On radial behaviour and balanced Bloch functions

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Abstract. A Bloch function $g$ is a function analytic in the unit disk such that $(1 - |z|^2) |g'(z)|$ is bounded. First we generalize the theorem of Rohde that, for every “bad” Bloch function, $g(r \zeta) (r \to 1)$ follows any prescribed curve at a bounded distance for $\zeta$ in a set of Hausdorff dimension almost one. Then we introduce balanced Bloch functions. They are characterized by the fact that $|g'(z)|$ does not vary much on each circle $\{|z| = r\}$ except for small exceptional arcs. We show e.g. that

$$\int_0^1 |g'(r \zeta)| dr < \infty$$

holds either for all $\zeta \in \mathbb{T}$ or for none.

1. Radial behaviour of Bloch functions.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{T} = \partial \mathbb{D}$. The function $g$ analytic in $\mathbb{D}$ is called a Bloch function if

$$(1.1) \quad \|g\|_B = \sup_{z \in \mathbb{D}} (1 - |z|^2) |g'(z)| < \infty.$$ 

This holds if and only if the Riemann image surface of $g$ as a cover of $\mathbb{C}$ does not contain arbitrarily large unramified disks. We denote the family of Bloch functions by $\mathcal{B}$. 

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First we generalize a surprising result of Steffen Rohde [Ro93]. Let $c_1, c_2, \ldots$ be positive absolute constants and let $\dim E$ denote the Hausdorff dimension [Fa85, p. 7] of $E \subset \mathbb{T}$. Note that $\dim \mathbb{T} = 1$.

**Theorem 1.1.** Let $G \subset \mathbb{C}$ be a domain with $0 \in G$ and let $g$ be a Bloch function with $\|g\|_B \leq 1$ and $g(0) = 0$. We assume that, for almost all $\zeta \in \mathbb{T}$,

$$\lim_{r \to 1} g(r \zeta) \text{ lies in } \mathbb{C} \setminus G \text{ or does not exist.}$$

Let $\Gamma$ be any halfopen curve in $G$ starting at 0. If

$$c_1 < R < \text{dist}(0, \partial G), \quad \text{dist}(\Gamma, \partial G) \geq 2R,$$

then there exists $E_\Gamma \subset \mathbb{T}$ with

$$\dim E_\Gamma \geq 1 - \frac{c_2}{R}$$

such that, for $\zeta \in E_\Gamma$, we can find a parametrization $\gamma_\zeta(r)$, $0 \leq r < 1$ of $\Gamma$ with $\gamma_\zeta(0) = 0$ such that

$$|g(r \zeta) - \gamma_\zeta(r)| \leq 2R, \quad \text{for } 0 \leq r < 1.$$

This theorem is due to Rohde [Ro93] for the case that $G = \mathbb{C}$. Thus the radial image follows any prescribed curve with a bounded deviation on a set of dimension almost 1. Now we apply this theorem to (injective) conformal maps $f$ of $\mathbb{D}$ into $\mathbb{C}$. It is well-known [DuShSh66], [Be72] that

$$f \text{ conformal implies } \| \log f' \|_B \leq 6,$$

$$\| \log f' \|_B \leq 1 \text{ implies } f \text{ conformal.}$$

If the radial limit $f(\zeta)$ exists and is finite (which holds for almost all $\zeta \in \mathbb{T}$), we write

$$\alpha(\zeta) = \lim \inf_{r \to 1} \arg (r \zeta - f(\zeta)),$$

$$\beta(\zeta) = \lim \sup_{r \to 1} \arg (f(r \zeta) - f(\zeta)).$$

We give a partial generalization of [CaPo97, Theorem 1].
Corollary 1.2. Let \( f \) map \( \mathbb{D} \) conformally into \( \mathbb{C} \) and suppose that
\[
\limsup_{r \to 1} |f'(r \zeta)| \geq 1, \quad \text{for almost all } \zeta \in \mathbb{T},
\]
\[
\liminf_{r \to 1} |f'(r \zeta_0)| = 0, \quad \text{for some } \zeta_0 \in \mathbb{T}.
\]
Then, for \( j = 1, 2, 3, 4 \), there exist sets \( E_j \subset \mathbb{T} \) with \( \dim E_j = 1 \), such that
\begin{enumerate}
  \item \( \alpha(\zeta) = -\infty, \beta(\zeta) = +\infty \), for \( \zeta \in E_1 \) (twist point),
  \item \( \alpha(\zeta) = \beta(\zeta) = +\infty \), for \( \zeta \in E_2 \) (spiral point),
  \item \( -\infty < \alpha(\zeta) < \beta(\zeta) = +\infty \), for \( \zeta \in E_3 \) (gyration point),
  \item \( -\infty < \alpha(\zeta) + 2\pi < \beta(\zeta) < +\infty \), for \( \zeta \in E_4 \) (oscillation point).
\end{enumerate}
Moreover \( f(\zeta) \) is well-accessible for \( \zeta \in E_j \) \( (j = 1, 2, 3, 4) \).

