To show that \( T_f \) maps BMO to itself, we observe that \( T_f h_1, T_f h_2 \) and \( T_f h_1 h_2 \) are SIO’s, because the kernel of \( Q_i h \) satisfies the same estimates as the kernel of \( P_i \).  
Since these operators are bounded on \( L^2 \) as well as \( T_f \), they also map \( L^\infty \) to BMO.  
Hence \( T_f \) is bounded on BMO.

The proof of Theorem 4 is complete.

9. Bicommutators of Calderón-Coifman type

In the classical situation, a standard kernel \( K \) is antisymmetric if \( K(x,y) = -K(y,x) \) for all \((x,y)\in\Omega\).  
Such a kernel induces automatically and SIO \( T \) defined for all \( f, g \in C_0^\infty(\mathbb{R}) \) by

\[
\langle g, T_f \rangle = \lim_{\epsilon \to 0} \iint_{|x-y| > \epsilon} g(x)K(x,y)f(y) \, dx \, dy.
\]

(9.1)

The existence of the limit is a consequence of the antisymmetry of the kernel \( K \) and of the smoothness of \( f \) and \( g \).  
Actually,

\[
\langle g, T_f \rangle = \frac{1}{2} \iint K(x,y)[g(x)f(y) - f(x)g(y)] \, dx \, dy,
\]

(9.2)

so that \( |\langle g, T_f \rangle| \leq C (\text{diam}\, \text{supp} \, g \, \text{supp} \, f)^\frac{1}{2} \|g\|_\infty \|f\|_\infty \).

This clearly implies that \( T \) has the WBP.  
Since \( T = -T^* \), \( T \) is bounded on \( L^2 \) if and only if \( T_1 \in \text{BMO} \), by Theorem 1.

The best known examples of CZO’s generated by antisymmetric kernels in the manner just described are the Calderón commutators associated to the kernels \( [(A(x) - A(y))/(x-y)]^k \cdot (x-y)^{-1} \) where \( k \geq 0 \) and \( A: \mathbb{R} \to \mathbb{C} \) satisfies \( A' = a \in L^\infty \).  
Calderón proved in [2] that \( |T_k|_{CZO} \leq C^k \).  
This estimate which has been improved in [7], can be easily obtained from Theorem 1 ([9]).
Actually, this can also be derived from a more general result on antisymmetric kernels.

Let $K$ be an antisymmetric standard kernel and $A: \mathbb{R} \to \mathbb{C}$ be such that $A' = A \in L^\infty$, and let $K_a$ be defined by $K_a(x, y) = K(x, y) [A(x) - A(y)] \cdot (x - y)^{-1}$ for all $(x, y) \in \Omega$. Clearly $K_a$ is also an antisymmetric standard kernel and defines an SIO $T_a$ having the WBP.

**Proposition 1.** If $T$ is a CZO, then $T_a$ is a CZO, and for all $\delta \in ]0, 1]$ there exists $C_{\delta} > 0$ such that

$$\| T_a \|_{CZO} \leq C_{\delta} \| a \|_{\infty} T \|_{\text{CZO}}.$$  \hfill (9.3)

This proposition can be generalized to the product setting.

Let $L: \Omega \times \Omega \to \mathbb{C}$ be a function such that for all $(x_1, y_1)$ and $(x_2, y_2) \in \Omega$

$$|L(x_1, y_1, x_2, y_2)| \leq \frac{C}{|x_1 - y_1||x_2 - y_2|}. \hfill (9.4)$$

If $L$ is antisymmetric in each couple it defines a continuous operator $T: C_0^0(\mathbb{R}) \otimes C_0^0(\mathbb{R}) \to [C_0^0(\mathbb{R}) \otimes C_0^0(\mathbb{R})]'$ by

$$\langle g_1 \otimes g_2, T f_1 \otimes f_2 \rangle = \lim_{\epsilon_1, \epsilon_2 \to 0} \iint_{|x_1 - y_1| > \epsilon_1, |x_2 - y_2| > \epsilon_2} g_1(x_1) g_2(x_2) L(x_1, y_1, x_2, y_2) f_1(y_1) f_2(y_2) \, dx_1 \, dx_2 \, dy_1 \, dy_2$$

for all $f_1, f_2, g_1, g_2 \in C_0(\mathbb{R})$.

As in the classical situation, the existence of this limit is a consequence of the antisymmetry of $L$, of (9.4) and of the smoothness of $f_1, f_2, g_1$ and $g_2$. It is easy to see that $T$ has two kernels $K_1$ and $K_2$ in the sense of Definition 8, in Section 3. These are given by

$$\langle g, K_1(x) f \rangle = \lim_{\epsilon \to 0} \iint_{|u| < \epsilon} g(u) K(x, y, u, v) f(v) \, du \, dv$$

and

$$\langle g, K_2(x) f \rangle = \lim_{\epsilon \to 0} \iint_{|u| < \epsilon} g(u) K(u, v, x, y) f(v) \, du \, dv$$

for all $f, g \in C_0^0(\mathbb{R})$.

