Almost classical solutions of Hamilton-Jacobi equations

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Abstract

We study the existence of everywhere differentiable functions which are almost everywhere solutions of quite general Hamilton-Jacobi equations on open subsets of $\mathbb{R}^d$ or on $d$-dimensional manifolds whenever $d \geq 2$. In particular, when $M$ is a Riemannian manifold, we prove the existence of a differentiable function $u$ on $M$ which satisfies the Eikonal equation $\|\nabla u(x)\|_x = 1$ almost everywhere on $M$.

1. Introduction

It has been proved by Z. Buczolich [4] that if $d \geq 2$, there exists $u : \mathbb{R}^d \to \mathbb{R}$, differentiable at every point, such that $\nabla u(0) = 0$ and $\|\nabla u(x)\| \geq 1$ almost everywhere, thus giving a negative answer to the gradient problem of C. E. Weil [10]. Malý and Zelený [8] gave an elegant proof of this result using a new mathematical game. Then Deville and Matheron [6], refining the methods introduced by the above authors, proved that if $\Omega$ is a bounded open subset of $\mathbb{R}^d$ with $d \geq 2$, there exists a function $u : \overline{\Omega} \to \mathbb{R}$, continuous on $\overline{\Omega}$, differentiable at every point of $\Omega$, such that $u(x) = 0$ for all $x \in \partial \Omega$, and such that $\|\nabla u(x)\| = 1$ almost everywhere on $\Omega$. Notice that because of Rolle’s theorem, there exists $x_0 \in \Omega$ such that $\nabla u(x_0) = 0$, so the function $u$ cannot be $C^1$-smooth. We shall call $u$ an almost-classical solution of the Eikonal equation $\|\nabla u\| = 1$. This equation has also a unique viscosity solution, which is the function $x \mapsto \text{dist}(x, \partial \Omega)$, where $\text{dist}(x, \partial \Omega) = \inf\{\|x - y\| ; y \in \partial \Omega\}$. The viscosity solution is not everywhere differentiable on $\Omega$. Therefore, an almost classical solution of the Eikonal equation is not equal to the viscosity solution of the Eikonal equation. Nevertheless in optimal control, where

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this equation arises naturally, the viscosity solution is the “right” solution of the Eikonal equation. We refer to [2] and [5] for an account on viscosity solutions of Hamilton-Jacobi equations.

The contents of the paper are as follows. In Section 2, we recall some technical results from [6] which will be needed in this paper. In Sections 3 and 4, we study the existence of almost-classical solutions for more general Hamilton-Jacobi equations on open subsets of $\mathbb{R}^d$. Finally, Section 5 is devoted to Hamilton-Jacobi equations on manifolds, and in particular we will consider the Eikonal equation $\|\nabla u(x)\|_x = 1$ on a Riemannian manifold. See e.g. [1], [7] and [9] for further information about Hamilton-Jacobi equations on Riemannian manifolds.

Now we introduce some terminology. Let $\Omega$ be an open subset of $\mathbb{R}^d$, and let $F : \mathbb{R} \times \Omega \times \mathbb{R}^d \to \mathbb{R}$ and $u_0 : \partial \Omega \to \mathbb{R}$ be continuous. As usual, we say that a continuous function $u : \Omega \to \mathbb{R}$ is a classical solution of $F(u(x), x, \nabla u(x)) = 0$ with Dirichlet condition $u|_{\partial \Omega} = u_0$ if for all $x \in \partial \Omega$, $u(x) = u_0(x)$, and for all $x \in \Omega$, $u$ is differentiable at $x$ and $F(u(x), x, \nabla u(x)) = 0$.

We say that $u$ is a classical subsolution of $F(u(x), x, \nabla u(x)) = 0$ if for all $x \in \Omega$, $u$ is differentiable at $x$ and $F(u(x), x, \nabla u(x)) \leq 0$.

Definition 1.1. We say that a continuous function $u : \overline{\Omega} \to \mathbb{R}$ is an almost classical solution of $F(u(x), x, \nabla u(x)) = 0$ with Dirichlet condition $u|_{\partial \Omega} = u_0$ if:

- $u(x) = u_0(x)$ for all $x \in \partial \Omega$,
- $u$ is a classical subsolution of $F(u(x), x, \nabla u(x)) = 0$,
- and $u$ satisfies $F(u(x), x, \nabla u(x)) = 0$ for almost every $x \in \Omega$ (in the sense of Lebesgue measure on $\mathbb{R}^d$).

Notice that a classical solution is an almost classical solution, and that if $u$ is an almost classical solution, then $u$ is continuous on $\overline{\Omega}$ and differentiable at every point of $\Omega$. In many natural examples, classical solutions of the Hamilton-Jacobi equation $F(u(x), x, \nabla u(x)) = 0$ exist only under very restrictive conditions on $F$. We prove the existence of almost classical solutions under quite general hypotheses on $F$. Observe that our results imply the existence of an almost classical solution $u$ of the Eikonal equation satisfying the boundary condition $u|_{\partial \Omega} = 0$. In particular, we have:

Theorem 1.2. Let $\Omega$ be an open subset of $\mathbb{R}^d$ with $d \geq 2$, and let $F : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a continuous function. Suppose that the following conditions hold:

(A) There exists a continuous function $u_0 : \overline{\Omega} \to \mathbb{R}$, which is $C^1$-smooth on $\Omega$ and such that $F(x, \nabla u_0(x)) \leq 0$, for every $x \in \Omega$. 


(B) For each compact subset $K \subset \Omega$, there exists $M_K > 0$ such that
$$\inf \{ F(x, p) : x \in K, p \in \mathbb{R}^d, \| p \| \geq M_K \} > 0.$$  

Then there exists an almost classical solution of $F(x, \nabla u(x)) = 0$, with Dirichlet condition $u|_{\partial \Omega} = u_0$.

The above result will actually follow from the more general Theorem 3.1 that will also provide other existence results of almost classical solutions of Hamilton-Jacobi equations. The proof of Theorem 3.1 will be given in Section 4.

In the last section, we consider Hamilton-Jacobi equations defined on a smooth manifold $M$ of dimension $d \geq 2$, which always will be assumed to be Hausdorff and second countable. As usual, $TM$ denotes the tangent bundle of $M$. A point in $TM$ will be $(x, v)$, where $x \in M$ and $v$ belongs to the tangent space $T_x M$. In the same way, $T^*M$ denotes the cotangent bundle of $M$. A point in $T^*M$ will be $(x, \xi)$, where $x \in M$ and $\xi \in T^*_x M$ is a linear form on the tangent space $T_x M$. If $u : M \to \mathbb{R}$ is differentiable at $x \in M$ we denote its differential at $x$ by $du(x)$. Under suitable hypotheses on $F : T^*M \to \mathbb{R}$, we obtain the existence of almost classical solutions of an equation of the form $F(x, du(x)) = 0$. In particular, we obtain:

**Theorem 1.3.** Let $M$ be a smooth manifold of dimension $d \geq 2$, and let $F : T^*M \to \mathbb{R}$ be a $C^1$-smooth function. Suppose that the following conditions hold:

(A) There exists a $C^1$ function $u_0 : M \to \mathbb{R}$ such that $F(x, du_0(x)) \leq 0$, for every $x \in M$.

(B) For each $x \in M$, the set $B(x) = \{ \xi \in T^*_x M : F(x, \xi) \leq 0 \}$ is compact, the set $S(x) = \{ \xi \in T^*_x M : F(x, \xi) = 0 \}$ is connected, and the function $F(x, \cdot)$ has maximal rank on the set $S(x)$.

Then there exists a differentiable function $u : M \to \mathbb{R}$ such that $F(x, du(x)) = 0$ for almost every $x \in M$.

If now we have a Riemannian manifold $(M, g)$ and $u : M \to \mathbb{R}$ is differentiable, for every $x \in M$ we identify in the usual way the differential $du(x)$ with the gradient $\nabla u(x)$ by means of the scalar product $g_x(\cdot, \cdot)$ on the tangent space $T_x M$. In this case we obtain the following analogue of Theorem 1.2:

**Theorem 1.4.** Let $(M, g)$ be a Riemannian manifold of dimension $d \geq 2$, and let $F : TM \to \mathbb{R}$ be a continuous function. Suppose that the following conditions hold:
(A) There exists a \( C^1 \) function \( u_0 : M \to \mathbb{R} \), such that \( F(x, \nabla u_0(x)) \leq 0 \), for every \( x \in M \).

