Periodic Quasiregular Mappings of Finite Order

David Drasin and Swati Sastry

Abstract

The authors construct a periodic quasiregular function of any finite order \(\rho\), \(1 \leq \rho < \infty\). This completes earlier work of O. Martio and U. Srebro.

1. Introduction

Let \(f\) be a (sense-preserving) quasiregular map on \(\mathbb{R}^m (m \geq 2)\). Thus \(f\) is \(ACL^m\) and there is a \(K < \infty\) with

\[ |f'(x)|^m \leq K J_f(x) \quad \text{a.e.,} \]

where the left side is the norm of the induced operator on the tangent space at \(x\), and the right side is the Jacobian determinant. The now-standard reference is Rickman’s monograph [4]. These mappings carry much of the geometric theory of analytic and meromorphic functions to higher dimensions. Suppose in addition that \(f\) is entire. We then set

\[ M(r, f) = \max_{|x| \leq r} |f(x)|, \]

and define the order \(\rho\) of \(f\) by

\[ \rho = \limsup_{r \to \infty} \frac{\log \log M(r, f)}{\log r}. \]

Perhaps the most important function in the theory is V. Zoric’s analogue of the exponential function, \(Z(x)\) (cf. [4, p.15]). It is not a local homeomorphism, has order one, and is periodic in \(m - 1\) of the variables. Using the Zoric function, O. Martio and U. Srebro [3] observed that there exist \((m - 1)\)-periodic mappings of order 1 and \(\infty\), and (Theorem 8.7) that 1 is a lower bound for the orders of such functions.

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They raise a question [3, p. 38] which is answered by our

**Theorem 1.1** Let \( \rho, 1 \leq \rho \leq \infty \) be given. Then there exists an \((m - 1)\)-periodic \( K(m) \)-quasiregular map \( g \) of exact order \( \rho \).

In view of [3], this theorem has significance only when \( \rho \in (1, \infty) \). The main step in our construction is Theorem 2.1, in which we associate an entire \( K \)-qr map \( f \) to any of a class of slowly increasing functions \( \nu(r) \) which satisfy (2.2) below; \( K \) will be independent of the specific choice of \( \nu \) and depend only on the dimension \( m \). For example, let \( \nu(r) = \rho (\log r)^{\rho - 1} \) for any fixed \( \rho > 1 \).

Not only will we have \( \log M(r, f) \sim (\log r)^\rho \), but for most large \( x \),

\[
\log |f(x)| \sim (\log |x|)^\rho,
\]

where the symbol \( \sim \) means that the ratio of the two sides is bounded above and below by positive constants. From this it is routine to see that

\[
g(x) = f \circ Z(x)
\]

is entire, \((m - 1)\)-periodic, \( K_1 \)-qr and of exact order \( \rho \). In the special case \( m = 2 \) and \( K = 1 \) (analytic functions), the functions of Theorem 2.1 exhaust the class of entire functions of very slow completely regular growth. These functions are discussed, for example, in [1, §6.7].

In [3, p. 38] Martio and Srebro raise another question, for which Theorem 1.1 yields a negative answer. So long as \( \rho > 1 \), the function \( f \) will have infinitely many zeros in \( \mathbb{R}^m \). Then (1.3) guarantees that \( g \) also has infinitely many zeros in each fundamental region \( \Omega \) of the function \( Z \) in \( \mathbb{R}^m \). Martio and Srebro had asked if \( \rho \) must always be infinite whenever \( g \) is quasiregular, \((m - 1)\)-periodic and some equation \( g(x) = a \) has infinitely many solutions in a fundamental region. They show in Theorem 8.7 that when \( \rho = 1 \) each \( a \in \mathbb{R}^m \) has only finitely many preimages in each \( \Omega \). Our Theorem 1.1 implies that their theorem is sharp: when \( f \) is chosen as in (1.2) and (1.3), then \( g \) assumes all values infinitely often in each \( \Omega \).

### 2. A generalization of the power mapping

**Theorem 2.1** Let \( \nu(r) \) be a positive increasing function such that \( \nu \to \infty \),

\[
rv'(r) < \frac{\nu(r)}{2}, \quad rv'(r) = o(\nu(r)) \quad (r \to \infty),
\]

and set

\[
A(r) = \exp \int_{1}^{r} \nu(t)t^{-1}dt.
\]
Then there exists an entire \( K = K(m) - qr \) map \( f \) on \( \mathbb{R}^m \) with
\[
M(r, f) \sim A(r) \quad (r \to \infty).
\]
Moreover, on \( S(r) = \{ x; |x| = r \} \), we have \((h_{m-1} \text{ is } (m-1)-\text{Hausdorff measure})\)
\[
|f(x)| > (1 + o(1))A(r) \quad (|x| \to \infty, \ x \in S(r) \setminus E(r)),
\]
where \( h_{m-1}(E(r)) = o(r^{m-1}) = o(h_{m-1}(S(r))) \).