The McMillan Twist Theorem [Mc69], [Po92, p. 142] states that, for almost all points \( \zeta \in \mathbb{T} \), either \( \zeta \) is a twist point or the angular derivative \( f'(\zeta) \neq 0, \infty \) exists. The three sets of points satisfying ii), iii) and iv) were introduced in [Do92] and [CaPo97]. The Twist Theorem shows that these sets have measure 0. If \( \lim_{r \to 1} f'(r \zeta) \) fails to exist on a set of positive measure then Plessner’s Theorem for Bloch functions [Po92, p. 140] shows that assumption (1.9) is automatically satisfied. The special case of Corollary 1.2 that \( \lim f'(r \zeta) \) exists almost nowhere is contained in [CaPo97, Theorem 1]. The boundary point \( f(\zeta) \) is called well-accessible [Po92, p. 251] if there is a curve \( z(t), 0 \leq t \leq 1 \) with \( z(0) = \zeta \) such that
\[
\text{diam} \{ f(z(\tau)) : \ t \leq \tau \leq 1 \} = O \left( \text{dist} \left( f(z(t)), \partial f(\mathbb{D}) \right) \right), \quad \text{as } t \to 1.
\]
It is known [CaPo97, (3.17)] that the condition
\[
-b \leq \log |f'(r \zeta)| \leq b, \quad b > 1,
\]
implies that \( f(\zeta) \) is well-accessible and [CaPo97, (3.18)] that
\[
|\arg f'(r \zeta) - \arg (f(r \zeta) - f(\zeta))| \leq c_3 b.
\]
Proof of Corollary 1.2. Let $n > c_1$; see (1.3). By (1.9) there exist $r_n < 1$ such that $a_n = \log f'(r_n \zeta_0)$ satisfies $\Re a_n < -16 n$. We define
\begin{equation}
(1.12) \quad \varphi_n(z) = \frac{z + r_n \zeta_0}{1 + r_n \zeta_0 z}, \quad f_n = f \circ \varphi_n, \quad g_n = \frac{1}{8} (\log f' \circ \varphi_n - a_n).
\end{equation}
Then $g_n \in B$ with $g_n(0) = 0$ and $\|g_n\|_B \leq 1$ by (1.6). We apply Theorem 1.1 with $G = \{\Re w < |\Re a_n|\}$, $R = n$ and curves
$$
\Gamma_j(t), \quad 0 \leq t < 1 \quad (j = 1, 2, 3, 4)
$$
such that $\Gamma_j(0) = 0$, $\Re \Gamma_j(t) = 0$ and, as $t \to 1$,

i) $\lim \inf \Im \Gamma_1(t) = -\infty$, $\lim \sup \Im \Gamma_1(t) = +\infty$,

ii) $\lim \Im \Gamma_2(t) = +\infty$,

iii) $-\infty < \lim \inf \Im \Gamma_3(t) < +\infty$, $\lim \sup \Im \Gamma_3(t) = +\infty$,

iv) $\lim \inf \Im \Gamma_4(t) = 0$, $\lim \sup \Im \Gamma_4(t) = 3\pi + 2 n + (c_3 b_n + |a_n|)/8$,

see (1.15) below. Then (1.3) is satisfied, and (1.2) holds by (1.8) because $|\Re a_n| > 16 n$. We conclude that there are sets $E_{jn} \subset \mathbb{T}$ with
\begin{equation}
(1.13) \quad \dim E_{jn} \geq 1 - \frac{c_2}{n}, \quad \text{for } j = 1, \ldots, 4 \text{ and } n > c_1,
\end{equation}
such that (1.5) holds for $\zeta \in E_{jn}$. We obtain from (1.12) that
\begin{equation}
(1.14) \quad \log f_n'(z) = a_n + \log ((1 - r_n^2) (1 + \overline{\zeta_0} r_n z)^{-2}) + 8 g_n(z).
\end{equation}
Since $\Re \gamma_{\zeta}(r) = 0$ it follows from (1.5) that
\begin{equation}
(1.15) \quad |\log |f_n'(r \zeta_0)|| \leq b_n := |\Re a_n| + \log \frac{1 + r_n}{1 - r_n} + 16 n
\end{equation}
so that $f_n(\zeta)$ is well-accessible; see (1.10). We obtain from (1.5), (1.11) and (1.15) that
\begin{equation}
\limsup_{r \to 1} |\arg (f_n(r \zeta) - f_n(\zeta)) - 8 \gamma_{\zeta}(r)|
\end{equation}
\begin{align}
(1.16) \quad &< 16 n + c_3 b_n + |\Im a_n| + 2 \\
&< \infty,
\end{align}
for $\zeta \in E_{jn}$. Finally we set

$$E_j = \bigcup_n \varphi_n(E_{jn}), \quad j = 1, 2, 3, 4.$$ 

Then $\dim E_j = 1$ by (1.13), and if $\zeta \in E_j$ then $\zeta = \varphi_n(\zeta_n)$ for some $\zeta_n \in E_{jn}$.

Hence $f(\zeta) = f_n(\zeta_n)$ is well-accessible, and by the Koebe distortion theorem it is easy to deduce from (1.16) and the choice of $\Im \Gamma_j(t)$ that $\alpha(\zeta)$ and $\beta(\zeta)$ have the required properties.

**Remark 1.** We assume now that $f(D)$ is bounded by a rectifiable curve. Then $f' \in H^1$ and thus [Du70, p. 24]

$$f'(z) = e^{i\alpha} \exp \left( \frac{1}{2\pi} \int_T \frac{\zeta + z}{\zeta - z} \log |f'(\zeta)||d\zeta| \right) \exp \left( - \int_T \frac{\zeta + z}{\zeta - z} \frac{d\mu(\zeta)}{\zeta - z} \right),$$

where $\mu \geq 0$ is a singular measure. By definition $f(D)$ is a Smirnov domain if $\mu = 0$. Hence (1.8) holds if $|f'(\zeta)| \geq 1$ for almost all $\zeta \in T$, and (1.9) holds if $f(D)$ is not a Smirnov domain. In particular Corollary 1.2 can be applied if $f(D)$ is a Keldish-Lavrentiev domain, that is a non-Smirnov domain for which $|f'(\zeta)| = 1$ for almost all $\zeta \in T$; see [DuShSh66].

**Remark 2.** There are local versions of Theorem 1.1 and Corollary 1.2. We can replace $T$ by an open subarc $A$ and restrict $\zeta$ and our sets $E$ to lie in $A$.

2. **The proof of Theorem 1.1.**

We use the martingale technique introduced by Makarov [Ma90] into the theory of Bloch functions. For $n = 0, 1, \ldots$ let $D_n$ be the family of dyadic arcs of length $2\pi/2^n$ on $T$, that is,

$$D_n = \left\{ e^{it} : \frac{2\pi k}{2^n} \leq t < \frac{2\pi (k + 1)}{2^n} \right\}, \quad 0 \leq k < 2^n.$$ 

If $I$ and $J$ are any dyadic arcs then $I \cap J = \emptyset$ or $I \subset J$ or $J \subset I$. Let $g \in B$ and $n = 0, 1, \ldots$. We define the *martingale associated to $g$* by

$$W_n(\zeta) \equiv W_n(I) = \lim_{r \to 1} \frac{1}{|I|} \int_I g(rs) |ds|, \quad \text{for } \zeta \in I \in D_n,$$
where $|\cdot|$ denotes the linear measure on $\mathbb{T}$. Let $c_1, c_2, \ldots$ denote suitable positive absolute constants. We need two known results. The first is due to Makarov [Ma90]; compare [Po92, p. 156].