Let $A: \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ be a function such that

$$\frac{\partial^2 A}{\partial x_1 \partial x_2} = a \in L^\infty$$
(in the distributional sense) and let $\tilde{A}: \Omega \times A \to C$ be defined by

$$\tilde{A}(x_1, y_1, x_2, y_2) = \frac{A(x_1, x_2) + A(y_1, y_2) - A(y_1, x_2) - A(x_1, y_2)}{(x_1 - y_1)(x_2 - y_2)}$$

for all $(x_1, y_1)$ and $(x_2, y_2) \in \Omega$. If $L$ is antisymmetric and satisfies (9.4), then $L\tilde{A}$ has the same properties. Hence $L\tilde{A}$ defines an operator $T_\sigma$ in the same manner as $L$ defines $T$. The first kernel $K_{a, 1}$ of $T_\sigma$ is defined by

$$\langle g, K_{a, 1}(x, y)f \rangle = \lim_{\varepsilon \to 0} \int_{|u - v| > \varepsilon} g(u)L(x, y, u, v)\tilde{A}(x, y, u, v)f(v)dvdu.$$

Notice now that because $T$ is a $\delta$-SIO, so is $T_\sigma$. This is an immediate consequence of Proposition 1 and the fact that for $x$ and $y$ fixed, $\tilde{A}(x, y, u, v)$ is $gf$ the form $|B(u) - B(v)|/(u - v)$, with $|B'|_\infty \leq |a|_\infty$ and $b^4(x, y, u, v)$ is of the form $|C(u) - C(v)|/(u - v)$, with $|C'|_\infty \leq |a|_\infty/(x - y)$.

**Proposition 2.** If $T$ is a CZO, then $T_\sigma$ is a CZO, and for all $\delta \in [0, 1]$ there exists $C_\delta > 0$ such that

$$\|T_\sigma\|_{\text{CZO}} \leq C_\delta \|a\|_{\text{CZO}}.$$

In particular the kernel $[((x_1 - y_1)(x_2 - y_2))^{-1/2}]^k$ defines a CZO $T_k$ of norm less than $C^k\|a\|_{\text{CZO}}^k$. The $L^2$-boundedness of $T_1$ was first proved by J. Aguirre in [1].

We now turn to the proofs and start with Proposition 1. For simplicity we shall assume $\delta = 1$. We know that it is enough to show that for $a \in L^\infty$, $T_\sigma 1 \in \text{BMO}$ and $\|T_\sigma 1\|_{\text{BMO}} \leq C\|a\|_{\text{BMO}}$. To show this inequality we are going to exhibit a CZO $S$ such that $T_\sigma 1 = Sa$, that is, for all $g \in C_0^\infty(\mathbb{R})\langle g, Sa \rangle = \langle g, T_\sigma 1 \rangle$. This equality determines $\langle g, Sa \rangle$ for all $g \in C_0^\infty(\mathbb{R})$ and $a \in C_0^\infty(\mathbb{R})$. But since $|K_\sigma(x, y)| \leq |C| |x - y|^{-2}$ when $a \in C_0^\infty(\mathbb{R})$, $T_\sigma 1$ acts not only on $C_0^\infty(\mathbb{R})$ but on $C^\infty(\mathbb{R})$, so that $(g, Sa)$ is well defined for all $a \in C_0^\infty(\mathbb{R})$ and $g \in C^\infty(\mathbb{R})$. Moreover,

$$\langle g, Sa \rangle = \lim_{\varepsilon \to 0} \int_{|x - y| > \varepsilon} g(x)K(x, y)\frac{A(x) - A(y)}{x - y}dx dy.$$

Let $g$ and $a$ have disjoint supports. Then if $x \in \text{Supp} g$ and $|y - x| \leq \text{dist} (\text{Supp} a, \text{Supp} g)$, $A(x) = A(y)$ so that the integral defining $\langle g, Sa \rangle$ is absolutely convergent. This permits us to compute the kernel of $S$, namely $K_\sigma(x, u) = \int_{-\infty}^{\infty} K(x, y)\frac{1}{x - y}dy$ if $x > u$ and $K_\sigma(x, u) = \int_{u}^{\infty} K(x, y)\frac{1}{x - y}dy$ if $x < u$. This kernel $K_\sigma$ is clearly a standard kernel and because of that we can apply Theorem 1 to show that $S$ is a CZO. We first notice that $S1 = T1$ and therefore
lies in $\text{BMO}$. This is formally obvious, since when $a = 1$, $[A(x) - A(y)] \cdot (x - y)^{-1} dx dy$, but it can be proved rigorously using Lemma 1. Next, we compute $S^* 1$. For $a \in C_0^\infty(\mathbb{R})$, $A(x) = 0$ for $x$ large enough. Moreover, since $\langle g, Sa \rangle$ can be rewritten as $\frac{1}{2} \int \int [g(x) - g(y)]K(x, y) [A(x) - A(y)] (x - y)^{-1} dx dy$ an application of Lemma 1 shows easily that $S^* 1 = 0$. Finally, to prove that $S$ has the WBP we choose $g$ and $a \in C_0^\infty(\mathbb{R})$ and suppose that the supports of $a$ and $g$ are contained in some interval $]x_0 - t, x_0 + t[$. We decompose the integral as $I_1 + I_2 + I_3$, where