When \( \nu(r) \equiv n \in \mathbb{Z}^+ \), the construction is a more complicated version of the power mapping as described in [4, Ch.1, §3.2]. The theorem can be reformulated to allow \( \nu \) to tend to a finite limit, but since \( \nu \to \infty \) in cases of interest, we impose this additional hypothesis.

The map \( f \) depends on a sequence \( \{ r_n \} \) with
\[
\nu(r_n) = n,
\]
and will be defined on the boundary of each \( m \)-cube \( Q_r \),
\[
Q_r = \{ x; \| x \|_\infty \leq r \}.
\]
Every \( \partial Q_r \) has \( 2m \) faces \( \{ F_j \} \), on each of which \( x_j \equiv \pm r \) for some \( 1 \leq j \leq m \).

Note from (2.2) and (2.5) that
\[
n \log \frac{r_{n+1}}{r_n} \to \infty,
\]
since \( 1 = \int_{r_n}^{r_{n+1}} t \nu'(t) dt / t = o(1) n \log(r_{n+1}/r_n) \). We choose \( \varepsilon_0 = \varepsilon_0(m) \) with
\[
0 < \varepsilon_0 < \frac{1}{2}, \quad \sin^{-1} \varepsilon_0 < \frac{1}{2} \sin^{-1} m^{-1/2}.
\]
Then (2.6) yields \( r_0 \) and \( n_0 = n_0(\varepsilon_0, \nu) \geq 4 \) so that
\[
(m + 1) r \nu'(r) / \nu(r) \leq \varepsilon_0 \quad (r > r_0), \quad \nu(r_0) = n_0 \in \mathbb{Z},
\]
\[
n \log \frac{r_{n+1}}{r_n} > (m + 1) \varepsilon_0^{-1} \quad (n \geq n_0).
\]
In this and the next two sections we construct \( f \) on \( \cup \partial Q_r \ (r \geq r_0) \), leaving the simpler range \( 0 \leq r \leq r_0 \) to §5.

With the \( \{ r_n \} \) as in (2.5), let \( J_n \ (n \geq n_0) = [r_n, r_{n+1}] \). We partition \( J_n \) into \( m + 1 \) intervals \( J_{n, \ell} = [r'_{n, \ell}, r''_{n, \ell}] \ (0 \leq \ell \leq m) \), subject to \( r'_{n,0} = r_n, r''_{n,m} = r''_{n,m+1} \); (2.9) shows that we may suppose
\[
\varepsilon_0 \log \left( \frac{r''_{n,\ell}}{r'_{n,\ell}} \right) = \log \left( \frac{n + 1}{n} \right), \quad (1 \leq \ell \leq m, \ n \geq n_0).
\]
Thus for each $1 \leq \ell \leq m$, $r_{n,\ell}^n = (1 + o(1))r_{n,\ell}^n (n \to \infty)$, while $r_{n,1}/r_n \to \infty$. Since $n \geq n_0$ is usually fixed in §2–4, we often ignore it in our notations.

In §3 we construct $f$ on

$$\bigcup_{n \geq n_0} \bigcup_{r \in J_n^0} Q_r,$$

where we set $J^0 = J_n^0 = [r_{n,0}, r_{n,0}'] \equiv [r_0', r_0''] n \geq n_0$. The situation is simpler here since the combinatorics on each $\partial Q_r$ does not change with $r$, while in §4 we modify this approach on the $\{J_n^k\}, n \geq n_0, k \geq 1$.