**Proposition 2.1** (Makarov). Let $g \in B$, $\|g\|_B \leq 1$ and let $W_n$ be the associated martingale. Then

$$\text{(2.3)} \quad |g(r \zeta) - W_n(\zeta)| < c, \quad \text{for } \zeta \in \mathbb{T}, \quad 1 - \frac{1}{2^n} \leq r \leq 1 - \frac{1}{2^{n+1}},$$

$$\text{(2.4)} \quad |W_{n+1}(\zeta) - W_n(\zeta)| < c, \quad \text{for } \zeta \in \mathbb{T}.$$

We also need the following technical result [ON95], [Do97]; compare [Ro93, p. 493].

**Proposition 2.2** (O’Neill, Donaire). Let $W_n$ be the martingale associated to $g \in B$ and let $\|g\|_B \leq 1$, $0 < \alpha < \pi/2$. Let $I \in \mathcal{D}_m$ and $R > c_1(\alpha)$. If the stopping time

$$\text{(2.5)} \quad \tau_I(\zeta) = \inf \{n > m : |W_n(\zeta) - W_m(\zeta)| \geq R\}$$

is finite for almost all $\zeta \in I$, then

$$\text{(2.6)} \quad |\{\zeta \in I : |\arg (W_{\tau_I(\zeta)}(\zeta) - W_m(\zeta)) - \vartheta| < \alpha\}| \geq c_2(\alpha) |I|,$$

for every $\vartheta$. Here $c_1(\alpha)$ and $c_2(\alpha)$ only depend on $\alpha$.

**Proof of Theorem 1.1.** a) Let $\Gamma(t)$, $0 \leq t < 1$ be some parametrization of our given curve $\Gamma$. Let $\mathcal{F}_0 = \{\mathbb{T}\}$ and $t_0 = 0$. We shall recursively construct families $\mathcal{F}_j$ of dyadic arcs such that each arc in $\mathcal{F}_j$ is contained in some arc of $\mathcal{F}_{j-1}$, furthermore stopping times

$$\text{(2.7)} \quad t_j(\zeta) \equiv t_j(I) \in [0, 1], \quad \text{for } \zeta \in I \in \mathcal{F}_{j-1}$$

constant on $I$ such that $t_{j-1}(\zeta) \leq t_j(\zeta)$ and

$$\text{(2.8)} \quad \text{dist}(W_m(I), \mathbb{C} \setminus G) > R + c, \quad \text{for } I \in \mathcal{F}_j \cap \mathcal{D}_m,$$

where $c$ is the constant of Proposition 2.1.
b) Suppose that $F_j$ and $t_j$ have already been defined. Let $\zeta \in I \in F_j$. Then $I \in D_m$ for some $m$. If $t_j(\zeta) = 1$ then we define $t_{j+1}(\zeta) = 1$, otherwise

\[(2.9) \quad t_{j+1}(\zeta) \equiv t_{j+1}(I) = \inf \{ t > t_j(\zeta) : |\Gamma(t) - W_m(I)| \geq R \}, \]

if this set is empty we define $t_{j+1}(I) = 1$ and $A_j(I) = I$.

Now let $t_{j+1}(I) < 1$. Plessner’s theorem for Bloch functions [Po92, p. 140] says that, for almost all $\zeta \in \mathbb{T}$, either the radial limit $g(\zeta)$ exists or the limit set of $g(r \zeta)$ as $r \to 1$ is equal to $\hat{\mathbb{C}}$. Hence it follows from assumption (1.2) that

\[
\liminf_{r \to 1} \text{dist} \left( g(r \zeta), \mathbb{C}\setminus G \right) = 0, \quad \text{for almost all } \zeta \in \mathbb{T},
\]

so that, by (2.3),

\[
\liminf_{n \to \infty} \text{dist} \left( W_n(\zeta), \mathbb{C}\setminus G \right) \leq c, \quad \text{for almost all } \zeta \in \mathbb{T}.
\]

Therefore we obtain from (2.4) and (2.8) that, for almost all $\zeta \in I$, the stopping time $\tau_1(\zeta)$ defined in (2.5) is finite. By (2.4) we then have

\[(2.10) \quad R \leq |W_{\tau_1(\zeta)}(\zeta) - W_m(\zeta)| < R + c. \]

Thus we can apply Proposition 2.2 with $\alpha = 1/4$. We see from (2.6) that, for $R > c_3 = \max \{4c, c_1\}$, the set

\[
(2.11) \quad A_j(I) = \left\{ \zeta \in I : |\arg (W_{\tau_1(\zeta)}(\zeta) - W_m(\zeta)) - \arg (\Gamma_{t_{j+1}(I)} - W_m(I))| < \frac{1}{4} \right\}
\]

satisfies $|A_j(I)| \geq c_2 |I|$. Note that $A_j(I)$ is the union of dyadic arcs $J \in D_n$ with $n > m$.

We define $F_{j+1}$ as the family of the dyadic arcs $J$ of $A_j(I)$ for all $I \in F_j$. Then

\[(2.12) \quad \sum_{\substack{J \subset I \quad J \in F_{j+1}}} |J| = |A_j(I)| \geq c_2 |I|. \]

Furthermore it follows from (2.4) and (2.10) that $\tau_1(\zeta) \geq m + R/c$. Hence

\[(2.13) \quad J \in F_{j+1}, \ J \subset I \in F_j \implies |J| \leq 2^{-R/c} |I|. \]
Now we verify (2.8) for \( j + 1 \), that is, we shall show that

\[(2.14) \quad \text{dist} (W_n(J), C \setminus G) > R + c, \]

for \( J \in \mathcal{F}_{j+1}, \zeta \in I \in \mathcal{F}_j, n = \tau_I(\zeta) \); see (2.11). This is trivial by (2.8) if \( t_{j+1}(I) = 1 \) and thus \( A_j(I) = I \). Therefore let \( t_{j+1}(I) < 1 \). Since \( \Gamma(t) \) is continuous we see from (2.9) that \( |\Gamma(t_{j+1}(I)) - W_m(I)| = R \). Hence it follows from (2.10) and (2.11) that the quantity

\[ q = \frac{W_n(\zeta) - W_m(\zeta)}{\Gamma(t_{j+1}(I)) - W_m(I)} \]

satisfies \( 1 \leq |q| \leq 1 + c/R \) and \( |\arg q| < 1/4 \). Since \( R > c_3 \geq 4c \) we deduce that \( |q - 1| < 1/2 \). Hence

\[ |W_n(\zeta) - \Gamma(t_{j+1})| = |\Gamma(t_{j+1}) - W_m(\zeta)||q - 1| < \frac{R}{2} \]

and it follows by assumption (1.3) that

\[ \text{dist} (W_n(\zeta), C \setminus G) \geq \text{dist} (\Gamma, \partial G) - \frac{R}{2} \geq \frac{3R}{2} > R + c. \]

