The Essential Singularity of the Solution of a Ramified Characteristic Cauchy Problem

By

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§0. Introduction

J. Leray [L] and L. Gårding, T. Kotake and J. Leray [G-K-L] have studied the singularities of the solution of a Cauchy problem with holomorphic data, when the initial surface includes some characteristic points. They have proved that the solution may be ramified around a hypersurface $K$.

Y. Hamada [H] has studied another class of characteristic Cauchy problem. In his case, the solution may have an essential singularity, although the data are regular.

Let $Pu = v$ be our equation. We already know that we must allow $u$ to be ramified or to have an essential singularity. Now that we understand this necessity, it would be desirable to allow $v$ to be singular without introducing a larger class for $u$.

[D] and [O-Y] are studies in this direction. They are generalizations of [L] and [G-K-L].

In the present paper, we consider a problem similar to the one in [H]. Although we impose a stronger condition on the operator $P$ than in [H], we assume a weaker condition on $v$: it is allowed to be singular. Moreover, by employing a symbol calculus like the one in [D], we can explain easily why $u$ has an essential singularity even for a holomorphic $v$.

§1. Statement of the Results

Let $S$ and $K$ be the hypersurfaces in $\mathbb{C}_x^2$ defined by $x_1 = x_2^q$ and $x_1 = 0$ respec-
tively, where \( q \) is an integer \( \geq 2 \). We introduce a class of the stalk of ramified functions at \( x=0 \), denoted by \( \mathcal{N}_{q,K} \). It is defined by

\[
f(x) \in \mathcal{N}_{q,K} \iff f(x) = \sum_{j=0}^{q-1} f_j(x) x_1^{j/q}, \quad f_j \text{ is holomorphic near } x=0.
\]

We set

\[
\mathcal{N}_{q,K}^l = \{ f(x) \in \mathcal{N}_{q,K} ; \ f \text{ vanishes on } S \text{ up to order } l \} \quad (l \geq 0).
\]

Moreover, we set

\[
\mathcal{N}_{q,K} = \sum_{j=0}^{q-1} x_1^{j/q} \lim_{x \to 0} \sigma(x) \chi(K).
\]

A function in \( \mathcal{N}_{q,K} \) may be ramified and have an essential singularity.

To formulate a Cauchy problem, we introduce

\[
\tilde{\mathcal{N}}_{q,K}^l = \{ f \in \mathcal{N}_{q,K} ; \ f \text{ vanishes on } S \text{ up to order } l \} \quad (l \geq 0).
\]

We have

**Theorem 1.** Let \( P(x, D) \) be a differential operator near the origin

\[
P(x, D) = D_1^{A_1} D_2^{A_2} \sum_{|\alpha| < A_1 + A_2} D^\alpha a_\alpha(x), \quad A_1 \geq 0, A_2 \geq 0
\]

where \( a_\alpha(x) \) is holomorphic near the origin and is a polynomial in \( x_1 \) and \( x_2 \). Then, for any element \( v(x) \) of \( D_1^{A_1} \mathcal{N}_{q,K}^{A_1} \), there exists a unique element \( u(x) \) of \( \tilde{\mathcal{N}}_{q,K}^{A_1+A_2} \) such that

\[
P u = v
\]

holds.

**Remark.** If \( \sum_{|\alpha| < A_1 + A_2} D^\alpha a_\alpha(x) \) is of order less than \( A_1 \) with respect to \( D_1 \), then \( P \) belongs to the class treated in \([O-Y]\) and the solution \( u \) is in \( \mathcal{N}_{q,K}^{A_1+A_2} \).

**Theorem 2.** ([O-Y]) Assume that \( A_1 \geq 1 \). Then

(A) \( x_1^{-A_1} \mathcal{N}_{q,K} \subset D_1^{A_1} \mathcal{N}_{q,K}^{A_1} \). Equality holds if \( A_1 = 1 \).
(B) \( x^{q+1} \notin D_1^{A_1} N_{q, x}^{A_1} \) if \( l \geq q \).

The proof of Theorem 2 is given in [O-Y]. In the following, we are going to prove Theorem 1.

§2. The Inverse of a Microdifferential Operator

We review the definition of microdifferential operators and formal norms. For details, see [K-K-K].