The map $f$ has to evolve in $J = J_n$ subject to:

(A) on $\partial Q_{r_n}$ $f$ is (a constant multiple of) a power-type map of ‘degree’ $n$ (cf. [4, p. 14]). Thus each of the $2m$ facets of $\partial Q_{r_n}$ is first divided into $(2n)^{m-1}$ congruent $(m-1)$-‘boxes’ $K$, where a box is the product of $m$ closed intervals: $K = I_1 \times \ldots \times I_m$, with one $I_i = \{+r\}$ or $\{-r\}$ and $|I_i| = r/n$ when $i \neq j$. With $S_{m-1} = 2^{m-1}(m-1)!$ as determined below (3.1), we then divide each $K$ into $S_{m-1} (m-1)$-simplices $\Lambda_r$. The map $f$ is defined on each $\Lambda_r$ by (3.6), so that $f$ is $K$-qc on $\Lambda_r, K$–qr on $Q_r$, with $|f(x)| \sim A(r_n)$ for $x \in \partial Q_r$;

(B) situation (A) holds on $\partial Q_{r_{n+1}}$, with $n + 1$ in place of $n$;

(C) the process is such that $f$ is $K$–qr and $|f(x)| \sim A(|x|)$ for most $x$ on every $\partial Q_r, r \geq r_0$.

We conclude this section with a PL version of the sphere $S^m$. While Rickman’s map is based on the manifold $S^m$ being in the range (and is a so-called Alexander map) our construction in §4 seems to require the polyhedron $P$ of Proposition 2.12. Let $S' = \{|x'| = 1\} \cap \{x_m = 0\}$ be the unit $(m-2)$–sphere. Depending on the context, we may view $\alpha \in S'$ as a vector in $\mathbb{R}^{m-1}$ or one in $\mathbb{R}^m$ whose final coordinate is zero. Choose $m$ points $\alpha^0, \ldots, \alpha^{m-1} \in S'$ so that the vectors $\alpha^j - \alpha^0 (1 \leq j \leq m-1)$ form a basis of $\mathbb{R}^{m-1}$ which is $L(m)$–bilipschitz equivalent to the standard basis, the origin is in the convex hull of the $\{\alpha^i\}$, and the map $(\alpha^j - \alpha^0) \to e^j$ is sense-preserving; the $\{e^j\}$ are the standard basis of $\mathbb{R}^{m-1}$. Let $\Delta$ be the convex hull of the $\{\alpha^i\}$, and $s\Delta = \{sp; p \in \Delta\}$. For $s > 0$ and $q = s \sum \lambda_i \alpha^i \in \Delta_s$, consider the function

$$\lambda(q) = \lambda_s(q) = m s \inf_i \lambda_i \quad (q \in \Delta_s).$$

(The factor $m$ ensures that $\max_{\Delta_s} \lambda(q) = s$.)
Proposition 2.12 For each $s > 0$, the graph of the function $\lambda_s(q), q \in \Delta_s$, is a polyhedron $P^+ = P_s^+ \subset \{x_m \geq 0\}$. If we define $P^-$ as the graph of $-\lambda_s(q)$, then

$$P = P^+ \cup P^-$$

is a polyhedron composed of subsets of a finite number of hyperplanes with 0 in its interior. If $q \in \partial \Delta_s$, then $\lambda(q) = 0$.

The ray from 0 to the point $(q, \pm \lambda(q)) \in P$ makes an angle $\Phi$ with $P$ such that

$$(2.13) \quad |\sin \Phi| > 3\tau > 0,$$

where $\tau$ depends only on the specific choice of the $\{\alpha^i\}$.

Proof. It suffices to consider $s = 1$. Then $P$ determined by $2m$ hyperplanes each of which contains $m - 1$ of the $\{\alpha^i\}$ and one of the points $(\alpha, \pm 1)$, where $\alpha = \sum \alpha^i/m$ is the barycenter of $\Delta$, so it is clear that 0 is interior to $P$. The normal to each of these hyperplanes has a nonzero component orthogonal to the hyperplane $\{x_m = 0\}$, so the result follows by elementary linear algebra. ■

3. The first stage

Recall the $\{J_n\} = \{\cup_{0 \leq \ell \leq m} J^\ell_n\}, n \geq n_0$, from the discussion of (2.10). Let $r \in J^0_n$, and consider a face $F \subset \partial Q_r$ on which $x_j = \epsilon r$, for $\epsilon = \pm 1$. Then for $1 \leq i \leq n, i \neq j$, the planes

$$(3.1) \quad \Pi_p(n) = \{x_i = pr/n\}, \quad |p| \leq n,$$

divide $F$ into $(2n)^{m-1}(m-1)$-boxes $K$, and barycentric subdivision of each box in turn partitions $F$ into a union of $(m-1)$-simplices $\Lambda_r$, which are positively or negatively oriented with respect to the standard orientation $\partial Q_r$ inherits from $\mathbb{R}^m$. As $r \in \cup_{n \geq n_0} J^0_n$ and $1 \leq j \leq m$ vary, note that each vertex $b(r)$ of $\Lambda_r$ may be associated to a vector $p \in \mathbb{Z}^m$:

$$(3.2) \quad b(r) = \left(\frac{p_1}{2n}, \frac{p_2}{2n}, \ldots, \frac{p_m}{2n}\right)r,$$

with $|p_i| \leq 2n$; on $F$, $p_j = 2\epsilon n$. Each $\Lambda_r$ is $L$-bilipschitz equivalent to the standard $(m-1)$-simplex, up to the scaling factor (cf. (2.3))

$$\frac{r}{\nu(r)} = A(r) \frac{A(r)}{A'(r)},$$
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with \( L = L(m) \). Thus

\[
L^{-1} \frac{r}{\nu(r)} \leq |b^i(r) - b^j(r)| \leq L \frac{r}{\nu(r)} \quad (i \neq j).
\]

The vertices of \( \cup_{\partial Q_r} \Lambda_r \) are put into \( m \) classes \( b^i \), \( 0 \leq i \leq m - 1 \), using the standard model \( \Delta \) of Proposition 2.12. On some face \( F \subset \partial Q_r \) choose a positively oriented simplex \( \Lambda_0^r \), and label its vertices \( b^i(r) \), \( 0 \leq i \leq m - 1 \), the ordering taken so that the map

\[
\sum \lambda_i b^i(r) \rightarrow \sum \lambda_i \alpha^i \quad (\lambda_1 \geq 0, \sum \lambda_i = 1)
\]

from \( \Lambda_0^r \) to \( \Delta \) has positive Jacobian. We may then consistently assign classes \( b^i \) to any of the vertices of all \( \Lambda_r \subset \partial Q_r \), so that if \( \Lambda_r \) and \( \Lambda_r' \) share a lower dimensional subsimplex, the vertices common to both simplexes belong to the same class. Note that the mapping (3.4) when defined on each simplex \( \Lambda_r \) is sense preserving if \( \Lambda_r \) is positively oriented, and sense reversing otherwise.

With \( s = A(r) \) (\( r \in J_n^0 \)) from (2.3), let \( p = \sum \lambda_i b^i(r) \in \Lambda_r \subset \partial Q_r \), set

\[
p' = s(\sum \lambda_i \alpha^i) \quad (s = A(r)),
\]

and, recalling the function \( \lambda(p') \) of (2.11), define

\[
f(p) = (p', \pm \lambda(p')) = \left( s \sum \lambda_i \alpha^i, \pm \lambda(p') \right) \quad (s = A(r)).
\]

The first entry on the right side of (3.6) is an \((m-1)\)-vector, and the second is a scalar, and the \( \pm \) sign is taken according to whether (3.4) preserves or reverses orientation. Thus (3.6) is always sense preserving.

**Lemma 3.7** Let \( B : e^1, \ldots, e^m \) be the standard basis of \( \mathbb{R}^m \). Then there is a \( K_1 < \infty \) such that at almost each point \( p \) and \( f(p) \) exist bases \( V = \{ v^i \} \) and \( W = \{ w^i \} \) of the tangent spaces \( T_p \) and \( T_{f(p)} \) such that the linear maps determined by

\[
e^i \leftrightarrow v^i, \quad e^i \leftrightarrow w^i
\]

are \( K_1 \)-quasiconformal. Moreover, if \( J_f \) is the Jacobian matrix relative to the bases \( V \) and \( W \), then

\[
J_f = A'(r)I.
\]

Hence, if \( K_2 \) is the dilatation of the map (3.4), then \( f \) is \( K = K_1^2 K_2 \)-quasiregular.
Hence (2.13) ensures that \( w \) is positively oriented with respect to \( r \). Situation to \( \{ \nu \} \) implies \( |w| \leq \{ \nu \} \).

First consider \( p \) (the vector from 0 to) \( r \). There is a sign in (3.6), and \( \lambda_k = \min_i \lambda_i \) in a neighborhood of \( p \). The basis for \( T_p \) consists of \( \mathcal{V} = \{ v^1, \ldots, v^m \} \) such that \( v^m = \sum \lambda_i (b^i)(r) \), and for \( 1 \leq t \leq m - 1 \), the \( \{ v^t \} \) are the vectors \( (v(r)/r)(b^{\sigma(t)} - b^k) \), where the \( \{ \sigma(t) \}_{i=1}^{m-1} \) exhaust the range \( 1 \leq t \leq m \), \( \sigma \neq k \), ordered so that \( V \) is positively oriented with respect to \( B \). At \( f(p) = (p', \lambda(p)) \) the basis of \( T_{f(p)} \) will be normalized \( Df \)-images of \( \mathcal{V} \), so that when \( t < m \), \( w^t = (\alpha^{t+1} - \alpha^k, -m) \). When \( r \in J^n \) (\( n \geq n_0 \)) the final basis vector \( w^m \) in \( \mathcal{V} \) is \( w^m = (\sum \lambda_i \alpha^i, m \lambda_k) \), but this will be modified in Lemma 4.7 for the situation \( r \in \bigcup_{i \geq 1} J^n_i \), \( n \geq n_0 \).