(B) There exists a locally bounded function \( \rho : M \to (0, \infty) \) such that, for every \( x \in M \), the set \( B(x) = \{ v \in T_xM : F(x, v) \leq 0 \} \) is contained in the ball of center 0 and radius \( \rho(x) \) in \( T_xM \).

Then there exists a differentiable function \( u : M \to \mathbb{R} \) such that \( F(x, \nabla u(x)) = 0 \) for almost every \( x \in M \).

Thus if for a Riemannian manifold \( (M, g) \) we consider the function \( F : TM \to \mathbb{R} \) given by

\[
F(x, v) = \|v\|_x - 1 = (g_x(v, v))^{1/2} - 1,
\]

it is clear that the constant functions \( u_0 \equiv 0 \) and \( \rho \equiv 1 \) satisfy the above requirements. Therefore we obtain that there exists a differentiable function \( u \) on \( M \) which satisfies the Eikonal equation \( \| \nabla u(x) \|_x = 1 \) almost everywhere on \( M \). Whenever the manifold \( M \) is compact, there exists a point \( x_0 \in M \) such that \( \nabla u(x_0) = 0 \). Therefore, there is no classical solution of this equation, and an almost classical solution \( u \) of this equation cannot be \( C^1 \)-smooth. So almost classical solutions of Hamilton-Jacobi equations are often exotic.

2. Preliminary results

We recall three lemmas from [6] that we shall use here. The first lemma is a criterium of differentiability for the sum of a series of \( C^1 \)-smooth functions. We shall use the following notation: if \( X \) and \( Z \) are Banach spaces and \( f : X \to Z \), then the oscillation of \( f \) with respect to \( \delta > 0 \) is defined by

\[
osc(f, \delta) = \sup \{ \| f(x_1) - f(x_2) \| : x_1, x_2 \in X, \| x_1 - x_2 \| \leq \delta \}
\]

Lemma 2.1. Let \( (u_n)_{n \geq 1} \) be a sequence of \( C^1 \) functions between two Banach spaces \( X \) and \( Y \). Assume that:

(a) the series \( \left( \sum \nabla u_n(x) \right) \) is pointwise convergent;

(b) the sequence \( \left( \nabla u_n \right) \) converges uniformly to 0;

(c) \( \|u_{n+1}\|_\infty = o(\|u_n\|_\infty) \);

(d) \( \lim_{n \to \infty} osc \left( \sum_{k=1}^{n} \nabla u_k, \|u_{n+1}\|_\infty \right) = 0 \).

Then the series \( \left( \sum u_n \right) \) is uniformly convergent, the function \( u := \sum_{n=1}^{\infty} u_n \) is everywhere differentiable, and \( \nabla u(x) = \sum_{n=1}^{\infty} \nabla u_n(x) \) for all \( x \in X \).
We say that a subset $Q$ of $\mathbb{R}^d$ is a cube if $Q = \prod_{i=1}^d [a_i, b_i]$, where each $[a_i, b_i]$ is a closed and bounded interval of $\mathbb{R}$. And we say that $Q$ is a closed cube if $Q = \prod_{i=1}^d [a_i, b_i]$. A function $v$ defined on a cube $Q$ is said to be \textit{piecewise constant} if there is a finite partition $Q$ of $Q$ into cubes such that $v$ is constant on every cube of the partition $Q$. The following result gives the existence of a $C^\infty$-smooth function $u : \mathbb{R}^d \to \mathbb{R}$, which vanishes in a neighbourhood of the exterior of a cube $Q$ and such that its derivative is equal to $a$ or $-a$ (where $a$ is a given non zero vector in $\mathbb{R}^d$) on a subset of $Q$ of measure almost equal to the measure of $Q$. The Lebesgue measure on $\mathbb{R}^d$ will be denoted $\lambda_d$.

Lemma 2.2. Let $a \in \mathbb{R}^d$ be a non zero vector, let $Q$ be a cube in $\mathbb{R}^d$, and let $\varepsilon > 0$. Then, there exists a bounded, $C^\infty$-smooth function $u : \mathbb{R}^d \to \mathbb{R}$ satisfying the following properties:

(a) $u$ vanishes in a neighbourhood of $\partial Q$ and $\|u\|_\infty \leq \varepsilon$;
(b) $\lambda_d(\{x \in Q : \nabla u(x) = -a \text{ or } \nabla u(x) = a\}) \geq (1 - \varepsilon)\lambda_d(Q)$;
(c) one can write $\nabla u = v + w$ with $\|w\|_\infty < \varepsilon$; the set \{v(x) : x \in Q\} is included in the segment $[-a, a]$, and the function $v$ is piecewise constant on $Q$.

The last lemma relies on ideas due to J. Maly and M. Zeleny [8], and is also from [6]. The mapping $t$ is defined using that a suitable game has a winning strategy.

Lemma 2.3. Let $B$ be a closed ball of $\mathbb{R}^d$. Then, there exists a map $t : B \to \mathbb{R}^d$ such that if a sequence $(\sigma_n) \in B$ satisfies $\langle t(\sigma_n), \sigma_{n+1} - \sigma_n \rangle \geq 0$ for all $n$, then $(\sigma_n)$ converges.

3. Almost classical solutions on open subsets of $\mathbb{R}^d$

For a wide class of Hamilton-Jacobi equations, we give an existence theorem of almost classical solutions defined on the closure of on an open subset of $\mathbb{R}^d$, and satisfying an homogeneous Dirichlet condition.

Theorem 3.1. Let $\Omega$ be an open subset of $\mathbb{R}^d$ with $d \geq 2$, and $F : \mathbb{R} \times \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a continuous function. Suppose that the following conditions hold:

(A) $F(0, x, 0) \leq 0$, for every $x \in \Omega$; that is, the function $u_0$ identically equal to 0 is a classical subsolution of $F(u(x), x, \nabla u(x)) = 0$.

(B) For each compact subset $K \subset \Omega$, there exist $\alpha_K > 0$ and $M_K > 0$ such that for all $x \in K$, for all $u \in [0, \alpha_K]$ and for all $p \in \mathbb{R}^d$ satisfying $\|p\| \geq M_K$, we have $F(u, x, p) > 0$. 


Then there exists a function \( u \geq 0 \) on \( \overline{\Omega} \) which is an almost classical solution of \( F(u(x), x, \nabla u(x)) = 0 \), with Dirichlet condition \( u \big|_{\partial \Omega} = 0 \). Moreover, the extension \( \tilde{u} \) of \( u \) to \( \mathbb{R}^d \) satisfying \( \tilde{u}(x) = 0 \) if \( x \notin \overline{\Omega} \) is differentiable at every point of \( \mathbb{R}^d \).

The proof of Theorem 3.1 will be postponed until Section 4. Along this section, we will obtain several consequences of this result.

**Remark 3.2.** It will be useful to note that condition (B) in Theorem 3.1 is equivalent to condition \((B')\) and also to condition \((B'')\) below:

\[(B') \quad \text{For each compact subset } K \subset \Omega, \text{ there exists } \alpha_K > 0 \text{ such that the set}\]
\[B(K; \alpha_K) = \{(u, x, p) \in [0, \alpha_K] \times K \times \mathbb{R}^d : F(x, u, p) \leq 0\}\]
\[\text{is compact in } \mathbb{R} \times \Omega \times \mathbb{R}^d.\]

\[(B'') \quad \text{For each } x_0 \in \Omega, \text{ there exist a compact neighborhood } V^{x_0} \text{ and } \alpha > 0, \text{ such that the set}\]
\[B(V^{x_0}; \alpha) = \{(u, x, p) \in [0, \alpha] \times V^{x_0} \times \mathbb{R}^d : F(x, u, p) \leq 0\}\]
\[\text{is compact in } \mathbb{R} \times \Omega \times \mathbb{R}^d.\]

We now consider the case of general Dirichlet conditions.