$$I_1 = \frac{1}{2} \int \int \frac{[g(x) - g(y)]K(x, y)}{x - y} A(x) - A(y) \, dx dy,$$

$$I_2 = \frac{1}{2} \int \int g(x)K(x, y) \frac{A(x) - A(y)}{x - y} \, dx dy,$$

and $I_3 = I_2$ because of the antisymmetry of $K$. Clearly $|I_1| \leq C\|g\|_\infty |a|_\infty t^2$ and $|I_3| \leq C\|g\|_\infty |a|_\infty t$. These estimates imply that $S$ has the WBP. Theorem 1 can be applied to $S$, which is a CZO. This proves Proposition 1.

We shall denote by $U$ the linear mapping that sends a CZO $T$ defined by an antisymmetric kernel to the operator $S$ we have just considered. From the proof of Proposition 1 it follows that $(9.6) \|U(T)\|_{\text{CZO}} \leq C|\|T\|_{\text{CZO}}$.

The proof of Proposition 2 follows the same lines as the proof of Proposition 1. Notice first that an SIO $T$ defined from an antisymmetric kernel by (9.5) has the WBP. This can be seen exactly as in the classical situation. Moreover, such an operator satisfies $T_1 = T^* 1 = -T_1 = -T^* 1$. Hence, to prove that it is bounded on $L^2$, it is enough to show that $T_1 \in \text{BMO}$, by Theorem 4, and this is necessary by Theorem 3.

We now wish to prove that if an antisymmetric kernel $L$ defines a CZO, $T$ then the SIO $T_a$ defined by $L \hat{a}$ satisfies $T_a \in \text{BMO}$. To do this we consider the operator $W: L^\infty(\mathbb{R} \times \mathbb{R}) \to [C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R})]'$ defined by $Wa = T_a$. If $a \in C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R})$ then $T_a$ is actually an element of $[C_0^\infty(\mathbb{R}) \otimes C_0^\infty(\mathbb{R})]'$ because of the decay properties of $L \hat{a}$ at $\infty$. Hence $\langle g_1 \otimes g_2, Wa_1 \otimes a_2 \rangle$ can be defined for $g_1, g_2, a_1, a_2 \in C_0^\infty(\mathbb{R})$ by

$$\lim_{\epsilon_1 \to 0 \epsilon_2 \to 0} \iint_{|x_1 - y_1| > \epsilon_1, |x_2 - y_2| > \epsilon_2} g_1(x_1)g_2(y_2)L(x_1, y_1, x_2, y_2)\hat{a}(x_1, y_1, x_2, y_2) \, dx_1 dy_1 dx_2 dy_2.$$

We are left with proving that $W$ is a CZO. To compute the kernels $X_1$ and $X_2$ of $W$ we notice that if $a = a_1 \otimes a_2$. 
\[ \tilde{A}(x_1, y_1, x_2, y_2) = \frac{A_1(x_1) - A_1(y_1)}{x_1 - y_1} \frac{A_2(x_2) - A_2(y_2)}{x_2 - y_2}, \]

where \( A_1 = a_1 \) and \( A_2 = a_2 \). The computation we did to compute \( K_z \) shows that if \( x_1 > u_1 \) and \( g_2, a_2 \in C_0^\infty(\mathbb{R}) \), then

\[ \langle g_2, X_t(x_1, u_1) a_2 \rangle = \lim_{\varepsilon_2 \to 0} \int_{|x_2 - y_2| > \varepsilon_2} g_2(x_2) \frac{L(x_1, y_1, x_2, y_2)}{x_1 - y_1} \frac{A_2(x_2) - A_2(y_2)}{x_2 - y_2} dx_2 dy_2 dy_1. \]

This is also equal to

\[ \left\langle g_2, \left(U \int_{-\infty}^{u_1} K_t(x_1, y_1) \frac{1}{x_1 - y_1} dy_1 \right) a_2 \right\rangle. \]

Similar formulas hold for the case \( x_1 < u_1 \) and for \( X_2 \). From (9.6) it follows that \( X_1 \) and \( X_2 \) are both 1CZ-1-standard kernels. The WBP of \( W \) can also be checked easily. Indeed, for \( g_1, a_1 \in C_0^\infty(\mathbb{R}) \),

\[ \langle g_1, W^1 a_1 \rangle = U \left[ \lim_{\varepsilon \to 0} \int_{|x_1 - y_1| > \varepsilon} g_1(x_1) K_t(x_1, y_1) \frac{A_1(x_1) - A_1(y_1)}{x_1 - y_1} dx_1 dy_1 \right], \]

so that the proof in the classical case extends immediately. Moreover, this equality implies that for \( a_1, g_1 \in C_0^\infty(\mathbb{R}) \langle g_1, W^1 a_1 \rangle^* 1 = 0 \), or equivalently, \( W^2 1 = 0 \). Similarly \( W^* 1 = 0 \). By Lemma 3 this implies \( W^* 1 = W^1 = W^* 1 = 0 \). Finally, \( W^1 = T^1 \) can be proved using Lemma 3. Therefore \( W \) is a CZO and Proposition 2 is proved.