Since \( \lambda(p') \) is also determined by the coefficient \( \lambda_k \) of \( b^k \) for \( p' \) near \( p \), (3.6) shows that \( f \) is linear near \( p \). Hence if \( t < m \) and \( h \) is small,

\[
 p + hv^t = b^k + \sum_{i \neq \sigma(t), k} \lambda_i b^i + (\lambda_{\sigma(t)} + h(\nu(r)/r))(b^{\sigma(t)} - b^k),
\]
and (2.3), (2.11), (3.5) and (3.6) yield for \( 1 \leq t \leq m - 1 \) that

\[
 (3.8) \quad Df(v^t) = \frac{f(p + hv^t) - f(p)}{h} = \frac{\nu(r)}{r} A(r)(\alpha^{t+1} - \alpha^k, -m) \equiv A'(r)w^t.
\]

Next, consider \( Df(v^m) \). Let \( r' = r + h \) and consider the image of \( p + hv^m = \sum \lambda_i (b^i + h(b^i)) \). By (3.1),

\[
 p + hv^m = \sum \lambda_i (b^i(r) + h(b^i)'(r)) = \sum \lambda_i b^i(r') \quad (r' = r + h),
\]
so that \( f(p + hv^m) - f(p) = (A(r') - A(r))(\sum \lambda_i \alpha^i, m \lambda_k) \), and

\[
 (3.9) \quad Df(v^m) = A(r')w^m.
\]

We check that the bases \( \mathcal{V} \) and \( \mathcal{W} \) satisfy the assertions of Lemma 3.7. First consider \( p \in \Lambda_r \). The explicit form of the simplices \( \Lambda_r \) and the arrangement of the \( \{ \sigma(t) \} \) show that the first \( m - 1 \) vectors \( v^t \) form part of such a basis at \( T_p \) and lie parallel to that face \( F \) of \( \partial Q_r \) which contains \( p \), while (3.3) implies \( |v^t| \sim 1 \). In addition, we deduce from (3.1) that \( |v^m| \sim 1 \), and that (the vector from 0 to) \( p \) makes an angle \( \Theta \) with \( F \) such that \( |\sin \Theta| > m^{-1/2} \), so \( \Theta \) is uniformly bounded away from 0. Thus \( \mathcal{V} \) is related to \( \mathcal{B} \) as claimed in the Lemma.

Now consider \( \mathcal{W} \). That \( |w^i| = |(\alpha^i - \alpha^k, -m)| \sim 1 \) for \( i < m \) follows from properties of the \( \{ \alpha^i \} \). In addition, we have that \( |w^m| = |(\sum \lambda_i \alpha^i, m \lambda_k)| \sim 1 \). This follows from (2.11) and (3.6) when \( \lambda_k = \min \lambda_i > \eta > 0 \), but when \( \lambda_k \) is small, then \( \sum \lambda_i \alpha^i \) lies near \( \partial \Delta \), and so \( \sum \lambda_i \) already has magnitude at least \( h \) for some fixed \( h > 0 \). To check that the \( \{ w^j \} \) span \( \mathbb{R}^m \) appropriately, note that the \( \{ w^j \} (j < m) \) span the tangent plane at \( f(p) \in A(r)P \). Hence (2.13) ensures that \( w^m \) has a uniformly nontrivial normal component to \( A(r)P \) at \( f(p) \).
4. Interpolation

In order to define \( f \) on \( \partial Q_r \) for \( r \in J^k_n, n \geq n_0 \) we follow the scheme of §3, but need to arrange new simplices (or partial simplices) so that (B) in §2 holds when \( r = r_{n+1} \). We do this by working with the \((m-1)\) free coordinates on a given face \( F \) one at a time, and when \( r \in J^k_n \), this will be \( x_\ell \).