This completes our construction.

c) We define

\[(2.15) \quad E_\Gamma = \bigcap_{j \geq 1} \bigcup_{I \in \mathcal{F}_j} I. \]

It follows from (2.12) and (2.13) by a theorem [Po92, p. 226] of Hungerford [Hu88] and Makarov [Ma90] that

\[ \dim E_\Gamma \geq \frac{\log (c_2 2^{R/c})}{\log 2^{R/c}} = 1 - \frac{c \log \left( \frac{1}{c_2} \right)}{R \log 2}, \]

which proves (1.4).

Now let \( \zeta \in E_\Gamma \). There are two cases.

i) First we assume that \( t_j(\zeta) < 1 \) for all \( j \). Let \( I_j \in \mathcal{F}_j \) be the arc containing \( \zeta \). Then \( I_j \in \mathcal{D}_{n_j} \) for some \( n_j \). We define \( \varphi_\zeta : [0, 1] \rightarrow [0, 1] \) by \( \varphi_\zeta(2^{-n_j}) = t_j(\zeta) \) and linear in between. We parametrize \( \Gamma \) by
\( \gamma_\zeta(r) = \Gamma(\varphi_\zeta(r)), \) \( 0 \leq r < 1. \) If \( 1 - 2^{-n_j} \leq r \leq 1 - 2^{-n_{j+1}} \) then \( t_j(\zeta) \leq \varphi_\zeta(r) \leq t_{j+1}(\zeta) \) and thus

\[
|g(r \zeta) - \gamma_\zeta(r)| \leq |g(r \zeta) - W_{n_j}(\zeta)| + |\Gamma(\varphi_\zeta(r)) - W_{n_j}(I_j)| \leq c + R \leq 2R
\]

by (2.3) and (2.9).

ii) Now we suppose that \( t_j(\zeta) < 1 \) for \( j \leq k \) and \( t_j(\zeta) = 1 \) for \( j > k. \) Then we define \( \varphi_\zeta \) as in (i) for \( j < k \) but linear in \([1 - 2^{-n_k}, 1]. \) If \( 1 - 2^{-n_k} \leq r < 1 \) then (see (2.9))

\[
|\Gamma(\varphi_\zeta(r)) - W_n(\zeta)| < R, \quad \text{for } n \geq n_k
\]

and (1.5) follows as above.


Let \( \triangle(\zeta, \rho) \) denote the non-euclidean disk of center \( \zeta \in \mathbb{D} \) and radius \( \rho. \) For \( g \in \mathcal{B} \) we define

\[
(3.1) \quad \mu_g(r) = \sup_{r \leq |z| < 1} (1 - |z|^2) |g'(z)|, \quad 0 \leq r < 1.
\]

Using the maximum principle for \( |z| \leq r, \) we see that

\[
(3.2) \quad |g'(z)| \leq \max \left\{ \frac{\mu_g(r)}{1 - r^2}, \frac{\mu_g(r)}{1 - |z|^2} \right\}, \quad \text{for } z \in \mathbb{D}, \ 0 \leq r < 1.
\]

By definition we have \( g \in \mathcal{B}_0 \) if \( \mu_g(r) \to 0 \) as \( r \to 1. \)

We call \( g \) a balanced Bloch function if there exist \( a > 0 \) and \( \rho < \infty \) such that

\[
(3.3) \quad \sup_{z \in \triangle(\zeta, \rho)} (1 - |z|^2) |g'(z)| \geq a \mu_g(|\zeta|), \quad \text{for } \zeta \in \mathbb{D}.
\]

This condition is trivially satisfied if \( 0 < \alpha \leq |g'(z)| \leq \beta < \infty \) for \( z \in \mathbb{D}. \) Balanced Bloch functions for the case \( g \notin \mathcal{B}_0 \) were first considered by P. Jones [Jo89]; see e.g. also [Ro91], [BiJo97]. Jones showed that if \( J = \partial f(\mathbb{D}) \) is a quasicircle, then \( \log f' \) is balanced and not in \( \mathcal{B}_0 \) if and only if

\[
\inf_{w_1, w_2 \in J} \sup \left\{ \frac{|w_1 - w| + |w - w_2|}{|w_1 - w_2|} : \ w \in J \text{ between } w_1 \text{ and } w_2 \right\} > 1.
\]
Curves with this property are called uniformly wiggly. The prototype of balanced Bloch functions are sufficiently regular series with Hadamard gaps.

**Theorem 3.1.** Suppose that

\begin{equation}
1 < \lambda \leq \frac{n_{k+1}}{n_k} \leq \lambda' < \infty, \quad \text{for } k = 0, 1, \ldots
\end{equation}

\begin{equation}
\frac{1}{M} \left( \frac{n_i}{n_k} \right)^{\alpha} |b_j| \leq |b_k| \leq M |b_j|, \quad \text{for } 0 \leq j \leq k
\end{equation}

with constants $M$ and $\alpha < 1$. Then

\begin{equation}
g(z) = \sum_{k=0}^{\infty} b_k z^{n_k}, \quad z \in \mathbb{D},
\end{equation}

is a balanced Bloch function.