Of course this result extends to an arbitrary finite product of copies of \( \mathbb{R} \), by a simple induction on the number of factors. We omit the details.

10. A Littlewood-Paley inequality for arbitrary rectangles

We wish to prove the following extension of Theorem 2.

**Theorem 5.** Let \( \{ R_k \}_{k \in \mathbb{N}} \) be a collection of disjoint rectangles in \( \mathbb{R}^n \) with sides parallel to the axes and let \( S_k \) be the Fourier multiplier of symbol \( x_{R_k} \). Let \( \tilde{\Delta} \) be defined on \( L^2 \) by \( |\tilde{\Delta} f| = \left[ \sum_k (S_k f)^2 \right]^{1/2} \). Then \( \tilde{\Delta} \) is bounded on \( L^p \) for all \( p \in [2, +\infty[ \).

We shall assume the reader to be familiar with [15] where this theorem is proved in the case \( n = 1 \). In this paper it is shown that the theorem for \( n = 1 \) is a consequence of the following.
Lemma 15. Let $\psi$ be fixed in $S(\mathbb{R})$ such that $\chi_{[-2, 2]} \leq \psi \leq \chi_{[-3, 3]}$ and let $\Psi_k^j$ be the convolution operator of symbol $\hat{\psi}(\frac{\cdot}{2^k} - j)$. Let $\chi: \mathbb{Z} \times \mathbb{Z} \to \{0, 1\}$ be such that the operator $T_\chi$ is bounded on $L^2$ where $T_\chi f(x) = \chi(j, k)[\Psi_k^j f(x)]_{j,k}$ takes its values in $L^2(\mathbb{Z} \times \mathbb{Z})$. Then $T_\chi$ is bounded from $L^\infty$ to BMO$_{\mathbb{R}^2(\mathbb{Z} \times \mathbb{Z})}$.

Observe that, by Plancherel's theorem, the $L^2$-boundedness of $T_\chi$ is equivalent to

$$\sum_{j,k} |\chi(j, k)[\Psi_k^j(x - j)|^2 \in L^\infty(d\xi).$$

The reduction of Theorem 2 to Lemma 15 is done by means of standard Littlewood-Paley theory, $A_2$ weights and interpolation between $L^2 \to L^2$ and $L^\infty \to \text{BMO}$. All these arguments go through in the $n$-dimensional setting without any problem. Finally the main ingredient in the proof of Lemma 15 is the following.

Lemma 16. Let $(x, t) \in \mathbb{R}^2_+$ and $a \in L^2_{\text{loc}}$ be supported out of $|x - 2t, x + 2t|$. Then for all $\eta > 0$, there exists $C_\eta > 0$ such that

$$(10.1) \quad \sum_{k,j} |(Q_k \psi_j^j(a)(x)|^2 \leq C_\eta \int a^2(z) \left( \frac{t}{|x - z|} \right)^{5/3 - \eta} \frac{dz}{t}.$$  

Note that the summation is over all $(k, j) \in \mathbb{Z} \times \mathbb{Z}$. This lemma is actually a reformulation of the Lemma 4.1 of [15] and we leave the translation to the reader. From (10.1) it follows by a standard argument that if $a \in L^\infty$ then

$$\sum_{k,j} \chi(j, k)[(Q_k \psi_j^j(a)]^2 \frac{dt}{t} \frac{dx}{x}$$

is a Carleson measure, or equivalently that $[\chi(j, k)[\psi_j^j(a)]_{j,k}$ lies in BMO$_{\mathbb{R}^2(\mathbb{Z} \times \mathbb{Z})}(\mathbb{R})$. Thus Lemma 16 implies Lemma 15.

As we said, Theorem 5 follows from an appropriate analogue of Lemma 15 in the $n$-dimensional context.

Lemma 17. Let $\Psi$ and $\Psi_k^j$ be as in Lemma 15 and let $\chi: \mathbb{Z}^n \to \{0, 1\}$ be such that the operator $T_\chi: L^2(\mathbb{R}^n) \to L^2_{\text{loc}}(\mathbb{R}^n)$ is bounded, where $T_\chi f(x)$ takes its values in $L^2(\mathbb{Z}^n)$ and is given by $[\chi(j, k)[\Psi_k^j \otimes \cdots \otimes \Psi_k^j(a)](x)]_{j,k}$. Then $T_\chi$ is bounded from $L^\infty$ to BMO$_{\mathbb{R}^2(\mathbb{Z} \times \mathbb{Z})}$, or equivalently,

$$(10.2) \quad \sum_{j,k} \chi(j, k)[Q_k \psi_j^j \otimes \cdots \otimes Q_k \psi_j^j a(x)]^2 \frac{dt}{t} \frac{dx}{x}$$

is a Carleson measure on $[\mathbb{R}^2]^n$ if $a \in L^\infty$.
To avoid any convergence problems we suppose that \( \chi \) is finitely supported but we shall obtain estimates independent of this assertion. We shall use a variant of Lemma 5. It can be shown that (5.3) remains true if \( \| [Q,T]_{x_j x_j} \|_{C_0} \) is replaced by \( \| [t_j/(x_j - z_j)]^{1++} \| \), the point being that Lemma 8 remains true if in (5.4) \( \| Q, T |_{z_j z_j} \|_{C_0} \) is replaced by \( |t_j/(x_j - z_j)]^{1++} \). Let us rewrite the variant of (4.3):

\[
(10.3) \quad \int_\Omega |E_{x_j x_j}^\prime| \frac{t_j^{j-1}}{(x_j - z_j)^{1++}} dz_j dt_j dx_j \leq C_n, \Omega |\Omega|.
\]