Consider, for example, the face \( F \subset \partial Q_r \) on which \( x_j \equiv r \). For each \( 1 \leq i \leq m, i \neq j, F \) again is partitioned by \((m-1)\) -planes orthogonal to the \( x_\ell \)-axis. This has already been described when \( r \in J^0_n \), so consider a fixed \( \ell \geq 1 \). Then for each \( i < \ell, i \neq j \), the planes

\[
(4.1) \quad \Pi^i_p(n + 1) = \{ x_i = p r/(n + 1) \}, \quad |p| \leq n + 1
\]

divide \( F \) into \( 2(n + 1) \) congruent slices, and when \( i > \ell, i \neq j \), the \( \{ \Pi^i_p(n) \} \), \( |p| \leq n \) of (3.1) divide \( F \) into \( 2n \) congruent slices.

We next consider \( i = \ell \), and recall \( \varepsilon_0 \) in (2.7) and that \( J^\ell_n = [r_\ell', r_\ell'' \ell] \). Then use (2.10) to define \( \nu_\ell(r) \) with

\[
(4.2) \quad \frac{d(\log \nu_\ell(r))}{d(\log r)} = \frac{r \nu_\ell'(r)}{\nu_\ell(r)} = \frac{1}{\log(r''_\ell/r'_\ell)} \equiv \varepsilon_0 \quad (r'_\ell \leq r \leq r''_\ell),
\]

and partition \( F \) by planes \( \Pi^\ell_p(\nu_\ell) \equiv \{ x_\ell = p r/\nu_\ell(r), p \in \mathbb{Z}, 0 \leq |p| \leq n \} \). As \( r \) increases in \( J^\ell_n \), each \( \Pi^\ell_p(\nu_\ell) \) recedes from \( \{ x_\ell = \pm r \} \) and so for the appropriate choice of \( n^* \in \{ n, n + 1 \} \), the \( \{ \Pi^\ell_p(n^*) \} \) \( i \neq j, \ell \), and \( |p| \leq n^* \), \( \{ \Pi^\ell_p(\nu_\ell) \} \) and \( \{ x_\ell = \pm r \} \) create new boxes \( K \subset F \), which when \( r = r''_\ell \) are all congruent. Boxes whose boundary is disjoint from \( \{ x_\ell = \pm r \} \) are called interior boxes, and the others are boundary boxes.

As in §3, these boxes must be divided into simplices, and \( f \) defined simplex by simplex. If \( K_0 \) is an interior box, its barycentric subdivision leads at once to oriented simplexes \( \Lambda_\ell \) as in §3, with vertices \( b(r) \) having coordinates \( b_i(r) \), such that for \( i \neq j, i < \ell \), we have \( b_i = (2p_i r)/(2(n + 1)) \) \((|p_i| \leq n + 1)\), while \( b_\ell = (2p_\ell r)/(2\nu_\ell(r)) \) \((|p_\ell| \leq n)\) and \( b_i = (2p_i r)/(2n) \), \(|p_i| \leq n \) when \( i > \ell, i \neq j \). On \( F \) we have \( b_j \equiv r \). This again allows the simplex structure and orientation to be transferred to the interior boxes. The only new feature is that the coordinate \( b_\ell \) of each vertex satisfies

\[
(4.3) \quad r b'_\ell = b_\ell \left( 1 - \frac{r \nu_\ell'}{\nu_\ell} \right) \equiv b_\ell (1 - \varepsilon_0),
\]

instead of what appears in (3.2). Since \( n \leq \nu_\ell(r) \leq n + 1 \), these simplexes \( \Lambda_\ell \) are \((1 + o(1))-\text{bilipschitz} \) equivalent to those \( \Lambda_\ell \) for \( r \in J^\ell_n \), and so the mappings (3.4) are uniformly \((1 + o(1))K_2-\text{qc} \) (perhaps sense reversing).
We next consider the boundary boxes, and partition them into what we call partial simplices \( \Lambda^*_r \). It suffices to work in \( \{ x_\ell \geq 0 \} \cap Q_r \). The \( x_\ell \)-coordinates \( (i \neq \ell) \) of these boxes are the same as those corresponding to vertices of interior boxes, while the \( x_\ell \)-coordinate, \( b_\ell \), is either \( (n/\nu_\ell(r))r \) or \( r \). Let

\[
r^* = \frac{1}{2} \left( 1 + \frac{n}{\nu_\ell(r)} \right) r = \left( \frac{n + \nu_\ell(r)}{2\nu_\ell(r)} \right) r,
\]