A typical example of a balanced Bloch function is

\begin{equation}
g(z) = \sum_{k=1}^{\infty} k^{-\gamma} z^{2^k}, \quad 0 \leq \gamma < \infty.
\end{equation}

**Proof.** Let $M_1, M_2, \ldots$ denote constants that depend only on $\lambda, \lambda', \alpha$ and $M$. If $1 - 1/n_j \leq r \leq 1 - 1/n_{j+1}$ and $|z| = r$ then, by (3.6),

\begin{align*}
|z g'(z)| & \leq \sum_{k=0}^{j} n_k |b_k| + \sum_{k=j+1}^{\infty} n_k |b_k| \exp \left( - \frac{n_k}{n_{j+1}} \right) \\
& \leq M n_j^{\alpha} |b_j| \sum_{k=0}^{j} n_k^{1-\alpha} + \lambda' M n_j |b_j| \sum_{k=j+1}^{\infty} n_k |b_k| \exp \left( - \frac{n_k}{n_{j+1}} \right)
\end{align*}

by (3.5) and (3.4). Since $t e^{-t}$ is decreasing for $t \geq 1$ we therefore obtain from (3.4) that

\begin{align*}
|z g'(z)| & \leq M_1 n_j |b_j| + \lambda' M n_j |b_j| \sum_{\nu=0}^{\infty} \lambda^{\nu} \exp \left( - \lambda^{\nu} \right) \leq M_2 \frac{|b_j|}{(1 - r^2)}.
\end{align*}

Using the maximum principle near $z = 0$, we thus see from (3.1) that

\begin{equation}
\mu_g(r) \leq \sup_{k \geq j} M_3 |b_k| \leq M_4 |b_j|, \quad \text{for } 1 - \frac{1}{n_j} \leq r \leq 1 - \frac{1}{n_{j+1}}.
\end{equation}
Now we apply a standard method [Bi69] to estimate the coefficients of gap series. It follows from (3.4), (3.5) and [GHPo87, Theorem 2] that

\[ n_j |b_j| \leq M_5 \sup \{ |g'(z)| : z \in \Delta(\zeta, \rho) \}, \]

for \( 1 - M_6/n_j \leq |\zeta| \leq 1 - M_7/n_j \). Hence

\[ \sup_{z \in \Delta(\zeta, \rho)} (1 - |z|^2) |g'(z)| \geq M_8^{-1} (1 - |\zeta|^2) n_j |b_j| \geq M_9^{-1} \mu_g(r) \]

by (3.7).

Further examples of balanced Bloch functions come from automorphic forms. Let \( \Gamma \) be a Fuchsian group with compact fundamental domain \( F \) in \( \mathbb{D} \). Let \( h \) be an analytic automorphic form of weight 1, corresponding to a differential on the Riemann surface \( \mathbb{D}/\Gamma \). Then \( \gamma' h \circ \gamma = h \) for \( \gamma \in \Gamma \) and

\[ g(z) = \int_0^z h(\zeta) \, d\zeta, \quad z \in \mathbb{D} \]

is a balanced Bloch function because \( \overline{F} \subset \mathbb{D} \). Note that \( \inf \mu_g(r) > 0 \).

Now we prove two results on real convex functions needed for the next section.

**Lemma 3.2.** Let the real-valued functions \( \varphi \) and \( \psi \) be continuous and convex in the interval \( I \subset \mathbb{R} \). If the function

\[ \chi(s) = \sup_{t \geq s} (\varphi(t) - \psi(t)) + \psi(s), \quad s \in I \]

is finite, then it is also continuous and convex in \( I \).

**Proof.** The function \( \sup \{ \varphi(t) - \psi(t) : t \in I, t \geq s \} \) is decreasing in \( s \in I \). Let \( I_k = [s_k, t_k] \) be its intervals of constancy with values \( c_k \). We define

\[ \chi_k(s) = \begin{cases} \varphi(s), & \text{for } s \in I \backslash I_k, \\ c_k + \psi(s), & \text{for } s \in I_k. \end{cases} \]

Since \( \varphi(s) - \psi(s) \leq c_k \) for \( s \in I_k \), we have

\[ \varphi(s) \leq c_k + \psi(s) = \chi_k(s), \quad \text{for } s_k \leq s \leq t_k, \]
with equality for \( s = s_k \) and \( s = t_k \). The convex function \( \varphi \) has left and right derivatives \( D^\pm \varphi \) in \( I \) and \( D^\pm \varphi \) is increasing [HLP67, p. 91-94]. If \( s < s_k \) then

\[
D^+ \chi_k(s) = D^+ \varphi(s) \leq D^+ \varphi(s_k) \leq D^+ \psi(s_k) = D^+ \chi_k(s_k)
\]

by (3.10). If \( s_k \leq s < t_k \) then

\[
D^+ \chi_k(s_k) = D^+ \psi(s_k) \leq D^+ \psi(s) = D^+ \chi_k(s)
\]

by (3.9). Since \( D^- \psi(t_k) \leq D^- \varphi(t_k) \) by (3.10), we furthermore have

\[
D^+ \chi_k(s) \leq D^- \psi(t_k) \leq D^- \varphi(t_k) \leq D^+ \varphi(t_k) = D^+ \chi_k(t_k).
\]

Using again that \( D^+ \varphi \) and \( D^+ \psi \) are increasing, we deduce that \( D^+ \chi_k \) is increasing in \( I \). Since \( \chi_k \) is locally absolutely continuous it follows by integration that \( \chi_k \) is convex. Finally \( \chi = \sup_k \chi_k \) by (3.9) and (3.10), so \( \chi \) is also convex.

**Lemma 3.3.** The function

\[
\chi(s) = \log \mu_g(e^s) - \log (1 - e^{2s}), \quad -\infty < s < 0
\]

is convex and the function \( u(z) = \chi(\log |z|) \) with \( u(0) = \log \mu_g(0) \) is continuous and subharmonic in \( \mathbb{D} \).

**Proof.** Let \( M(r) = \max \{ |g'(z)| : |z| = r \} \). It follows from (3.1) that (3.8) holds with

\[
\varphi(s) = \log M(e^s), \quad \psi(s) = -\log (1 - e^{2s}).
\]

The function \( \varphi \) is convex by the Hadamard three circles theorem [Co78, p. 137], and \( \chi \) is convex because \( \psi''(s) = 4e^{2s}(1-e^{2s})^{-2} > 0 \). Therefore \( \chi \) is convex by Lemma 3.2. It follows that \( u \) is subharmonic [HaKe76, Theorem 2.2].