For technical reasons we need to assume that the \( T_i \)'s constructed in Section 5 take their values in the set \( \{ 2^k, K \in \mathbb{Z} \} \). This is of course not a restriction. Replacing \( T_i \) by \( \inf \{ 2^k, T_i \leq 2^k \} \) yields (4.3) and (10.3) a fortiori and (4.2) with the constant \( 2^n C_n \). We shall use this family of functions \( \{ T_i, i \in [1, n] \} \) with various sets playing the role of \( \Omega \) and even various dimensions. Let \( I \) be a set of indices in \( [1, n] \) and \( \omega \) a bounded open subset of \( \mathbb{R}^l \). Then \( \{ T_i(x, t, \omega), i \in I, (x, t) \in S(\omega) \} \) will refer to the family of functions constructed in dimension \( |I| \) with \( \omega \) playing the role of \( \Omega \).

Let \( \Omega \) be an open subset of \( \mathbb{R}^n \) and let \( \Omega_n, T_i(x, t, \Omega), 1 \leq i \leq n \) be as in Sections 4 and 5. By the same argument as in Section 4, and with the same notations, we are reduced to proving an estimate similar to (4.5), namely for \( a \in L^\infty(\mathbb{R}^n) \) and \( \| a \|_\infty < 1 \):

\[
(10.4) \quad \int_{S(\Omega)} \Sigma(\kappa, \lambda) \left| Q, \Psi \left( \alpha_{x, t}, \omega(x) \right) \right|^2 \frac{dx dt}{t} \leq C_n |\Omega|.
\]

This inequality will be a consequence of the following.

**Lemma 18.** Suppose that the function \( a_{x, t} \) is of the form \( a_{x, t} \chi \), where \( E_{x, t} \leq \mathbb{R}^n \) is defined by the following set of conditions.

Let \( i \in [1, n] \). For all \( j \in [1, n] \), \( x_j, z_j, j \leq l_j \in \mathbb{Z} \), be such that \( 2^{l_j} \leq |x_j - z_j| < 2^{l_j + 1} \). Let \( S_1(x, t), S_2(x, t, l_1), \ldots, S_l(x, t, l_1, \ldots, l_{i-1}) \) be functions taking their values in the set \( \{ 2^k, k \in \mathbb{Z} \} \) and larger than \( 2t_1, 2t_2, \ldots, 2t_l \), respectively. Let \( F_{x_1 \ldots x_l} \) be a subset of \( \mathbb{R}^n \). Then \( (z_1, \ldots, z_n) \in E_{x, t} \) if and only if

\[
(z_1, z_1, \ldots, z_n) \in F_{x_1 \ldots x_l} = \mathbb{F}_{x_1 \ldots x_l},
\]

\[
H_{2t_1} \geq |x_1 - z_1| \geq 2^{l_1} \geq S_1(x, t, l_1) \geq 2t_1,
\]

\[
H_{2t_2} \geq |x_2 - z_2| \geq 2^{l_2} \geq S_2(x, t, l_1, l_2) \geq 2t_2,
\]

\[
H_{2t_{i-1}} \geq |x_i - z_i| \geq 2^{l_{i-1}} \geq S_i(x, t, l_1, \ldots, l_{i-1}) \geq 2t_i.
\]

Let \( l = [1, i] \) and \( x_j, t_j, l_j \) be fixed, and let \( D_{x_j t_j} = \bigcup \prod_{q > j}|x_q - t_q, x_q + t_q| \), where the union is extended to those \( (x_{j+1}, t_{j+1}, \ldots, x_n, t_n) \) such that
\[ 2^{i_1} \geq S_1(x, t), \ldots, 2^{i_n} \geq S_i(x, t, l_1, \ldots, l_{i-1}). \text{ If for all } \epsilon > 0 \]

\[ \left(10.5\right) \int \int_{|x_i - z_j| \geq 2t_f} \left| D_x x_i \right| \frac{dx_{i_1} dt_f dz_f}{t_f^{1-\epsilon}|x_i - z_j|^{1+\epsilon}} \leq C_i |\Omega|, \]

then

\[ \left(10.6\right) \int_{x(0)} \sum_{k,j} |Q \Psi \| a_{x,k}(x)|^2 \frac{dx dt}{t} \leq C |\Omega|. \]