**Corollary 3.3.** Let \( \Omega \) be an open subset of \( \mathbb{R}^d \) with \( d \geq 2 \), and \( F : \mathbb{R} \times \Omega \times \mathbb{R}^d \to \mathbb{R} \) be a continuous function. Suppose that the following conditions hold:

\[(A) \quad \text{There exists a continuous function } u_0 : \overline{\Omega} \to \mathbb{R}, \text{ which is } C^1\text{-smooth on } \quad \Omega \text{ and such that } F(u_0(x), x, \nabla u_0(x)) \leq 0, \text{ for every } x \in \Omega.\]

\[(B) \quad \text{For each compact subset } K \subset \Omega, \text{ there exist } M_K > 0 \text{ and } \alpha_K > 0 \text{ such that for all } x \in K, \text{ for all } u \in [0, \alpha_K] \text{ and for all } p \in \mathbb{R}^d \text{ satisfying}\]
\[\|p\| \geq M_K, \text{ we have } F(u_0(x) + u, x, p) > 0.\]

Then there exists an almost classical solution \( u \) of \( F(u(x), x, \nabla u(x)) = 0 \), with Dirichlet condition \( u \big|_{\partial \Omega} = u_0 \). Moreover, if \( u_0 \) is \( C^1\)-smooth on \( \mathbb{R}^d \), the function \( u \) can be extended to a differentiable function on \( \mathbb{R}^d \).

**Proof.** Define \( G(u, x, p) = F(u + u_0(x), x, p + \nabla u_0(x)) \). Conditions \((A)\) and \((B)\) of Theorem 3.1 are satisfied for \( G \). Thus, there exists an almost classical solution \( v \) of \( G(v(x), x, \nabla v(x)) = 0 \), with Dirichlet condition \( v \big|_{\partial \Omega} = 0 \), and furthermore \( v \) can be extended to a differentiable function on \( \mathbb{R}^d \). The function \( u : \overline{\Omega} \to \mathbb{R} \) defined by \( u(x) = u_0(x) + v(x) \) is then an almost classical solution of \( F(u(x), x, \nabla u(x)) = 0 \), with Dirichlet condition \( u \big|_{\partial \Omega} = u_0 \). \( \blacksquare \)
Almost classical solutions of Hamilton-Jacobi equations

Notice that Theorem 1.2 is a straightforward consequence of Corollary 3.3. Another easy consequence of Corollary 3.3 is the following existence result of almost classical solutions for stationary Hamilton-Jacobi equations:

**Corollary 3.4.** Let \( \Omega \) be an open subset of \( \mathbb{R}^d \) with \( d \geq 2 \), and let \( F : \Omega \times \mathbb{R}^d \to \mathbb{R} \) be a continuous function. Suppose that the following conditions hold:

(A) There exists a continuous function \( u_0 : \overline{\Omega} \to \mathbb{R} \), which is \( C^1 \)-smooth on \( \Omega \) and such that \( u_0(x) + F(x, \nabla u_0(x)) \leq 0 \), for every \( x \in \Omega \).

(B) For each compact \( K \subset \Omega \), there exists \( M_K > 0 \) such that

\[
\inf \{ u_0(x) + F(x, p); x \in K, p \in \mathbb{R}^d, \|p\| \geq M_K \} > 0
\]

Then there exists an almost classical solution of \( u(x) + F(x, \nabla u(x)) = 0 \), with Dirichlet condition \( u \mid_{\partial \Omega} = u_0 \).

Next we give a further application of Theorem 3.1. We shall need the following notions. If \( A \) is a subset of \( \mathbb{R}^d \), we denote its complement by \( A^c = \mathbb{R}^d \setminus A \). Let us recall the definition of the Hausdorff distance between closed sets of a metric space. If \( X \) is a metric space, for each \( A \subset X \) and \( r > 0 \) we denote \( B(A, r) = \{ x \in X : \text{dist}(x, A) < r \} \). We denote \( C(X) \) the set of all closed bounded subsets of \( X \). If \( C \) and \( D \) are in \( C(X) \), the Hausdorff distance between them is

\[
d_H(C, D) = \inf \{ r \in (0, \infty] : C \subset B(D, r) \text{ and } D \subset B(C, r) \}.
\]

**Theorem 3.5.** Let \( \Omega \) be an open subset of \( \mathbb{R}^d \) with \( d \geq 2 \). For each \( x \in \Omega \) let \( U(x) \) be an open bounded subset of \( \mathbb{R}^d \) containing 0. Assume that the set-valued mapping \( x \mapsto \partial U(x) \) from \( \Omega \) into \( (C(\mathbb{R}^d), d_H) \) is continuous on \( \Omega \). Then there exists a differentiable function \( u : \mathbb{R}^d \to \mathbb{R} \) such that:

1. \( u \mid_{\Omega^c} \equiv 0 \) and \( \nabla u \mid_{\Omega^c} \equiv 0 \).
2. \( \nabla u(x) \in \overline{U(x)} \) for every \( x \in \mathbb{R}^d \).
3. \( \nabla u(x) \in \partial U(x) \) for almost every \( x \in \Omega \).

**Proof.** Consider the function \( F : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) defined by \( F(u, x, p) = -\text{dist}(p, \partial U(x)) \) if \( x \in U(x) \), and \( F(u, x, p) = \text{dist}(p, \partial U(x)) \) otherwise. Since the mapping \( x \mapsto \partial U(x) \) from \( \Omega \) into \( (C(\mathbb{R}^d), d_H) \) is continuous on \( \Omega \), it is easy to see that \( F \) is continuous on \( \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \). The function identically
equal to 0 is a classical subsolution of $F(u, x, p) = 0$. On the other hand, for each compact subset $K \subset \mathbb{R}^d$, there exists $R > 0$ such that

$$\bigcup_{x \in K} \partial U(x) \subset B(0, R),$$

and therefore

$$\bigcup_{x \in K} \overline{U(x)} \subset B(0, R).$$

Hence for all $u \in \mathbb{R}$, for all $p \in \mathbb{R}^d$ satisfying $\|p\| \geq 2R$ and for all $x \in K$, we have $F(u, x, p) > 0$. The two hypothesis of Theorem 3.1 are then satisfied. The almost classical solution of $F(u, x, p) = 0$ given by Theorem 3.1 satisfies the required properties.

4. Proof of Theorem 3.1.

In order to prove Theorem 3.1, we first consider the case of a cube, on which almost classical solutions will be obtained as the sum of a series of $C^\infty$-smooth functions, and the general case will then follow easily.

Lemma 4.1. Assume that the hypotheses of Theorem 3.1 are satisfied, and let $C$ be a cube such that $\overline{C}$ is contained in $\Omega$. Then there exists a differentiable function $u_C : \mathbb{R}^d \rightarrow \mathbb{R}$ such that:

1. $u_C \geq 0$ and $u_C(x) = 0$ for all $x$ which is not in the interior of $C$.
2. $F(u_C(x), x, \nabla u_C(x)) \leq 0$ for every $x \in C$.
3. $F(u_C(x), x, \nabla u_C(x)) = 0$ for almost every $x \in C$.

Proof of Theorem 3.1. We first fix an increasing sequence $(K_n)_{n \geq 1}$ of compact subsets of $\Omega$ such that the union of all $K_n$’s is equal to $\Omega$. We also assume that each $K_n$ is the closure of a finite union of cubes. By assumption (B), for each $n \geq 1$ there exist $M_n > 0$ and $\alpha_n > 0$ such that, for all $x \in K_n$, for all $u \in [0, \alpha_n]$ and for all $p \in \mathbb{R}^d$ satisfying $\|p\| \geq M_n$, we have $F(u, x, p) > 0$. We consider a decomposition

$$\Omega = \bigcup_{j=1}^{\infty} C_j,$$

where $(C_j)_{j \geq 1}$ is a locally finite family of cubes such that:

(a) $C_j \cap C_k = \emptyset$ if $j \neq k$.

(b) for each $j$, there exists $n$ such that $C_j \subset \overline{K_n \setminus K_{n-1}}$. 