and \( H : \{ x_\ell = r^* \} \). Then \( H \) lies midway between \( \Pi^\ell_n(\nu_\ell) \) and \( \{ x_\ell = r \} \), and each boundary box \( K \) is divided by \( H \) into two congruent subboxes \( K_\pm \). Let \( K_- = K \cap \{(nr/\nu_\ell) \leq x_\ell \leq r^* \} \) and \( K_+ \), the reflection of \( K_- \) in \( H \). In an obvious sense \( K_- \) may be considered as a subset of a (phantom) box \( K' \), which is bounded by the hyperplanes \( \Pi^\ell_n(\nu_\ell) \) and \( \Pi^\ell_{n+1}(\nu_\ell) \equiv \{ x_\ell = r(n + 1)/\nu_\ell(r) \} \), as well as the various hyperplanes \( \Pi^\ell_p(n^*) \ (i \neq j, \ell, \ n^* \in \{ n, n + 1 \} \) which meet \( \partial K \). In particular, \( K_- \) may be divided into oriented simplices \( \Lambda^*_r \) generated by vertices in the classes \( b^i(r) \) exactly as with the interior boxes \( K \). The vertices \( \Lambda^*_r \) of \( K_- \) are of the form \( \Lambda^*_r = \Lambda_r \cap K' \), with inherited orientation. In the same way, we obtain simplices \( (\Lambda^*_r)^* \subset K_+ \); these are reflections of the \( \{ \Lambda^*_r \} \) across \( H \).

We place \( \Lambda^*_r \subset K' \) in groups according to how many vertices \( \Lambda_r \supset \Lambda^*_r \) does not have on \( \Pi^\ell_n(\nu_\ell) \). This number, \( t(\Lambda^*_r) \), is at least 1 and at most \( m - 1 \). If \( (\Lambda^*_r)^* \subset K_+ \) is the reflection of \( \Lambda^*_r \) across \( H \), set \( t(\Lambda^*_r)^* = t(\Lambda^*_r) \), and note that the vertices of \( \Lambda_r \) and \( \Lambda^*_r \) which contribute to the appropriate \( t \) are of the same classes \( \{ b^i \} \), while orientations of the simplices are reversed. Let \( T = T(\Lambda^*_r) \) be the vertices of \( \Lambda_r \) which contribute to \( t(\Lambda^*_r)^* \): we call these the phantom vertices.

The mapping \( f \) of (3.7) must be modified so that

\[
\text{f is L–bilipschitz and K-qc in each } \Lambda^*_r,
\]

\[
(f(x))_m \geq 0 \text{ on } \Lambda^*_r, \quad (f(x))_m = 0 \text{ on } \partial \Lambda^*_r,
\]

where \((\cdot)_m \) is the \( m \)-th coordinate. The important requirement is that \((f(x))_m \) vanish in \( \partial \Lambda^*_r \); otherwise reflection across the boundary (compare with (3.6)) will not be possible. Note that (3.6) cannot be used, since \((f(x))_m \) is usually nonzero when \( x \in K_+ \cap K_- = H \cap K \). To avoid this we use \( T \) to modify the function \( \lambda \) of (2.11). According to the definition of \( t(\Lambda) \), if \( p = \sum \lambda_i b^i(r) \subset \Lambda^*_r \), then

\[
0 \leq \sum_{i \in \mathcal{T}} \lambda_i \leq L(r) \equiv \frac{\nu_\ell(r) - n}{2},
\]

where the left equality holds when \( p \in \Pi^\ell_n(\nu_\ell) \) and the right when \( p \in H \).
Thus if \( K_s \) is the image of \( \Lambda_r^* \cap H \), we have
\[
p' = s \sum \lambda_i \alpha^i \in K_s \iff \sum \lambda_i = \frac{\nu_r(r) - n}{2} = L(r).
\]
Now with \( p' \) and \( \lambda(p') \) as in (3.5) and (2.11), we define \( \lambda^*_s \) to have the same effect relative to \( \Lambda_r^* \): if
\[
p' = s \left( \sum \lambda_i \alpha^i \right) \in \Delta_A(r)
\]
and \( L \) is from (4.4), set
\[
\lambda^*(p') = s \min \left( \lambda(p'), (L(r) - \sum \lambda_i) \right),
\]
so that now \( \lambda^* \equiv 0 \) on \( K_A(r) \). Then when \( r \in J_{n}^\ell \) and \( p \in \Lambda_r^* \) (1 \( \leq \ell \leq m \)), we modify (3.6) to
\[
f(p) = (p', \pm \lambda^*(p')) = (s \sum \lambda_i \alpha^i, \pm \lambda^*(p')) \quad (s = A(r)),
\]
signs chosen so that \( f \) is sense preserving. If \( p \in \partial \Lambda^*_r \) and \( L(r) - \sum \lambda_i = 0 \), then \( p \in H \), and the extension to the symmetric \( (\Lambda'_r)^* \) is by reflection across \( H \) and \( K \).