Let us see first why Lemma 18 implies (10.4). Observe that it is enough to prove (10.4) when I is of the form \([1, i]\). Indeed, if the construction of the \(T_i\)'s is non-symmetric in the various indices, the properties of the \(T_i\)'s which are used are expressed symmetrically and therefore we can reorder the coordinates in such a way that I is of the form \([1, i]\). Now we apply Lemma 18 with \(S_1(x, t) = T_1(x, t, \Omega), S_2(x, t, \Omega) = T_2(x, t, \Omega) \ldots S_i(x, t, l_1, \ldots l_{i-1}) = T_i(x, t, \Omega)\) and \(F_{i+1, \ldots, i+|\Omega|} = R^{d+1}\) (with the notations of Lemma 5). Indeed if the \(T_j\)'s take their values in \(2^{k}, k \in \mathbb{Z}\), then \(|x_j - z_j| \geq T_j \gg 2^{i} \geq T_f\). Thus (10.5) coincides with (10.3) and (10.6) coincides with (10.4). We are left with showing Lemma 18.

The proof of Lemma 18 uses a backward induction on \(i\). We start with the case \(i = n\). Then we can use the following.

**Lemma 19.** Let \((x, t) \in \mathbb{R}^n_+\) and \(b_{x,i} \in L^1_{loc}\) be such that \(b_{x,i}(z) = 0\) if \(|x_i - z_i| \leq 2t_i\) for some \(i \in [1, n]\). Then for all \(\epsilon > 0\) there exists \(C_\epsilon > 0\) such that

\[ \left(10.7\right) \sum_{(i,j) \in [2^n]} \left| (Q \Psi \| b_{x,i}(z)) (x) \right|^2 \leq \int [b_{x,i}(z)]^2 \left[ \prod_{1 \leq i \leq n} t_i \right]^{2/3 - \eta} \left[ \prod_{1 \leq i \leq n} |x_i - z_i| \right]^{\nu/3 - \eta} dz. \]

This lemma is the \(n\)-dimensional analogue of Lemma 16 and its proof is nothing but the \(n\)-th fold application of Lemma 16 successively in each coordinate.

As for Lemma 5, (10.5) has to be interpreted differently when \(I = [1, n]\). In this case it reads

\[ \left(10.8\right) \int \int_{|x_i - z_j| \geq 2t_f, H \text{satisfied}} \frac{dx_{i_1} dt_f dz_f}{t_f^{1-\epsilon} |x_i - z_j|^{1+\epsilon}} \leq C_\epsilon |\Omega|, \]

the restriction \(z \in F_{x_i} t_f\) being irrelevant. Now to obtain (10.6), choose \(\eta = 1/3, b_{x,i} = a_{x,i}\) and apply (10.7). Then integrate against \(dx_{i_1} dt_f / b_f\) and (10.8) with \(\epsilon = 1/3\).

We turn to the general case and choose \(i < n\).
Let \((a_{st}, (x, t) \in S(\Omega))\) be a family of functions satisfying the hypothesis of the lemma. We decompose \(a_{st}\) as \(\sum_l a_{s_l t}\), where \(l_t \geq 2t_1\) and

\[
a_{s_l t} = a_{st} \prod_{\ell \in I} \chi_{2^\ell \leq |s_\ell - z_\ell| < 2^{\ell + 1}}.
\]

Now for \(x_t, t_t, l_t\) fixed and \(D_{s_l t}\) as defined in Lemma 5 in dimension \((n - |I|)\). This yields \((n - |I|)\) functions \(\tilde{T}_j, j > 1\), where \(\tilde{T}_j(x_t, t_t) = T_j(x_t, t_t, D_{s_l t})\) for \((x_t, t_t) \in S(D_{s_l t})\). Let

\[
\tilde{D}_{s_l t} = \bigcup_{x_j, t_j \in I} \prod_{j > 1} |x_j - \tilde{T}_j, x_j + \tilde{T}_j|.
\]

From (4.2) we conclude \(|\tilde{D}_{s_l t} t_s| \leq C|D_{s_l t} t_s|\). We define \(\tilde{a}_{s_l t} = a_{s_l t} x_{D_{s_l t}} (z_1 + \ldots + z_n)\) and \(\tilde{a}_{s_l t} = \sum s_{s_l t}\), \(\tilde{a}_{s_l t} = \sum s_{s_l t}\).

Let \((x, t) \in S(\Omega), (k, j) \in [2^n]^2\) and \(\alpha > 0\) be given. By Cauchy-Schwarz,

\[
|Q_j \Psi_{j} \tilde{a}_{s_l t}(x)|^2 \leq \left[ \sum_l 2^{-\sum_{1 \leq s_l \leq l + 1}} \right] \left[ \sum_l 2^{\sum_{s_l \leq l + 1}} |Q_j \Psi_{j} \tilde{a}_{s_l t}(x)|^2 \right]
\]

which is less than

\[
(10.9) \quad \left[ \prod_{1 \leq s_l \leq l} \frac{1}{t_s} \right]^\alpha \left[ \sum_l 2^{\sum_{1 \leq s_l \leq l + 1}} |Q_j \Psi_{j} \tilde{a}_{s_l t}(x)|^2 \right].
\]