996  R. Deville and J.A. Jaramillo
Refining if necessary this decomposition, we can also assume:

(c) \( \text{diam}(C_j) \leq \frac{1}{2^n} d_H(K_n, \partial \Omega) \) whenever \( C_j \subset \overline{K_n \setminus K_{n-1}} \).

By Lemma 4.1, for each \( j \geq 1 \) there exists a differentiable function \( u_j : \mathbb{R}^d \to \mathbb{R} \) such that:

1. \( u_j \geq 0 \) and \( u_j(x) = 0 \) for all \( x \) which is not in the interior of \( C_j \).
2. \( F(u_j(x), x, \nabla u_j(x)) \leq 0 \) for every \( x \in C_j \).
3. \( F(u_j(x), x, \nabla u_j(x)) = 0 \) for almost every \( x \in C_j \).

Then we define \( u : \mathbb{R}^d \to \mathbb{R} \) by setting

\[
u = \sup_{j \geq 1} u_j.\]

By property (1) above, \( u = u_j \) on each \( C_j \). Then it is easy to see that \( u \) is differentiable on \( \Omega \), identically equal to 0 on \( \mathbb{R}^d \setminus \Omega \), satisfies \( F(u(x), x, \nabla u(x)) \leq 0 \) for every \( x \in \Omega \) and \( F(u(x), x, \nabla u(x)) = 0 \) for almost every \( x \in \Omega \). It remains to check that \( u \) is differentiable at each point of \( \partial \Omega \). Fix \( n \geq 1 \).

We know that \( u \) vanishes on the boundary of each cube \( C_j \), so, by the mean value theorem,

\[
\sup \{ u(x) : x \in C_j \} \leq \sup \{ \nabla u(x) : x \in C_j \} \cdot \text{diam}(C_j).
\]

If \( C_j \subset \overline{K_n \setminus K_{n-1}} \), then \( \sup \{ \nabla u(x) : x \in C_j \} \leq M_n \). In that case, using (c), we obtain:

\[
\sup \{ u(x) : x \in C_j \} \leq \frac{1}{2^n} d_H(K_n, \partial \Omega).
\]

So whenever \( x \in K_n \setminus K_{n-1} \), we have that \( 0 \leq u(x) \leq \text{dist}(x, \partial \Omega)/2^n \). This implies that for each point \( x \in \partial \Omega \), \( u \) is differentiable at \( x \) and \( \nabla u(x) = 0 \). \( \blacksquare \)

**Proof of Lemma 4.1.** Observe that if \( F(0, x, 0) = 0 \) for almost every \( x \in C \), we can take \( u_C = 0 \) and the above assertions are satisfied. From now on, we assume that

\[
\lambda_d(\{ x \in C : F(0, x, 0) < 0 \}) > 0
\]

By assumption (B), there exists \( \alpha > 0 \) such that

\[
r := \sup \{ \|p\| : F(u, x, p) \leq 0 \text{ for some } x \in \overline{C} \text{ and } u \in [0, \alpha] \}
\]

is finite. We fix a map \( t : B(0, 1 + r) \to \mathbb{R}^d \) satisfying the conditions of Lemma 2.3.
The function $u_C$ will be given by a series

$$u_C = \sum_{n=1}^{\infty} u_n,$$

where each $u_n$ is a $C^{\infty}$-smooth function on $\mathbb{R}^d$. For each $n$, we will write $\nabla u_n = v_n + w_n$, and we will denote

$$U_n = \sum_{k=1}^{n} u_k \quad \text{and} \quad \sigma_n = \sum_{k=1}^{n} v_k.$$

**Construction of the functions $u_n$:** The functions $u_n$ will be constructed together with a sequence $(Q_n)_{n \geq 0}$ of partitions of $C$ into cubes, where each $Q_{n+1}$ is a refinement of $Q_n$. We also fix a sequence $(\varepsilon_k)_{k \geq 1}$ of positive numbers, with $(\varepsilon_k) \downarrow 0$ and such that $\inf \{ F(0, x, 0) : x \in C \} < -\varepsilon_1$ and $\varepsilon_1 < 1$, and we construct an increasing sequence of integers $(N_k)_{k \geq 0}$ with $N_0 = 0$. The following conditions will be proved by induction:

(i) There exists $x_0 \in C$ such that, for each $n \geq 1$, $u_n(x_0) = 0$ and $\nabla u_n(x_0) = 0$.

(ii) For each $n \geq 1$ and $x \in C$, $F(U_n(x), x, \nabla U_n(x)) \leq 0$.

(iii) For each $n \geq 1$ and $x \in C$, we have

$$\|\sigma_n(x)\| \leq 1 + r \quad \text{and} \quad \langle t(\sigma_n(x)), \sigma_{n+1}(x) - \sigma_n(x) \rangle = 0,$$

(iv) $\|u_1\|_{\infty} \leq \alpha/2$, and, for each $n \geq 1$, we have

$$0 < \|u_{n+1}\|_{\infty} \leq 2^{-n} \|u_n\|_{\infty} \quad \text{and} \quad \text{osc}(\nabla U_n, \|u_{n+1}\|_{\infty}) \leq 1/2^n$$

(v) For each $k \geq 1$ and each $N_{k-1} < n \leq N_k$, we have $\|v_n\|_{\infty} \leq \varepsilon_k$.

(vi) For each $k \geq 1$,

$$\lambda_d \{ x \in C : F(U_{N_k}(x), x, \nabla U_{N_k}(x)) \leq -\varepsilon_k \} \leq 2^{-k} \lambda_d(C).$$

**Construction of $u_1$:** Fix a cube $Q_0 \subset C$ with

$$d_H(Q_0, \partial C) > 0 \quad \text{and} \quad \sup \{ F(0, x, 0) : x \in Q_0 \} \leq -\varepsilon_1.$$ 

This implies, using the uniform continuity of $F$ on compact sets, that there exists $0 < \delta_1 \leq \varepsilon_1$ such that, whenever $x \in Q_0$, $0 \leq h \leq \delta_1$ and $\|q\| \leq 2\delta_1$, then:

$$F(h, x, q) \leq 0.$$
Choose $a = a(Q_0) \in \mathbb{R}^d$ such that $\|a(Q_0)\| = \delta_1$ and $\langle t(0), a \rangle = 0$ (this is possible since $d \geq 2$). Now applying Lemma 2.2 to the cube $Q_0$, we obtain a $C^\infty$ function $u_1$ on $\mathbb{R}^d$, and a cube partition $Q_1$ of $C$ such that $Q_0$ is a union of some elements of $Q_1$, such that:

- $u_1$ vanishes on a neighborhood of $\partial Q_0$ and outside of $Q_0$.
- $0 < \|u_1\|_\infty \leq \min\{\delta_1, a/2\}$.
- $\nabla u_1 = v_1 + w_1$, where $\|w_1\|_\infty \leq \min\{\delta_1, 1/2\}$, $v_1$ is constant on each cube of $Q_1$, and $v_1(Q_0) \subset [-a(Q_0), a(Q_0)]$. 

Fix $x_0 \in \partial Q_0$: we have $u_1(x_0) = 0$ and $\nabla u_1(x_0) = 0$, so condition (0) is satisfied. Conditions (i),(iii) and (v) are clearly satisfied, and (ii) follows from (4.1). So we can start the induction.

**Inductive step:** Fix $k \geq 1$, assume that $N_{k-1}$ has been defined, and for some $n \geq N_{k-1}$ the partition $Q_n$ and the function $u_n$ have been constructed.

First, there exists $0 < \delta_k \leq \varepsilon_k$ such that whenever $x \in \overline{U}$, $u \in [0, \alpha]$, $p \in B(0, 1 + r)$, $0 \leq h \leq \delta_k$ and $\|q\| \leq 2\delta_k$, then

\begin{equation}
F(u, x, p) \leq -\varepsilon_k/2 \implies F(u + h, x, p + q) \leq 0
\end{equation}

Next, choose a cube partition $\hat{Q}_n$ of $C$ refining $Q_n$ such that:

- If we denote $\hat{R}_n$ the family of all cubes $Q \in \hat{Q}_n$ such that $d_H(Q, \partial C) > 0$, and $K_n = \cup\{Q : Q \in \hat{R}_n\}$, we have that
  \begin{equation}
  \lambda_d(C \setminus K_n) < 2^{-(k+1)}\lambda_d(C).
  \end{equation}

- For all $Q \in \hat{R}_n$ and every $x, y \in Q$, we have
  \begin{equation}
  |F(U_n(x), x, \nabla U_n(x)) - F(U_n(y), y, \nabla U_n(y))| < \varepsilon_k/2.
  \end{equation}

The second condition above can be obtained using the uniform continuity of the mapping $x \mapsto F(U_n(x), x, \nabla U_n(x))$ on the compact set $\overline{U}$.