Let \( \mu_g \) be defined by (3.1). We consider the open level sets

\[
A_g(\varepsilon) = \{ z \in \mathbb{D} : (1 - |z|^2) |g'(z)| < \varepsilon \mu_g(|z|) \},
\]
for $0 < \varepsilon \leq 1$. We see from (4.1) and (3.1) that

$$|g'(z)| \geq \frac{\varepsilon \mu_g(r)}{1 - r^2} \geq \varepsilon \max_{|\zeta| = r} |g' (\zeta)|, \quad \text{for } z \notin A_g(\varepsilon), \ |z| = r.$$  

If $g'$ is unbounded it follows that $T \subset \overline{A_g(\varepsilon)}$ for all $\varepsilon > 0$. Otherwise we would have $|g'(z)| \to \infty$ as $z \to I$ for some arc $I$ of $T$, which is impossible by the Privalov uniqueness theorem [Po92, p. 140].

Let $M_1, \ldots$ denote positive constants that depend only on $a$ and $\rho$ in the definition (3.3) of balanced Bloch functions. In particular, if $g'$ is unbounded then $A_g(\varepsilon)$ is nonempty for $0 < \varepsilon \leq 1$. By contrast, the example $g(z) \equiv z$ shows that $A_g(\varepsilon)$ can be empty if $g'$ is bounded and $\varepsilon < 1$.

**Proposition 4.1.** Let $g$ be a balanced Bloch function and let $z_0 \in \mathbb{D}$. Then the harmonic measure satisfies

$$\omega(z_1, \overline{\Delta(z_0, 2\rho)} \cap \overline{A_g(\varepsilon)}, \Delta(z_0, 2\rho) \setminus \overline{A_g(\varepsilon)}) \leq \frac{M_1}{\log \left( \frac{1}{\varepsilon} \right)},$$

for some $z_1 \in \Delta(z_0, \rho)$.

**Proof.** We write $r = |z_0|$, $\Delta_0 = \Delta(z_0, 2\rho)$ and $A = \overline{A_g(\varepsilon)}$. It follows from (3.2) that

$$|g'(z)| \leq \frac{M_2}{1 - r^2} \mu_g(r), \quad \text{for } z \in \overline{\Delta_0}.$$  

It follows from (4.1) that

$$|g'(z)| \leq \frac{M_2}{1 - r^2} \mu_g(r) \varepsilon, \quad \text{for } z \in \overline{\Delta_0} \cap A.$$  

Hence the two-constants theorem [Ah73, p. 39] implies that

$$|g'(z)| \leq \frac{M_2}{1 - r^2} \mu_g(r) \varepsilon \omega(z, \overline{\Delta_0} \cap A, A) \Delta_0 \setminus A),$$

for $z \in \Delta_0 \setminus A$. By (3.3) there exists $z_1 \in \Delta(z_0, \rho)$ such that

$$|g'(z_1)| \geq \frac{a}{1 - |z_1|^2} \mu_g(r) \geq \frac{M_3^{-1}}{1 - r^2} \mu_g(r).$$
Hence (4.2) follows from (4.4).

**Theorem 4.2.** Let \( g \) be a balanced Bloch function. Then there are \( \alpha > 0 \) and \( \varepsilon_0 > 0 \) such that every component of \( A_g(\varepsilon) \) \((0 < \varepsilon < \varepsilon_0)\) lies in some disk \( \Delta(z_0, \varepsilon^\alpha) \) \((z_0 \in \mathbb{D})\) and contains a zero of \( g' \).

**Proof.** a) Let \( B \) be a component of \( A_g(\varepsilon) \), let \( z_0 \in B \) and let \( B_0 \) be the component of \( B \cap \Delta(z_0, \rho/2) \) with \( z_0 \in B_0 \). Let \( \varphi \) map \( \Delta(z_0, 2\rho) \setminus \overline{B_0} \) conformally onto \( \{ r < |z| < 1 \} \) such that \( \partial \Delta(z_0, 2\rho) \) corresponds to \( \mathbb{T} \). Then

\[
\omega(z, \overline{\Delta(z_0, 2\rho) \setminus \overline{B_0}}) \cap \Delta(z_0, 2\rho) \setminus \overline{B_0} = \log \left( \frac{1}{\varphi(z)} \right) \log \left( \frac{1}{r} \right).
\]

Since \( B_0 \subset A_g(\varepsilon) \) it follows from Proposition 4.1 and the principle of majorization for harmonic measure \([Ah73, p.39]\) that

\[
\frac{\log \left( \frac{1}{|\varphi(z_1)|} \right)}{\log \left( \frac{1}{r} \right)} \leq \frac{M_1}{\log \left( \frac{1}{\varepsilon} \right)},
\]

for some \( z_1 \in \Delta(z_0, \rho) \). Since \( B_0 \subset \Delta(z_0, \rho/2) \) a normal family argument gives \(|\varphi(z_1)| < 1 - \alpha_1\) where \( \alpha_1 > 0 \) depends only on \( a \) and \( \rho \). Hence \( r \leq \varepsilon^{\alpha_2} \) and therefore

\[
B_0 \subset \Delta(z_0, \varepsilon^{\alpha_3}), \quad \text{for } 0 < \varepsilon < \varepsilon_0.
\]

Since \( B \) is connected and contains \( z_0 \), it follows that \( B = B_0 \) if \( \varepsilon^{\alpha} < \rho/2 \).

b) Now we prove that every component \( B \) of \( A_g(\varepsilon) \) with \( \overline{B} \subset \mathbb{D} \) contains a zero of \( g' \). Suppose that \( g'(z) \neq 0 \) for \( z \in B \) and thus for \( z \in \overline{B} \). Then \( \log |g'| \) is harmonic in \( B \) and continuous in \( \overline{B} \). Hence it follows from Lemma 3.3 that

\[
v(z) = \log \mu_{g'}(|z|) - \log (1 - |z|^2) - \log |g'(z)|
\]

is subharmonic in \( B \) and continuous in \( \overline{B} \). Since \( B \) is a component of \( A_g(\varepsilon) \) and since \( \overline{B} \subset \mathbb{D} \), we see from (4.1) that \( v(z) = \log (1/\varepsilon) \) for \( z \in \partial B \) and thus \( v(z) \leq \log (1/\varepsilon) \) for \( z \in B \) by the maximum principle for subharmonic functions. But this contradicts (4.1).
Theorem 4.3. Let \( g \) be a balanced Bloch function and suppose that
\[
\frac{\mu_g(r')}{\mu_g(r)} \geq \frac{1 - r'}{1 - r} \lambda \left( \frac{1 - r}{1 - r'} \right), \quad \text{for } 0 < r < r' < 1,
\]
where \( \lambda(x) \nearrow \infty \) as \( x \to \infty \). Then there exist \( \varepsilon > 0 \) and \( \rho^* < \infty \) such that every disk \( \Delta(\zeta, \rho^*) \) \( (\zeta \in \mathbb{D}) \) contains a component of \( A_g(\varepsilon) \).