Definition 1. Let \( \Omega \) be a conic open set of \( T^*C^*_x \). We denote by \( \xi \) the dual variable of \( x \). Let \( P(x, \xi) \) be a formal sum of the following form:

\[
P(x, \xi) = \sum_{k=0}^{\infty} p_{m-k}(x, \xi),
\]

where \( p_{m-k}(x, \xi) \) is holomorphic in \( \Omega \) and is homogeneous of degree \( m-k \) with respect to \( \xi \). Then \( P(x, \xi) \) is said to be a microdifferential operator of order \( m \) in \( \Omega \) if it satisfies the following growth condition:

For an arbitrary compact subset \( K \) in \( \Omega \), there exists a positive constant \( C_K \) such that

\[
|p_{m-k}(x, \xi)| \leq C_K^{k+1} k!.
\]

We sometimes write \( P(x, \xi) \) as \( P(x, D) \).

The correspondence

\( \Omega \mapsto \{P(x, D); \ P \text{ is a microdifferential operator of order } m \text{ in } \Omega\} \)

forms a sheaf on \( T^*C^*_x \), which we denote by \( \mathcal{B}(m) \).

In the calculus of microdifferential operators, formal norms defined in [Bou-Kr] are very useful.

Definition 2. In the situation of Definition 1, the formal norm \( N_m^K(P; t) \) is a formal sum defined as

\[
N_m^K(P; t) = \sum_{k, \alpha, \beta} \frac{2(2\pi n)^{-k} k!}{(k! + k)!} \sup_K \|D_\xi^\beta D_\xi^\alpha p_{m-k}(x, \xi)\| t^{2k+|\alpha+\beta|},
\]
where the sum is taken with respect to \( k \in \mathbb{N}_0 = \{0, 1, 2, \ldots \} \), \( \alpha, \beta \in \mathbb{N}_0^n \).

**Remark.** If \( N_m^K (P; \varepsilon) < \infty \) holds for some \( \varepsilon > 0 \), then the growth condition (G) is satisfied. Conversely, if (G) is satisfied, then \( N_m^{K'} (P; \varepsilon) < \infty \) for some \( K' \subset K \) and \( \varepsilon > 0 \).

We quote two lemmas from [Y].

**Lemma 1.** (Lemma 10 of [Y]) Let \( R(x, D) \) be a microdifferential operator of order \( \leq -j < 0 \) defined in a neighborhood of a compact set \( \omega \subset T^* \mathbb{C}^n_\omega \), where \( j \) is a positive integer. Then we have

\[
N_0^\omega (R; t) \ll \frac{(2n)^{-j}}{j!} t^{2j} N_{-j}^\omega (R; t).
\]

**Proof.** By definition,

\[
N_0^\omega (R; t) = \sum_{k, \alpha, \beta} \frac{2 (2n)^{-k} \alpha! \beta!}{(|\alpha| + k)! (|\beta| + k)!} \sup_\omega |D_\alpha^\beta D_\xi^\gamma r_{-k} (x, \xi)| t^{2k + |\alpha + \beta|}.
\]

where \( R = \sum_{k \geq 0} r_{-k} \) and \( r_{-k} \) is the homogeneous part of degree \(-k\). There is no contribution by the terms corresponding to \( k = 0, 1, 2, \ldots, j - 1 \). Hence, if we put \( l = k - j \),

\[
N_0^\omega (R; t) = \sum_{l \geq 0, \alpha, \beta} \frac{2 (2n)^{-l} \alpha! \beta!}{(|\alpha| + l + j)! (|\beta| + l + j)!} \sup_\omega |D_\alpha^\beta D_\xi^\gamma r_{-l} (x, \xi)| t^{2(l+j) + |\alpha + \beta|}.
\]

We have only to prove that

\[
\frac{2 (2n)^{-l} \alpha! \beta!}{(|\alpha| + l + j)! (|\beta| + l + j)!} \leq \frac{(2n)^{-j}}{j!} \frac{2 (2n)^{-l} \alpha! \beta!}{(|\alpha| + l)! (|\beta| + l)!}.
\]

This inequality is obtained by the calculation below.