We wish to show (10.6) with \(\tilde{a}_{s_l t}\) instead of \(a_{s_l t}\). Observe that the \(L^2\)-boundedness of \(T_s\) is equivalent to

\[
\sum_{j, k} \chi(j, k) \left[ \prod_{1 \leq r \leq n} \tilde{\psi}(2^{-k_r} - j_r) \right]^2 \leq C.
\]

Fix \((j, k) \in (Z^J)^2\) and set \(\xi_l = j_2^2.\) Since \(\tilde{\psi}(0) = 1\), we obtain

\[
\sum_{j, k} \chi(j, k) \left[ \prod_{1 \leq r \leq n} \tilde{\psi}(2^{-k_r} - j_r) \right]^2 \leq C,
\]

where \(j = (j_t, j_i)\) and \(k = (k_t, k_i)\). This implies that for \((j_t, k_t)\) fixed, the operator \(T_{j_t, k_t}\) defined by \(T_{j_t, k_t} f(x) = [\chi(j, k) \Psi_{j} \tilde{a}_{s_l t}(x)]_{j_t, k_t}\), is bounded from \(L^2(\mathbb{R}^d)\) to \(L^2(\mathbb{R}^d)\).

In order to estimate the \(l \cdot h \cdot s\) of (10.6) with \(\tilde{a}_{s_l t}\) we rewrite it as

\[
(10.10) \quad \int \sum_{j_t, k_t} \left[ \sum_{j_t, k_t} \chi(k, j) |Q_j \Psi_{j} \tilde{a}_{s_l t}(x)|^2 \right] \frac{dx_j dt_j}{t_j} \left[ \sum_{k_t, l_t} \chi(k, j) |Q_j \Psi_{j} \tilde{a}_{s_l t}(x)|^2 \right] \frac{dx_j dt_j}{t_j},
\]

and estimate first the part between brackets. By (10.9) this is less than

\[
\frac{1}{(t_j)^\alpha} \sum_l 2^{\sum_{1 \leq s_l \leq l + 1}} \int \sum_{k_t, l_t} \chi(k, j) |Q_j \Psi_{j} \tilde{a}_{s_l t}(x)|^2 \frac{dx_j dt_j}{t_j}.
\]
Observe that $\tilde{a}_{st}(z)$ is of the form $a(\xi)\chi_{G_{st}/t}(z)$ where $G_{st}t$ is some subset of $\mathbb{R}^n$ depending only on $x_t, t$, and $t$. Now for $x_t, t$, $t$, $j$, and $k$ fixed we can apply the boundedness of the operator $T_{j,k}$ to the function of $z_t$

$$[Q_{j,k} \Psi_{j,k} \tilde{a}_{st}] (x_t, z_t)$$

since this does not depend on $x_t$ and $t$. We obtain a majorization of the previous integral by

$$\frac{1}{(\gamma)^{2+\s}} \sum_{I_t} 2^{3(1+\s_0)} d\alpha \int_{\mathbb{R}^n} \left| [Q_{j,k} \Psi_{j,k} \tilde{a}_{st}] (x_t, z_t) \right|^2 d\alpha.$$

To estimate (10.10) we must sum in $(j, k)$, then in $t$, and finally integrate against $dx_t dt_t / t$. We fix $x_t, t$, $t$, and $z_t$. Observe that the function $\tilde{a}_{st}(z_t, z_t)$ vanishes if $|x_t - z_t| \leq 2t$ for some $s \in [1, i] = I$. Therefore we can apply Lemma 19 in dimension $i$ and obtain:

$$\sum_{j, k} \left| [Q_{j,k} \Psi_{j,k} \tilde{a}_{st}] (x_t, z_t) \right|^2 \leq \int_{\mathbb{R}^n} \left| \tilde{a}_{st}(z_t) \right|^2 \left( \frac{\prod_{1 \leq s \leq i} t_s}{\prod_{1 \leq s \leq i} |x_t - z_t|} \right)^{\frac{2}{3} - \s} d\alpha.$$

Now we integrate in $z_t$ and sum over $t$, keeping in mind that $2^{\s} \leq |x_t - z_t| \leq 2^{\s + 1}$. We are then reduced to integrating the following against $dx_t dt_t / t$:

$$\int_{\mathbb{R}^n} \left| \tilde{a}_{st}(z_t) \right|^2 \left( \frac{t_t}{|x_t - z_t|} \right)^{\frac{2}{3} - \s} d\alpha.$$

But this is less than

$$\|a\|^2 \int_{\mathbb{R}^n} \left| \tilde{D}_{st} (z_t) \right|^2 \left( \frac{|x_t - z_t|}{|x_t - z_t|} \right)^{\frac{2}{3} - \s} d\alpha.$$

Now we use $|D_{st} (z_t)| \leq C |D_{st} (z_t)|$, then integrate against $dx_t dt_t / t$ using (10.5) with $\eta = \alpha = \epsilon = \frac{5}{6}$, and we obtain the desired estimate for the expression (10.10).