Some (rather weak) condition like (4.5) is necessary as the balanced Bloch function \( g(z) \equiv z \) shows. Note that (4.5) implies that \( g' \) is unbounded.

Proof. We claim: Given \( \varepsilon > 0 \) there exists \( \rho' < \infty \) such that
\[
\Delta(\zeta, \rho') \cap A_g(\varepsilon) \neq \emptyset, \quad \text{for every } \zeta \in \mathbb{D}.
\]
This claim implies the assertion of Theorem 4.3 with \( \rho^* = \rho' + 2 \varepsilon \alpha \) and \( 0 < \varepsilon < \varepsilon_0 \) by Theorem 4.2.

Suppose our claim is false. Then, for \( 0 < \varepsilon < 1 \), there exist \( z_n \in \mathbb{D} \) such that
\[
(1 - |z|^2) |g'(z)| > \varepsilon \mu_g(|z|), \quad \text{for } z \in \Delta(z_n, n), \ n = 1, 2, \ldots
\]
We write \( r_n = |z_n| \) and consider the functions
\[
h_n(s) = \frac{1 - r_n^2}{\mu_g(r_n)} g' \left( \frac{s + z_n}{1 + \bar{z}_n s} \right), \quad s \in \mathbb{D}.
\]
It follows from (4.8) and (3.2) that \( |h_n(s)| \leq 4/(1 - |s|^2) \) for \( s \in \mathbb{D} \).
Therefore we may assume that \( h_n \to h \) as \( n \to \infty \) locally uniformly in \( \mathbb{D} \). Furthermore we may assume that \( z_n \to \zeta \in \mathbb{T} \).

Let \( |s| = \sigma < 1 \). By (3.1) and (4.5) we have
\[
\mu_g \left( \frac{s + z_n}{1 + \bar{z}_n s} \right) \geq \mu_g \left( \frac{\sigma + r_n}{1 + r_n \sigma} \right) \geq \frac{1 - \sigma}{1 + r_n \sigma} \lambda \left( \frac{1 + r_n \sigma}{1 - \sigma} \right) \mu_g(r_n).
\]
Hence it follows from (4.7) and (4.8) that
\[
|h_n(s)| \geq \frac{\varepsilon |1 + \bar{z}_n s|^2}{(1 + \sigma)(1 + r_n \sigma)} \lambda \left( \frac{1 + r_n \sigma}{1 - \sigma} \right).
\]
Since $h_n \to h$ and $\zeta_n \to \zeta$ as $n \to \infty$, we conclude that

$$|h(s)| \geq \frac{\varepsilon}{\lambda} \frac{(1 + \zeta s)^2}{(1 + \sigma)^2},$$

for $\Re(\zeta) > 0$. Hence

$$|h(s)| \to \infty, \quad \text{as } |s| \to 1,$$

$\Re(\zeta) > 0$ which contradicts the Privalov uniqueness theorem [Pr56, p. 208], [Po92, p. 140].

**Geometric interpretation.** Let $g$ be a balanced Bloch function that satisfies condition (4.5). Let $\varepsilon > 0$ be small but fixed. Then

$$|g'(z)| \geq \frac{\varepsilon \mu_g(|z|)}{1 - |z|^2} \to \infty, \quad \text{as } |z| \to 1, \quad z \in \mathbb{D}\setminus A_g(\varepsilon)$$

by (4.5). Theorem 4.2 says that the components of $A_g(\varepsilon)$ have small hyperbolic diameter, each containing a zero of $g'$, whereas Theorem 4.3 says that there are many components. Hence the surface

$$\{(x, y, u) : x + iy \in \mathbb{D}, \ u = |g'(x + iy)|\}$$

rises to infinity at $\partial \mathbb{D}$ except for very many very small but deep holes near the zeros of $g'$.  

Ruscheweyh and Wirths [RuWi82] have studied, for any Bloch function $g$, the set where $(1 - |z|^2)|g'(z)|$ attains its maximum and its relation to the zeros of $g'$.  

J. Becker [Be87], [PoWa82, Theorem 4.2] has shown that, for any $g \in \mathcal{B}$, the condition

$$(4.10) \quad \int_0^1 \mu_g(r)^2 \frac{dr}{1 - r} < \infty$$

implies that $g \in \text{VMOA}$ (vanishing mean oscillation) and thus has finite radial limits $g(\zeta)$ for almost all $\zeta \in \mathbb{T}$. It follows [Pr56, p. 208] that $\cap \{g(\zeta) : \zeta \in \mathbb{T}, \ g(\zeta) \neq \infty \text{ exists}\} > 0$.

Now we turn to a condition stronger than (4.10), namely

$$(4.11) \quad \int_0^1 \mu_g(r) \frac{dr}{1 - r} < \infty.$$
It follows from (3.1) by integration that \( \int_0^1 |g'(r \zeta)| \, dr < \infty \) for all \( \zeta \in \mathbb{T} \) and that \( g \) is continuous in \( \overline{D} \). We show now that exactly the opposite happens if \( g \in \mathcal{B} \) is balanced and condition (4.11) is false.

**Theorem 4.4.** Let \( g \) be a balanced Bloch function with

\[
(4.12) \quad \int_0^1 \mu_g(r) \frac{dr}{1-r} = \infty.
\]

If \( C \) is any curve in \( D \) ending on \( T \), then

\[
(4.13) \quad \int_C |g'(z)| \, |dz| = \infty.
\]

Furthermore \( g \) assumes every value in \( \mathbb{C} \) infinitely often in \( D \).