\[
\leq (2n)^{-j} \times \frac{1}{(|\alpha| + l + j) \cdots (|\alpha| + l + 1)} \times \frac{(|\beta| + l + j) \cdots (|\beta| + l + 1)}{(|\beta| + l)!}.
\]

\[
\leq (2n)^{-j} \times \frac{1}{|\beta| + l + 1} \times 1.
\]
Lemma 2. (A special case of Lemma 11 of [Y]) Let $Q$ be a microdifferential operator of order $\leq -1$. Then we have

$$N^\omega_0 (Q^j; t) \ll \frac{(2\pi)^{-j}}{j!} \mathbb{E} \{ N^\omega_1 Q; t \}^j.$$  

Proof. By [B-Kr], we have $N^\omega_{-j}(Q^j) \ll \{ N^\omega_1 (Q) \}^j$. Lemma 2 follows from Lemma 1. □

Now let us consider $P$ in Theorem 1. Define a microdifferential operator $\tilde{P}(x, D)$ by

$$\tilde{P}(x, D) = D^{-A_1} D^2^{-A_2} P(x, D).$$

Obviously we have

$$\tilde{P} = 1 - \sum_{|\alpha| < A_1 + A_2} D^{-A_1} D^2^{-A_2} D^\alpha a_\alpha(x),$$

and its adjoint $\tilde{P}^*$ is given by

$$\tilde{P}^*(x, D) = 1 - \sum_{|\alpha| < A_1 + A_2} a_\alpha(x) (-D_1)^{-A_1} (-D_2)^{-A_2} (-D)^\alpha.$$  

The summation is of order $\leq -1$. The inverse of $\tilde{P}^*$, which we denote by $R$, is calculated in terms of Neumann series:

$$R = (\tilde{P}^*)^{-1} = \sum_{j=0}^{\infty} Q(x, D)^j$$

where

$$Q(x, D) = \sum_{|\alpha| < A_1 + A_2} a_\alpha(x) (-D_1)^{-A_1} (-D_2)^{-A_2} (-D)^\alpha \in \mathcal{S}(-1).$$

Let $q_{jk}$ be the homogeneous term of degree $(-k)$ of $Q^j$: i.e.

$$Q(x, D)^j = \sum_{k=j}^{\infty} q_{jk}(x, D) \in \mathcal{S}(-j).$$
In fact, this is a finite sum as we will see later. By lemma 2 and the definition of the formal norm, we have

$$\frac{2(2n)^{-k} t^{2k}}{k!} \sup |q_{jk}| \leq \frac{(2n)^{-j}}{j!} t^{2j} (N_{-1}(Q; t))^j$$ if \( t > 0 \)

(For simplicity, we neglect to specify a compact set). Hence

$$|q_{jk}| \leq \frac{1}{2} (2n)^{-j+k} \frac{k!}{j!} t^{2(j-k)} (N_{-1}(Q; t))^j.$$

Next, we show the above-mentioned fact that \( Q^j = \sum_k q_{jk} \) is a finite sum. In fact, we have

**Lemma 3.** There exists a positive integer \( m \) independent of \( j \) such that \( Q^j \) consists of homogeneous terms of degree \(-j\), \(-(j+1), \ldots, -mj\).

**Proof.** A term of the form \( a(x) D_1^{r_1} D_2^{r_2} \cdots D_n^{r_n} \) is said to be of type \((s, -t)\), \( s \in \mathbb{N}_0, t \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}\), where \( a \) is a holomorphic function which is a polynomial in \( x_1 \) and \( x_2 \) of degree \( \leq s \) and \( \gamma_1 + \cdots + \gamma_n \geq -t \), \( \gamma_1 \in \mathbb{Z}, \gamma_2 \in \mathbb{Z}, \gamma_3 \in \mathbb{N}_0, \ldots, \gamma_n \in \mathbb{N}_0 \). (If \( s' \geq s \) and \( t' \geq t \), then a term of type \((s, -t)\) is also of type \((s', -t')\).

Let \( a_d \)'s be polynomials in \( x_1 \) and \( x_2 \) of degree \( \leq l \). Then \( Q \) consists of terms of type \((l, -A), A = A_1 + A_2\).

It is easy to see that if \( r_1(x, D) \) (resp. \( r_2(x, D) \)) is of type \((s_1, -t_1)\) (resp. \((s_2, -t_2)\)), then \( r_1(x, D) r_2(x, D) \) consists of terms of type \((s_1 + s_2, -t_1 - t_2)\), \((s_1 + s_2 - 1, -t_1 - t_2 - 1), \ldots, (0, -s_1 - s_2 - t_1 - t_2)\).