Now, each cube $Q$ of $\hat{Q}_n$ is contained in a cube $Q'$ of $Q_n$ and by (i), $\sigma_n$ is constant on $Q'$. We denote by $\sigma_n(Q)$ the constant value of $\sigma_n|_{Q'}$. Choose $a = a(Q) \in \mathbb{R}^d$ such that $\|a(Q)\| = \delta_k$ and $\langle t(\sigma_n(Q)), a \rangle = 0$. Now applying Lemma 2.2, for each cube $Q \in \hat{R}_n$ we obtain a $C^\infty$ function $u_Q$ on $\mathbb{R}^d$, and a cube partition $Q_{n+1}$ of $Q_0$ which is a refinement of $\hat{Q}_n$ (and therefore of $Q_n$), such that:

- $u_Q$ vanishes on a neighborhood of $\partial Q$.
- $0 < \|u_Q\|_\infty \leq \min\{2^{-n}\|u_n\|_\infty, \delta_k\}$ and $\text{osc}(\nabla U_n, \|u_Q\|_\infty) < 1/2^n$. 

Almost classical solutions of Hamilton-Jacobi equations 999
(c) \( \lambda_d \{ x \in Q : \nabla u_Q(x) = \pm \alpha(Q) \} \geq (1 - 2^{-k}) \lambda_d(Q) \).

(d) \( \nabla u_Q = v_Q + w_Q \), where \( \|w_Q\|_{\infty} = \min \{ \delta_k, 1/2^{n+2} \} \). \( v_Q \) is constant on each cube of \( Q_{n+1} \), and \( v_Q(Q) \subset [-a(Q), a(Q)] \). In particular, we have \( \|v_Q\|_{\infty} \leq \|a(Q)\| = \delta_k \leq \varepsilon_k \) and \( \|\nabla u_Q\|_{\infty} \leq 2\delta_k \).

Next we define the function \( u_{n+1} \) on \( \mathbb{R}^d \). We first choose for each \( Q \in \hat{R}_n \) a point \( x_Q \) in the closure of \( Q \) such that

\[
F(U_n(x_Q), x_Q, \nabla U_n(x_Q)) = \inf \{ F(U_n(x), x, \nabla U_n(x)) : x \in Q \}
\]

We define \( u_{n+1} \) on each cube of \( \hat{R}_n \) in the following way:

1. If \( F(U_n(x_Q), x_Q, \nabla U_n(x_Q)) > -\varepsilon_k \), we set \( u_{n+1} = 0 \) and \( v_{n+1} = w_{n+1} = 0 \) on \( Q \).

2. If \( F(U_n(x_Q), x_Q, \nabla U_n(x_Q)) \leq -\varepsilon_k \), we set \( u_{n+1} = u_Q \) on \( Q \). In this case, \( v_{n+1} = v_Q, w_{n+1} = w_Q \), and we have

\[
\lambda_d \{ x \in Q : \|\nabla u_{n+1}(x)\| = \delta_k \} \geq (1 - 2^{-k}) \cdot \lambda_d(Q).
\]

3. Finally, on \( (K_n)^c \), we set \( u_{n+1} = 0 \), and \( v_{n+1} = w_{n+1} = 0 \).

In this way we obtain that \( u_{n+1} \) is a \( C^\infty \) function on \( \mathbb{R}^d \), which vanish on a neighborhood of \( \partial Q \) for every \( Q \in \hat{R}_n \).

Next we are going to check conditions (0) to (vi) for \( n + 1 \). Since \( \hat{Q}_n \) is a refinement of \( Q_1 \) and \( x_0 \in \partial Q_0 \in Q_1 \), there exists \( Q \in \hat{Q}_n \) such that \( x_0 \in \partial Q \), so \( u_{n+1}(x_0) = 0 \) and \( \nabla u_{n+1}(x_0) = 0 \). This proves condition (0). Condition (i) is clearly satisfied. In order to prove condition (ii), fix \( x \in \Omega \). By induction hypothesis, \( F(U_n(x), x, \nabla U_n(x)) \leq 0 \). Let us prove that \( F(U_{n+1}(x), x, \nabla U_{n+1}(x)) \leq 0 \):

- If \( x \in (K_n)^c \), then \( u_{n+1} = 0 \) on a neighbourhood of \( x \) and \( \nabla U_{n+1}(x) = \nabla U_n(x) \), so \( F(U_{n+1}(x), x, \nabla U_{n+1}(x)) = F(U_n(x), x, \nabla U_n(x)) \leq 0 \).

- If \( x \in Q \in \hat{R}_n \) with \( F(U_n(x_Q), x_Q, \nabla U_n(x_Q)) > -\varepsilon_k \), then \( u_{n+1} = 0 \) on a neighborhood of \( Q \) and

\[
F(U_{n+1}(x_Q), x_Q, \nabla U_{n+1}(x_Q)) = F(U_n(x_Q), x_Q, \nabla U_n(x_Q)) \leq 0.
\]

- Finally, if \( x \in Q \in \hat{R}_n \) with \( F(U_n(x_Q), x_Q, \nabla U_n(x_Q)) \leq -\varepsilon_k \), by 4.4, we have for all \( x \in Q \), \( F(U_n(x), x, \nabla U_n(x)) \leq -\varepsilon_k/2 \). Since

\[
|U_n(x)| \leq \sum_{k=1}^{\infty} |u_k(x)| \leq \alpha,
\]
from the definition of \( r \) and the fact that \( F(U_n(x), x, \nabla U_n(x)) \leq 0 \), it follows that \( \| \nabla U_n(x) \| \leq r \). Since we have also \( \| u_{n+1} \|_\infty \leq \delta_k \) and \( \| \nabla u_{n+1} \|_\infty \leq 2\delta_k \), we deduce from 4.2 that

\[
F(U_{n+1}(x), x, \nabla U_{n+1}(x)) \leq 0
\]

In order to prove (iii), fix \( x \in C \). First, \( F(U_{n+1}(x), x, \nabla U_{n+1}(x)) \leq 0 \), so \( \| \nabla U_{n+1}(x) \| \leq r \), and

\[
\| \nabla U_{n+1}(x) - \sigma_{n+1}(x) \| \leq \sum_{k=1}^{n+1} \| w_k(x) \| \leq 1.
\]

Therefore \( \| \sigma_n(x) \| \leq 1+r \). Then, if \( v_{n+1}(x) = 0 \), we have \( \sigma_{n+1}(x) - \sigma_n(x) = 0 \), so \( \langle t(\sigma_n(x)), \sigma_{n+1}(x) - \sigma_n(x) \rangle = 0 \). On the other hand, if \( v_{n+1}(x) \neq 0 \), then \( x \in Q \) for some \( Q \in \hat{\mathcal{R}}_n \) and \( v_{n+1} = v_Q \). In this case \( \sigma_n(x) = \sigma_n(Q) \), and \( v_{n+1}(x) \in [-a(Q), a(Q)] \). Thus \( v_{n+1}(x) = \sigma_{n+1}(x) - \sigma_n(x) \) is proportional to \( a(Q) \), and therefore orthogonal to \( t(\sigma_n(Q)) \), and the condition \( \langle t(\sigma_n(x)), \sigma_{n+1}(x) - \sigma_n(x) \rangle = 0 \) is again satisfied.