**Lemma 4.7** Let \( p \in \partial Q_r \), \( r \in J_{n}^\ell \), \( \ell \geq 1, n \geq n_0 \). Then at almost every point \( p \) there are bases \( V \) and \( W \) of \( T_p \) and \( T_{f(p)} \) so that Lemma 3.7 holds.

**Proof.** Let \( p \) and \( p' = f(p) \) be as in Lemma 3.7, with \( \lambda_k \) the minimum \( \lambda \) near \( p \). Take \( V \) and \( \{w^1, \ldots, w^{m-1}\} \) exactly as in Lemma 3.7, but with the final basis vector, \( w^m \), replaced by a certain \( \hat{w}^m \). The first \( (m-1) \) components of \( \hat{w}^m \) are those of \( w^m \), but \( (\hat{w}^m)_m \) is modified to the bracketed term in (4.9) below (so that the factor \( A'(r) \) in (4.9) does not appear in \( \hat{w}^m \)).

When \( \lambda^*(p') = \lambda(p') \), the lemma reduces to Lemma 3.7, so we compute \( J_f \) when in a neighborhood \( \Omega \) of \( p \)
\[
\lambda^*(p') = s \left( L(r) - \sum \lambda_i \right) < \lambda(p'),
\]
so that the same set \( T \) is common to all \( p' \in \Omega \). The first \( (m-1) \) rows of \( J_f \) are unchanged, as are all but the diagonal entry of the bottom row. If \( p = \sum \lambda_i b^i(r) \), then \( p + hv^m = \sum \lambda_i b^i(r') \), \( r' = r + h \), so that once
again $\sum_{j} \lambda_j$ is invariant. Hence when (4.8) holds, (4.5) and (4.6) show that if $p \in \Omega$ and $h$ is small,

\[
(f(p + hv^m) - f(p))_m = (A(r') - A(r)) (L(r') - \sum_{j} \lambda_j) + A(r)(L(r') - L(r)),
\]

and hence (2.3), (4.2), (4.4) and (4.6) give that

\[
(Df(v^m))_m = A'(r) \left( L(r) - \sum_{j} \lambda_j \right) + A(r) \frac{\nu_k}{2} - \frac{1}{2} \left( \frac{\nu(r')}{r} \right) A(r) \left( \frac{\nu_k}{\nu} \right) \nu
\]

(4.9)

Thus if $Df(v^m) = \hat{w}^m$, the $m$th component, $(\hat{w})_m$, satisfies

\[
(\hat{w})_m = \max \left( (w^m)_m, \left( L(r) - \sum_{j} \lambda_j \right) + \frac{1}{2} \frac{\nu_k}{\nu} \right)
\]

(recall $w^m$ from (3.9)). But $1/2 \leq (L - \sum \lambda_j) \geq 0$ and $2\nu \geq \nu_k \geq (\nu/2)$ when $r \in J_n$. This implies that $1 \geq (\hat{w})_m \geq \epsilon_0/4$.

We check that these bases satisfy the assertions of Lemma 3.7, and so only need consider $\hat{w}^m$ in the situation that (4.8) holds near $p$. Now $\epsilon_0/4 \leq (\hat{w})_m \leq |w^m|$, while for $j < m$, $(w^j)_m = -m$. Hence $\hat{w}^m$ makes an angle with $\text{span}[w^1, \ldots, w^{m-1}]$ whose sine is uniformly bounded below. This proves the Lemma.

5. Completion of proof

To extend $f$ to $Q_{r_0}$, recall from §3 that

\[
f(x) = A(r_0)\Psi(x) \quad (x \in \partial Q_{r_0}),
\]

where $\Psi : \partial Q_{r_0} \to P_{A(r_0)}$, the polyhedron $P$ of Proposition 3.5. Then exactly as in [2, p. 14] $f$ is extended to the rest of $\mathbb{R}^m$: 

\[
f(x) = \left( \frac{r}{r_0} \right)^{n_0} A(r_0)\Psi \left( \frac{r_0}{r} x \right) \quad (x \in \partial Q_r, r \leq r_0).
\]
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David Drasin
Mathematics Department
Purdue University
West Lafayette, IN 47907, USA
drasin@math.purdue.edu

Swati Sastry
swatisastry@hotmail.com

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