By induction, we can prove that \( Q^j \) consists of terms of type \((jl, -jA), \ldots, (0, -jl - jA)\). Combining this with the fact that \( \text{ord} \ Q^j \leq -j \), we obtain the lemma. \( \square \)

Let \( r_k(x, D) \) be the homogeneous term of degree \(-k\) of the operator \( R(x, D) = \tilde{P}^* (x, D)^{-1} = \sum_{j=0}^{\infty} Q(x, D)^j \). Then \( R = \sum_{k=0}^{\infty} r_k(x, D) \) and, by the lemma above,

$$r_k = \sum_{j=[k/m]}^{k} q_{jk}, \quad \text{where } \left\lfloor \frac{k}{m} \right\rfloor = \min \{ n \in \mathbb{N}_0; n \geq \frac{k}{m} \}.$$

We employ the estimate (1) to obtain

$$|r_k| \leq \sum_{j=[k/m]}^{k} |q_{jk}| \leq \sum_{j=[k/m]}^{k} \frac{1}{2} (2n)^{-j+k} \frac{k!}{j!} t^{2(j-k)} (N_{-1}(Q; t))^j.$$
By using
\[ \frac{1}{j!} \leq \frac{1}{[\frac{k}{m}]! (j-\frac{k}{m})!} , \]
we see that
\[ |r_k| \leq \frac{1}{2} (2n)^k k! \frac{1}{[\frac{k}{m}]!} t^{-2k} (2n)^{-\frac{k}{m}} \{ t^2 N_{-1}(Q; t) \}^{\frac{k}{m}} \]
\[ \times \sum_{j=\frac{k}{m}}^k (2n)^{-(j-\frac{k}{m})} \frac{1}{(j-\frac{k}{m})!} \{ t^2 N_{-1}(Q; t) \}^{j-\frac{k}{m}} \]
\[ \leq \frac{1}{2} (2n)^k k! \frac{1}{[\frac{k}{m}]!} t^{-2k} (2n)^{-\frac{k}{m}} \{ t^2 N_{-1}(Q; t) \}^{\frac{k}{m}} \]
\[ \times \exp \left( \frac{1}{2n} t^2 N_{-1}(Q; t) \right) . \]

Therefore, for any compact set \( \omega \) of \( \{ x \in C^n; |x| \ll 1 \} \times \{ \xi; \xi_1 \neq 0, \xi_2 \neq 0 \} \subset T^*C^n \), there exists a positive constant \( C_\omega \) independent of \( k \) such that
\[ (2) \quad \sup_{\omega} |r_k(x, \xi)| \leq C_\omega^{k+1} \frac{1}{[\frac{k}{m}]!} . \]

Here \( |x| \ll 1 \) means that \( |x| \) is sufficiently small. Now set
\[ r_k(x, D) = \sum_{|\beta|=\CD, k} b_\beta(x) D^\beta \in \mathcal{D} (\{ |x| \ll 1 \} \times \{ \xi_1 \neq 0, \xi_2 \neq 0 \}) . \]

Let us obtain an estimate on \( b_\beta(x) \) when \( \beta_1 > 0 (\Rightarrow \beta_2 < 0) \). Remark that the partial sum
\[ \sum_{k \geq 0} \sum_{|\beta|=\CD, k \beta_1 \leq 0} b_\beta(x) D^\beta \]
belongs to the class \( \mathcal{E}_K \) of \([D]\), and it is already well understood.

Since
\[ b_\beta(x) = \frac{1}{(2\pi i)^{n-1}} \oint_{|\xi_1|=\delta} \oint_{|\xi_2|=\delta} \cdots \oint_{|\xi_n|=\delta} \xi_2^{-\beta_2-1} \xi_3^{-\beta_3-1} \cdots \xi_n^{-\beta_n-1} \]
\[ \times \sum_{k \beta_1 \leq 0} r_k(x; 1, \xi_2, \xi_3, \ldots, \xi_n) d\xi_2 d\xi_3 \cdots d\xi_n , \]
we obtain, owing to (2)
where $C_{\delta, \delta'}$ is a positive constant independent of $k$.