Now we are going to see that \( u_{n+1} \neq 0 \). Indeed, if \( Q \in \hat{\mathcal{Q}}_n \) is such that \( Q \subset Q_0 \) and \( x_0 \in \partial Q \), then \( d_H(Q, \partial C) \geq d_H(Q_0, \partial C) > 0 \), so \( Q \in \mathcal{R}_n \). Since \( U_n(x_0) = 0 \) and \( \nabla U_n(x_0) = 0 \), we have

\[
F(U_n(x_Q), x_Q, \nabla U_n(x_Q)) \leq F(U_n(x_0), x_0, \nabla U_n(x_0)) \leq -\varepsilon_1 \leq -\varepsilon_k,
\]

and therefore \( u_{n+1} = u_Q \neq 0 \) on \( Q \). Condition (iv) follows now from (b). Next, condition (v) also holds, although we still have to define the integer \( N_k \).

Finally, let us prove that (vi) is satisfied. Suppose, to the contrary, that for every \( n > N_{k-1} \) we have:

\[
\lambda_d \{ x \in C : F(U_n(x), x, \nabla U_n(x)) \leq -\varepsilon_k \} > 2^{-k} : \lambda_d(C).
\]

By (4.3), we obtain that

\[
\lambda_d \{ x \in K_n : F(U_n(x), x, \nabla U_n(x)) \leq -\varepsilon_k \} > 2^{-(k+1)} : \lambda_d(K_n).
\]

Suppose now that

\[
F(U_n(x_Q), x_Q, \nabla U_n(x_Q)) \leq -\varepsilon_k.
\]

As we have noticed, in this case

\[
\lambda_d \{ x \in Q : \| \nabla U_{n+1}(x) - \nabla U_n(x) \| \geq \delta_k \} \geq \lambda_d \{ x \in Q : \| \nabla u_{n+1}(x) \| = \delta_k \} \geq (1 - 2^{-k}) : \lambda_d(Q).
\]
Now the proportion of cubes $Q$ in $\hat{R}_n$ satisfying this has to be at least $2^{-(k+1)}$. Therefore

$$\lambda_d \{ x \in K_n : \| \nabla U_{n+1}(x) - \nabla U_n(x) \| \geq \delta_k \} \geq (1 - 2^{-k})2^{-(k+1)} > 0.$$ 

This will be a contradiction with Lemma 2.3, since we are going to prove that the sequence $(\nabla U_n)_{n \geq 1}$ is pointwise convergent. Indeed, for each $x \in \mathbb{R}^d$, it follows from (i) that the sequence $(\sum_{k=1}^{n} w_k(x))_{n \geq 1}$ converges, and from (iii) and Lemma 2.3 that the sequence $(\sigma_n(x))_{n \geq 1}$ converges. Since

$$\nabla U_n(x) = \sigma_n(x) + \sum_{k=1}^{n} w_k(x)$$

we have that the sequence $(\nabla U_n(x))_{n \geq 1}$ is convergent. This contradiction shows that there exists an integer $N_k > N_{k+1}$ satisfying (vi). This concludes the inductive step.

**The function $u_C$:** We now define

$$u_C = \sum_{n=1}^{\infty} u_n.$$

By (vi) the series is uniformly convergent on $\mathbb{R}^d$, so that $u : \mathbb{R}^d \to \mathbb{R}$ is a continuous function, and it is clear that $u_C$ vanishes outside $C$. In order to see that $u_C$ is differentiable on $\mathbb{R}^d$, we check the conditions of Lemma 2.1. For each $n \geq 1$ let $k_n$ be an integer with $N_{k_n - 1} < n \leq N_{k_n}$. From (i) and (v), we have that

- $\| \nabla u_n \|_\infty \leq \| v_n \|_\infty + \| w_n \|_\infty \leq \varepsilon_{k_n} + 2^{-n} \to 0,$

and from (iv), we obtain:

- $\| \nabla u_{n+1} \|_\infty = o(\| \nabla u_n \|_\infty),$

- $osc(\nabla U_n, \| \nabla u_n \|_\infty) \leq 2^{-n} \to 0.$

Moreover, applying as before Lemma 2.3, the sequence $(\nabla U_n)_{n \geq 1}$ is pointwise convergent, that is,

$$\sum_{n=1}^{\infty} \nabla u(x)$$
is convergent for every $x \in \mathbb{R}^d$. Applying Lemma 2.1, we obtain that $u_C$ is
everywhere differentiable and $\nabla u_C$ is the pointwise limit of
\[ \sum_{k=1}^{n} u_k. \]
Now (ii) implies that $F(u_C(x), x, \nabla u_C(x)) \leq 0$ for every $x \in \Omega$. Finally, let
us prove that $F(u_C(x), x, \nabla u_C(x)) = 0$
almost everywhere on $C$. Consider $x \in C$ such that $F(u_C(x), x, \nabla u_C(x)) < 0$.
Taking into account that $\nabla u_C(x) = \lim_k \nabla U_{N_k}(x)$, we can find some inte-
ger $k_0$ such that
\[ F(U_{N_k}(x), x, \nabla U_{N_k}(x)) < -\varepsilon_{k_0} \leq -\varepsilon_k, \]
for every $k \geq k_0$. Therefore the set $\{ x \in C : F(u_C(x), x, \nabla u_C(x)) < 0 \}$ is
contained in the set
\[ \limsup_k \{ x \in C : F(U_{N_k}(x), x, \nabla U_{N_k}(x)) < -\varepsilon_k \}. \]
Since
\[ \lambda_d \{ x \in C : F(U_{N_k}(x), x, \nabla U_{N_k}(x)) < -\varepsilon_k \} \leq 2^{-k} \lambda_d(C) \to 0, \]
by Borel-Cantelli lemma $\lambda_d(\{ x \in C : F(u_C(x), x, \nabla u_C(x)) < 0 \}) = 0$. That
is, $F(u_C(x), x, \nabla u_C(x)) = 0$ for almost every $x \in C$. ■

5. Almost classical solutions on Riemannian manifolds

In order to obtain our results for smooth manifolds, we will use the concept of
triangulation, as it is given by Whitney in [11] (see also [3]). In what follows
we assume that every smooth manifold is Hausdorff and second countable. If
$M$ is a smooth $d$-dimensional manifold, a triangulation of $M$ is a pair $(K, \pi)$,
where $K$ is a simplicial complex and $\pi : K \to M$ is a homeomorphism, such
that for each $d$-dimensional simplex $S$ of $K$ there exists a local chart $(W, \varphi)$
of $M$, where $W$ is a neighborhood of $\pi(S)$ and $\varphi \circ \pi$ is affine on $S$. According
to Whitney [11], every smooth manifold admits a triangulation.

Our first result is an extension of Theorem 3.1 to the setting of smooth
manifolds. The Riemannian structure is not needed here. We consider an
open subset $\Omega$ of the manifold $M$ and we denote $T^*\Omega$ the corresponding co-
tangent bundle. Then we consider equations of the form $F(u(x), x, du(x)) = 0$, where $F : \mathbb{R} \times T^*\Omega \to \mathbb{R}$ is a continuous function. We obtain the following:
Theorem 5.1. Let $M$ be a smooth manifold of dimension $d \geq 2$, consider an open subset $\Omega$ of $M$, and let $F : \mathbb{R} \times T^* \Omega \to \mathbb{R}$ be a continuous function. Suppose that the following conditions hold:

(A) There exists a $C^1$ function $u_0 : M \to \mathbb{R}$ such that $F(u_0(x), x, du_0(x)) \leq 0$, for every $x \in \Omega$.

(B) For each $x_0 \in \Omega$, there exist a compact neighborhood $V^{x_0}$ in $\Omega$ and $\alpha > 0$, such that the set $B(V^{x_0}; \alpha)$ is compact in $\mathbb{R} \times T^* M$, where

$$B(V^{x_0}; \alpha) = \{(u, x, \xi) \in \mathbb{R} \times T^* M : u \in [0, \alpha]; x \in V^{x_0}; F(u + u_0(x), x, \xi) \leq 0\}.$$

Then there exists a differentiable function $u : M \to \mathbb{R}$ such that:

1. $u \geq u_0$ on $M$, $u = u_0$ on $\Omega^c$ and $du = du_0$ on $\Omega^c$.

2. $F(u(x), x, du(x)) \leq 0$ for every $x \in \Omega$.