To complete the proof of (10.6) we must prove it also when $a_{s-t}$ is replaced by $a_{s-t} - \tilde{a}_{s-t}$ in the $l \cdot h \cdot s$. This is where we are going to use the induction hypothesis, namely that Lemma 18 is true for $k \in [t, n]$. Recall that $a_{st} - \tilde{a}_{st}$ is given by

$$[a_{st} - \tilde{a}_{st}] (z_t, z_t) = \sum_{t} a_{st}(z_t, z_t) \chi_{D_{st} (z_t)} (z_t).$$

By the definition of $\tilde{D}_{st} (z_t)$, we can write, if $z_t \in \tilde{D}_{st} (z_t)$,
1 = \sum_{K \subseteq J \atop K \neq \emptyset} \prod_{r \in K} \chi_{|s_r - z_i| > \tau_r}.

Therefore

\hat{a}_{st} - \hat{a}_{st} = \sum_{K \subseteq J \atop K \neq \emptyset} \hat{a}_{s,t,K},

where

\hat{a}_{s,t,K} = \left[ \sum_{l_j} a_{s,t,l_j} \chi_{s_j,t_j}^* (z_j) \prod_{r \in K} \chi_{|s_r - z_i| > \tau_r} \right] .

Now we apply the induction hypothesis to each function \( \hat{a}_{s,t,K} \). It is enough to show that we can do so when \( K \) is of the form \([i + 1, k]\), the general case being deduced by a reordering of the coordinate indices. Let \( k \geq i + 1 \) be fixed and \( K = [i + 1, k] \). Then \( \hat{a}_{s,t,K} \) satisfies the assumptions of Lemma 18 for \( k \), with \( S_1 \ldots S_i \), as before, \( S_{i+1} = T_{i+1} (x_f, t_f, D_{s,t,f}) \) \( \ldots \) \( S_k = T_k (x_f, t_f, D_{s,t,f}) \). The set \( F_{s,t,K}^{f,k} \) is equal to

\[ F_{s,t,f} \cap \bigcup_{l_j} \left[ \prod_{s \in I} [z_s, 2^{l_s} \leq |s - z_i| < 2^{l_s+1}] \times D_{s,t,f}^l \right] . \]

Finally (10.5) with \( I \cup K \) instead of \( I \) is a consequence of Lemma 5 applied to \( D_{s,t,f}^l \) and more particularly of (10.3) which in this case says that

\[
\int_D |D_{s,t,K}^l| \frac{t_k^{-1}}{|x_K - z_K|} dx_k dt_k \geq C_{n,t} |D_{s,t,f}^l| ,
\]

\( D_{s,t,K}^l \), \( D_{s,t,f}^l \) being defined as in Lemma 18. The conclusion is that we can indeed apply the induction hypothesis to all the functions \( \hat{a}_{s,t,K} \) for \( K \subseteq [1, n] \setminus I \) and \( K \neq \emptyset \) and therefore we obtain (10.6) for \( a_{st} - \hat{a}_{st} \). Thus Lemma 18 is proved, from which follows Lemma 17. Theorem 5 can now be proved by the same arguments as developed in [15] to deduce Theorem 2 from Lemma 15.

We omit the details.

This proof shows the limits of the underlying philosophy of this paper, also implicitly contained in [11]: take a good class of operators, look at the tensor products of them, write all the quantitatives properties you can about those tensor products and look at the class of all operators that satisfy the same quantitative properties; then you can work on this new class. From what we just did, it seems that working with the class obtained by starting from vector-valued singular integral operators satisfying (1.11) is not so simple. Indeed to prove Theorem 5 we had to use the very special structure of the operator under consideration, in particular that the summation in Lemma 16 could be taken over all \((k, j) \in \mathbb{Z}^2\) independently of the function \( \chi \) and that the operator
$T_x$ is a «local tensor product», which corresponds to the fact that it could be written under the form $(\Psi_f^i \otimes T_k,_{i,l})_{k,l,p}$, with the $T_k,_{i,l}$ essentially of the same form that $T_x$.

Let us conclude with a remark along the same lines. Starting with a class of symbols $S_{\rho}^0$ on $\mathbb{R} \times \mathbb{R}$, one can do the same «tensor product manipulation» and define a class of symbols $[S_{\rho \alpha}^0]^n$ on $\mathbb{R}^n \times \mathbb{R}^n$ by the conditions

$$\left| \frac{\partial^\alpha + \beta}{\partial \xi^\alpha \partial \xi^\beta} \hat{a}(x, \xi) \right| \leq C_{\alpha, \beta} \prod_{1 \leq i \leq n} (1 + |\xi_i|)^{\delta_i - \rho \alpha_i}.$$

Are the corresponding $\phi dO$'s bounded when $0 \leq \delta < \rho < 1$ or when $0 \leq \delta = \rho < 1$, for instance on $L^2$ or on some $L^p$? A partial answer is the following: if $\rho = 1$ then the corresponding $\phi dO$'s are CZO's in our sense. This can be seen by the same arguments as in [12]. Otherwise the problem seems entirely open.

References


Université de Strasbourg
Département de Mathématique
7 rue René Descartes
67 Strasbourg, France