**Geometric interpretation.** Let \( g \) be a balanced Bloch function that satisfies (4.10) and (4.12). The Riemann image surface of \( g \) over \( C \) then has many accessible boundary points; their projection to \( \mathbb{C} \) has positive capacity. But (4.13) shows that none of these boundary points is accessible through a curve of finite length.

**Proof.** Let \( c_1, c_2, \ldots \) denote suitable positive constants. Since \( C \) goes to \( T \), we can find \( z_n \in C \), \( r_n \wedge 1 \) and disks \( \Delta_n \) such that

\[
(4.14) \quad \Delta_n = \Delta(z_n, 2 \rho) \subset \{ r_n < |z| < r_{n+1} \}, \quad \frac{1-r_{n+1}}{1-r_n} > c_1.
\]

Let \( \varphi_n \) map \( \Delta_n \) conformally onto \( D \) such that \( \varphi_n(z_n) = 0 \). By Proposition 4.1 there exist \( \varepsilon > 0 \) and \( z_n^* \in \Delta(z_n, \rho) \) such that

\[
\frac{M_1}{\log \left( \frac{1}{\varepsilon} \right)} > \omega(z_n^*, \overline{\Delta_n \cap A_g(\varepsilon)}, \Delta_n \setminus A_g(\varepsilon)) = \omega(s_n^*, A_n, D \setminus A_n),
\]

where \( s_n^* = \varphi_n(z_n^*) \) and \( A_n = \varphi_n(\overline{\Delta_n \cap A_g(\varepsilon)}) \). If \( p_n \) denotes the circular projection onto the radius from 0 to \( T \) opposite to \( s_n^* \), then [Ah73, p. 43], [Ne53, p. 108]

\[
\omega(s_n^*, p_n(A_n), D \setminus p_n(A_n)) < \frac{M_1}{\log \left( \frac{1}{\varepsilon} \right)}.
\]
Since $s_n^* \in \varphi_n(\Delta(z_n, \rho)) = \{ |z| < \rho^* \}$ with $\rho^* < 1$ depending only on $\rho$, we see that the linear measure satisfies $|p_n(A_n)| < M_4/\log(1/\varepsilon)$. Since $\varphi_n(C \cap \Delta_n)$ connects 0 and $\mathbb{T}$, we conclude that

$$|\varphi_n(C \cap \Delta_n) \setminus A_n| \geq 1 - |p_n(A_n)| > 1 - \frac{M_4}{\log(1/\varepsilon)} > \frac{1}{2}$$

if $\varepsilon$ is chosen sufficiently small. It is easy to deduce that

$$|(C \cap \Delta_n) \setminus A_g(\varepsilon)| > c_1 (1 - |z_n|) > c_1 c_2 (1 - r_n)$$

by (4.14). Hence it follows from (4.1) that

$$\int_{C \cap \Delta_n} |g'(z)||dz| \geq \frac{\varepsilon}{2} \frac{\mu_g(r_{n+1})}{1 - r_n} |(C \cap \Delta_n) \setminus A_g(\varepsilon)| > \frac{\varepsilon}{2} c_2 \mu_g(r_{n+1}).$$

Since $\mu_g(r)$ is decreasing we have

$$\sum_n \mu_g(r_n) \geq c_1 \sum_n \int_{r_n}^{r_{n+1}} \frac{\mu_g(r)}{1 - r} dr = \infty$$

by (4.14) and (4.12). This implies (4.13).

The last assertion is an immediate consequence of (4.13) and the following proposition, where $g$ need not be a Bloch function.

**Proposition 4.5.** Let $g$ be analytic in $\mathbb{D}$ and suppose that (4.13) holds for any curve $C$ in $\mathbb{D}$ ending on $\mathbb{T}$. Then $g$ assumes every finite value infinitely often in $\mathbb{D}$.

**Proof.** a) For $w \in \mathbb{C}$ let $N(w) \leq \infty$ denote the number of zeros (with multiplicity) of $g - w$ in $\mathbb{D}$. Let $w, w' \in \mathbb{C}$ and let $L$ be a rectifiable Jordan arc from $w$ to $w'$ that does not meet $\{ g(z) : z \in \mathbb{D}, g(z) = 0 \}$ except possibly in $w$ and $w'$. At each point $z_k$ of $g^{-1}(\{ w \})$, we consider the maximal Jordan arcs $C_k$ in $g^{-1}(L)$ with initial point $z_k$; the number of these arcs is equal to the multiplicity of the zero $z_k$ of $g - w$. Therefore there are $N(w)$ arcs $C_k$ altogether.

The maximal arc $C_k$ ends either at some point $z'_k \in \mathbb{D}$ with $g(z'_k) = w'$ or approaches $\mathbb{T}$. The second case cannot arise by our assumption because $|g(C_k)| \leq |L| < \infty$. The number of points $z'_k$ that coincide is
equal to the multiplicity of $g - w'$ in $z_k'$. Hence $N(w') \geq N(w)$ and thus $N(w') = N(w)$ by symmetry. Thus we have shown

$$N(w) \equiv m \leq \infty, \quad \text{for } w \in \mathbb{C}. \tag{4.15}$$

b) Now we give a proof of the known fact that, for any function $g$ analytic in $D$, it is not possible that (4.15) holds with $m < \infty$. Let

$$r(\rho) = \sup \{|z| : |g(z)| = \rho\}, \quad 0 < \rho < \infty. \tag{4.16}$$

We claim that $r(\rho) < 1$. Otherwise there would exist $w$ with $|w| = \rho$ and points $z_n \in D$ with $|z_n| \to 1$ such that $g(z_n) \to w$. But $w$ is assumed $m$ times in $D$ so that there exist distinct $z_{nk}$ ($k = 1, \ldots, m$) with $g(z_{nk}) = g(z_n)$ and $z_{nk} \neq z_n$ for large $n$, which would imply $N(w) > m$.

It follows from (4.16) that $|g(z)| \neq \rho$ in $R(\rho) = \{r(\rho) < |z| < 1\}$. Since $g(R(\rho))$ is an unbounded domain we conclude that $|g(z)| > \rho$ for $z \in R(\rho)$ for any $\rho > 0$. Hence $|g(z)| \to \infty$ as $|z| \to 1$, which contradicts the Privalov uniqueness theorem.

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On radial behaviour and balanced Bloch functions


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