3. $F(u(x), x, du(x)) = 0$ for almost every $x \in \Omega$.

Proof. We will consider two cases.

First Case: Suppose first that $u_0 \equiv 0$ on $M$. Let $(K, \pi)$ be a triangulation of $M$, where $K$ is a simplicial complex and $\pi : K \to M$ is a homeomorphism, and consider the family $\{S_i\}_{i \in I}$ of all $d$-dimensional simplices of $K$. For each $i \in I$, denote $T_i = \pi(S_i)$. Then

$$M = \bigcup_{i \in I} T_i,$$

each $\partial T_i$ has measure zero in $M$, and $\text{int}(T_i) \cap \text{int}(T_j) = \emptyset$ if $i \neq j$. Since $M$ is locally compact and $\pi$ is a homeomorphism, we have that the simplicial complex $K$ is locally compact, and therefore locally finite. Thus the family $\{T_i\}_{i \in I}$ is locally finite. Since $M$ is also $\sigma$-compact, we obtain that the index set $I$ is countable. For each $i \in I$, denote $\Omega_i = \Omega \cap \text{int}(T_i)$. Then the set $\Omega \setminus (\bigcup_{i \in I} \Omega_i)$ has measure zero in $M$.

For each $i \in I$ there is a chart $(W_i, \varphi_i)$ in $M$ with $T_i \subset W_i$. Associated to this chart there is a natural diffeomorphism

$$\Phi_i : \mathbb{R} \times T^* W_i \to \mathbb{R} \times \varphi_i(W_i) \times \mathbb{R}^d$$

of the form $\Phi_i(u, x, \xi) = (u, \varphi_i(x), h_i(x, \xi))$, where $h_i(x, \xi) \in \mathbb{R}^d$ satisfies that, for every $p \in \mathbb{R}^d$:

$$\langle h_i(x, \xi), p \rangle = \xi \circ d\varphi_i(x)^{-1}(p).$$
If $\varphi_i(\Omega_i) \neq \emptyset$, consider the function $G_i = F \circ \Phi_i^{-1} : \mathbb{R} \times \varphi_i(W_i) \times \mathbb{R}^d \to \mathbb{R}$. In order to apply Theorem 3.1 to the function $G_i$ note that the following conditions hold:

(A) $G_i(0, z, 0) = F(0, \varphi_i^{-1}(z), 0) \leq 0$, for each $z \in \varphi_i(\Omega_i)$.

(B) For each compact subset $H$ of $\varphi_i(\Omega_i)$, there exists $\alpha_H > 0$ such that the set

$$B(H; \alpha_H) = \{(u, z, p) \in [0, \alpha_H] \times H \times \mathbb{R}^d : G_i(u, z, p) \leq 0\}$$

is compact in $\mathbb{R} \times \varphi_i(W_i) \times \mathbb{R}^d$.

Therefore, taking into account Remark 3.2, we obtain that there exists a differentiable function $v_i : \varphi_i(W_i) \to \mathbb{R}$ such that:

1. $v_i |_{\varphi_i(\Omega_i)} \equiv 0$ and $\nabla v_i |_{\varphi_i(\Omega_i)} \equiv 0$;
2. $G_i(v_i(z), z, \nabla v_i(z)) \leq 0$ for every $z \in \varphi_i(\Omega_i)$.
3. $G_i(v_i(z), z, \nabla v_i(z)) = 0$ for almost every $z \in \varphi_i(\Omega_i)$.

Then the function $u_i = v_i \circ \varphi_i : W_i \to \mathbb{R}$ is differentiable on $W_i$, and for each $x \in W_i$ we have that

$$F(u_i(x), x, du_i(x)) = F(u_i(x), x, dv_i(\varphi_i(x)) \circ d\varphi_i(x))$$
$$= F(\Phi_i^{-1}(v_i(\varphi_i(x)), \varphi_i(x), \nabla v_i(\varphi_i(x))))$$
$$= G_i(v_i(\varphi_i(x)), \varphi_i(x), \nabla v_i(\varphi_i(x)))$$

As a consequence, we obtain that

1. $u_i |_{\Omega_i} \equiv 0$ and $\nabla u_i |_{\Omega_i} \equiv 0$.
2. $F(u_i(x), x, du_i(x)) \leq 0$ for every $x \in \Omega_i$.
3. $F(u_i(x), x, du_i(x)) = 0$ for almost every $x \in \Omega_i$.

On the other hand, if $\varphi_i(\Omega_i) = \emptyset$, we set $u_i = 0$. Now we define $u : M \to \mathbb{R}$ by setting $u = u_i$ on each $T_i$. Then $u$ is well-defined, since $\partial T_i \subset \Omega_i^c$ for each $i \in I$. Taking into account that the family $\{T_i\}_{i \in I}$ is locally finite, we see that $u$ is differentiable on $M$, and it satisfies the required conditions.

**General Case:** In general, consider the continuous function $G : \mathbb{R} \times T^*\Omega \to \mathbb{R}$ defined by:

$$G(u, x, \eta) = F(u + u_0(x), x, \eta + du_0(x)).$$
We know that, or each \( x_0 \in \Omega \), there exist a compact neighborhood \( V^{x_0} \) in \( \Omega \) and \( \alpha > 0 \), such that

\[
B_F(V^{x_0}; \alpha) = \{(u, x, \xi) \in [0, \alpha] \times T^*\Omega : x \in V^{x_0}; F(u + u_0(x), x, \xi) \leq 0\}
\]

is compact in \( \mathbb{R} \times T^*\Omega \). Since the mapping \( \tau : \mathbb{R} \times T^*\Omega \rightarrow \mathbb{R} \times T^*\Omega \) given by

\[
\tau(u, x, \xi) = (u, x, \xi - du_0(x))
\]

is continuous, we have that the set

\[
B_G(V^{x_0}; \alpha) = \{(u, x, \eta) \in [0, \alpha] \times T^*\Omega : x \in V^{x_0}; G(u + u_0(x), x, \eta) \leq 0\} = \tau(B_F(V^{x_0}; \alpha))
\]

is compact in \( \mathbb{R} \times T^*\Omega \). Thus by the first case we obtain that there exists a differentiable function \( v : M \rightarrow \mathbb{R} \) such that:

1. \( v |_{\Omega^c} \equiv 0 \) and \( dv |_{\Omega^c} \equiv 0 \).
2. \( F(v(x), x, dv(x)) \leq 0 \) for every \( x \in \Omega \).
3. \( F(v(x), x, dv(x)) = 0 \) for almost every \( x \in \Omega \).

Now it is easy to see that the function \( u = u_0 + v \) satisfies the required properties.

Our next Corollary, which is analogous to Corollary 3.4, is an easy consequence of Theorem 5.1.

**Corollary 5.2.** Let \( M \) be a smooth manifold of dimension \( d \geq 2 \), consider an open subset \( \Omega \) of \( M \), and let \( F : T^*\Omega \rightarrow \mathbb{R} \) be a continuous function. Suppose that the following conditions hold:

(A) There exists a \( C^1 \) function \( u_0 : M \rightarrow \mathbb{R} \) such that \( u_0(x) + F(x, du_0(x)) \leq 0 \), for every \( x \in \Omega \).

(B) For each \( x_0 \in \Omega \), there exists a compact neighborhood \( V^{x_0} \) in \( \Omega \) such that the set \( B(V^{x_0}) = \{(x, \xi) \in T^*M : x \in V^{x_0}; u_0(x) + F(x, \xi) \leq 0\} \) is compact in \( T^*M \).

Then there exists a differentiable function \( u : M \rightarrow \mathbb{R} \) such that:

1. \( u \geq u_0 \) on \( M \), \( u = u_0 \) on \( \Omega^c \) and \( du = du_0 \) on \( \Omega^c \).
2. \( u(x) + F(x, du(x)) \leq 0 \) for every \( x \in \Omega \).
3. \( u(x) + F(x, du(x)) = 0 \) for almost every \( x \in \Omega \).
Also as a consequence of Theorem 5.1 we obtain the following result, which extends Theorem 1.3:

**Theorem 5.3.** Let $M$ be a smooth manifold of dimension $d \geq 2$, consider an open subset $\Omega$ of $M$, and let $F : T^*\Omega \to \mathbb{R}$ be a $C^1$-smooth function. Suppose that the following conditions hold:

(A) There exists a $C^1$-smooth function $u_0 : M \to \mathbb{R}$ such that $F(x, du_0(x)) \leq 0$, for every $x \in \Omega$.

(B) For each $x \in \Omega$, the set $B(x) = \{ \xi \in T^*_x M : F(x, \xi) \leq 0 \}$ is compact, the set $S(x) = \{ \xi \in T^*_x M : F(x, \xi) = 0 \}$ is connected, and the function $F(x, \cdot)$ has maximal rank on the set $S(x)$.

Then there exists a differentiable function $u : M \to \mathbb{R}$ such that:

1. $u \geq u_0$ on $M$, $u = u_0$ on $\Omega^c$ and $du = du_0$ on $\Omega^c$.
2. $F(x, du(x)) \leq 0$ for every $x \in \Omega$.
3. $F(x, du(x)) = 0$ for almost every $x \in \Omega$.

**Proof.** We are going to see that the conditions of Theorem 5.1 are satisfied. Fix $x_0 \in \Omega$, and consider a chart $(W, \varphi)$ in $M$ with $x_0 \in W$. Associated to this chart, consider as before the natural diffeomorphism

$$\Phi : T^*W \to \varphi(W) \times \mathbb{R}^d$$

of the form $\Phi(x, \xi) = (\varphi(x), h(x, \xi))$, where $h(x, \xi) \in \mathbb{R}^d$ satisfies that, for every $p \in \mathbb{R}^d$:

$$\langle h(x, \xi), p \rangle = \xi \circ d\varphi(x)^{-1}(p).$$

Denote $z_0 = \varphi(x_0)$. We take into account that $\Phi(S(x_0))$ is compact, and that $F \circ \Phi^{-1}$ has maximal rank on $\{z_0\} \times \Phi(S(x_0))$, and we apply the Implicit Function Theorem. Then we can find a neighborhood $U^{z_0}$ contained in $\varphi(W)$ and a finite family $V_1, \ldots, V_m$ of open subsets of $\mathbb{R}^d$ with compact closure such that $\Phi(S(x_0)) \subset V_1 \cup \cdots \cup V_m$ and, for each $j = 1, \ldots, m$, the set of points $(z, p) \in U^{z_0} \times V_j$ satisfying $F \circ \Phi^{-1}(z, p) = 0$ coincides, up to a permutation in the coordinates of $p$, with the graph of a $C^1$-smooth mapping $g_j : U^{z_0} \times W_j \to \mathbb{R}$, where $W_j$ is an open subset of $\mathbb{R}^{d-1}$.

We claim that there exists a compact neighborhood $V^{z_0}$ such that, for every $x \in V^{z_0}$, we have that $\Phi(S(x)) \subset V_1 \cup \cdots \cup V_m$. Indeed, if this is not the case, there exist a sequence $(z_n)_n \subset U^{z_0}$ converging to $z_0$ and a sequence $(p_n)_n \subset (V_1 \cup \cdots \cup V_m)^c$ such that $F \circ \Phi^{-1}(z_n, p_n) = 0$ for every $n$. Since each
$S(x_n)$ is connected and $\Phi(S(x_n)) \cap (V_1 \cup \cdots \cup V_m) \neq \emptyset$ for every $n$, we can assume that, in fact, $(p_n)_n \subset \partial(V_1 \cup \cdots \cup V_m)$, which is a compact set. Then, taking a subsequence, we can assume that $(p_n)_n$ is convergent to some point $p_0 \in \partial(V_1 \cup \cdots \cup V_m)$. Now $F \circ \Phi^{-1}(z_n, p_n) = \lim_n F \circ \Phi^{-1}(z_n, p_n) = 0$, that is, $p_0 \in \Phi(S(x_0))$, and this contradicts the fact that $\Phi(S(x_0)) \subset V_1 \cup \cdots \cup V_m$.

Then there exists $R > 0$ such that $\Phi(S(x)) \subset B(0, R)$ for every $x \in V^{x_0}$. Since $\Phi(S(x))$ is the boundary of $\Phi(B(x))$ we have that in fact $\Phi(B(x)) \subset B(0, R)$ for every $x \in V^{x_0}$. Thus the set

$$B(V^{x_0}) = \{(x, \xi) \in T^*\Omega: x \in V^{x_0}, F(x, \xi) \leq 0\}$$

is compact in $T^*\Omega$, and the requirements of Theorem 5.1 are satisfied.

In our next result we consider a Riemannian manifold $(M, g)$. As we mentioned before, if $u : M \to \mathbb{R}$ is differentiable, for every $x \in M$ we identify in the usual way the differential $du(x)$ with the gradient $\nabla u(x)$ by means of the scalar product $g_x(\cdot, \cdot)$ on the tangent space $T_xM$. In this case we obtain the following extension of Theorem 1.4:

**Theorem 5.4.** Let $(M, g)$ be a Riemannian manifold of dimension $d \geq 2$, consider an open subset $\Omega$ of $M$, and let $F : T\Omega \to \mathbb{R}$ be a continuous function. Suppose that the following conditions hold:

(A) There exists a $C^1$ function $u_0 : M \to \mathbb{R}$, such that $F(x, \nabla u_0(x)) \leq 0$, for every $x \in \Omega$.

(B) There exists a locally bounded function $\rho : \Omega \to (0, \infty)$ such that, for every $x \in \Omega$, the set $B(x) = \{v \in T_xM : F(x, v) \leq 0\}$ is contained in the ball of center $0$ and radius $\rho(x)$ in $T_xM$.

Then there exists a differentiable function $u : M \to \mathbb{R}$ such that:

1. $u \geq u_0$ on $M$, $u = u_0$ on $\Omega^c$ and $\nabla u = \nabla u_0$ on $\Omega^c$.
2. $F(x, \nabla u(x)) \leq 0$ for every $x \in \Omega$.
3. $F(x, \nabla u(x)) = 0$ for almost every $x \in \Omega$.

**Proof.** We are going to see that the conditions of Theorem 5.1 are satisfied. Fix $x_0 \in \Omega$ and consider a chart $(W, \varphi)$ in $M$ with $x_0 \in W$. Associated to this chart, consider the natural diffeomorphism

$$\Phi : TW \to \varphi(W) \times \mathbb{R}^d$$

given by $\Phi(x, v) = (\varphi(x), d\varphi(x)(v))$. Choose a compact neighborhood $V^{x_0}$ of $x_0$ contained in $W$ and $R > 0$ such that for every $x \in V^{x_0}$ the set
Almost classical solutions of Hamilton-Jacobi equations

\[ B(x) = \{ v \in T_x M : F(x, v) \leq 0 \} \] is contained in the closed ball of center 0 and radius \( R \) in \( T_x M \). Now set
\[
r = \inf \| v \|_x : x \in V^{x_0}, \| d\varphi(x)(v) \|_{\mathbb{R}^d} = 1 \}.
\]
By compactness, it is clear that \( r > 0 \). For every \( x \in V^{x_0} \) and every \( v \in T_x M \), we have
\[
\| d\varphi(x)(v) \|_{\mathbb{R}^d} \leq \frac{1}{r} \| v \|_x.
\]
Therefore
\[
B(V^{x_0}) = \{ (x, v) \in TM : x \in V^{x_0}; F(x, v) \leq 0 \}
\subset \{ (x, v) \in TM : x \in V^{x_0}; \| v \|_x \leq R \} \subset \Phi^{-1} \left( \varphi(V^{x_0}) \times B\left(0; \frac{R}{r}\right) \right).
\]
It follows that \( B(V^{x_0}) \) is a compact subset of \( TM \).

**Corollary 5.5.** Let \( (M, g) \) be a Riemannian manifold of dimension \( \geq 2 \) and let \( \Omega \) be an open subset of \( M \). Then there exists a differentiable function \( u : M \rightarrow \mathbb{R} \) such that \( u|_\Omega \equiv 0 \) and \( \| \nabla u(x) \|_x = 1 \) for almost every \( x \in \Omega \).

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**References**


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