Before concluding this section, we remark that

$$\tilde{P}^{-1} = R^* = \sum_{k=0}^{\infty} \{v_k(x, D)\}^* = \sum_{k=0}^{\infty} \sum_{|\beta|=k} (-D)^\beta v_\beta(x).$$

### §3. Some Preparation

**Lemma 4.**

$$\left(\frac{1}{z^{g-1}}D_z\right)^j = \frac{1}{z^j} \{\theta - q(j-1)\} \cdots \{\theta - q\} \theta, \quad j \geq 1$$

where $\theta = zD_z$.

**Proof.** One has

$$\theta \frac{1}{z^{k}} z^{k} = \frac{1}{z^{k}} \theta - z \frac{k}{z^{k+1}} = \frac{1}{z^{k}} (\theta - k).$$

The lemma is proved by induction. \(\square\)

**Lemma 5.** Let $j$ be a positive integer. We have for $0 < y < 1$,

$$\sum_{k=0}^{\infty} \frac{(k+q(j-1)) \cdots (k+q) ky^k}{\text{j factors}} \leq \frac{j^j y^j}{(1-y) (y^{j-1} (1-y))^j}$$

**Proof.** In fact,

$$\sum_{k=0}^{\infty} \frac{(k+q(j-1)) \cdots (k+q) ky^k}{\text{j factors}} \leq \sum_{k=0}^{\infty} \frac{(k+q(j-1)) \cdots (k+q(j-1)) (q-1) y^k}{\text{j factors}}$$

$$= \frac{1}{y^{\delta'-\delta-j}} \frac{d^j}{dy^j} \sum_{k=0}^{\infty} y^{k+q(j-1)}$$
Lemma 6. Let \( f(z) \) be a holomorphic function in \( \{ z \in \mathbb{C}; |z| < r + \varepsilon \}, \ r > 0, \ \varepsilon > 0 \). If \( |f(z)| \leq M \) holds in \( \{ z \in \mathbb{C}; |z| \leq r \} \) then we have, in \( \{ z \in \mathbb{C}; 0 < |z| < r \} \),

\[
\left| \left( \frac{1}{z^{q-1}D_z} \right)^j f(z) \right| \leq M \frac{j! \left( \frac{|z|}{r} \right)^q}{\left( 1 - \frac{|z|}{r} \right)^j \left( \frac{|z|}{r} \right)^{q-1} \left( 1 - \frac{|z|}{r} \right)^j}.
\]

Proof. Let the Taylor expansion of \( f \) be

\[
f(z) = \sum_{k=0}^{\infty} f_k z^k.
\]

Then we have

\[
f_k = \frac{1}{2\pi i} \oint_{|z|=r} \frac{f(z)}{z^{k+1}} \, dz, \quad |f_k| \leq Mr^{-k}.
\]

By using Lemma 4 we see that

\[
\left( \frac{1}{z^{q-1}D_z} \right)^j f(z) = \sum_{k=0}^{\infty} f_k \frac{1}{z^j} \left( kq(j-1) \right) \cdots \left( k-q \right) |z|^k.
\]

The series in the right hand side is estimated by Lemma 5. We obtain

\[
\left| \left( \frac{1}{z^{q-1}D_z} \right)^j f(z) \right| \leq \sum_{k=0}^{\infty} Mr^{-k} \frac{1}{|z|^j} \left( k+q(j-1) \right) \cdots \left( k+q \right) |z|^k
\]

\[
= \frac{M}{|z|^j} \frac{j! \left( \frac{|z|}{r} \right)^q}{\left( 1 - \frac{|z|}{r} \right)^j \left( \frac{|z|}{r} \right)^{q-1} \left( 1 - \frac{|z|}{r} \right)^j}.
\]
§4. The Action of a Microdifferential Operator on a Ramified Function

For the study of $\mathcal{N}_{q,K}$, we introduce a singular coordinate change $z = x_1^{1/q}$.

We denote by $\tilde{S}$ the hypersurface of $C^n_{x,x_2}$ defined by $z = x_2$. Here $x' = (x_3, \cdots, x_n)$.

The singular coordinate change induces an isomorphism

$$\mathcal{N}_{q,K} \cong \mathcal{O}_{(x,x_2,x')} = 0$$

$$f(x) = \sum_{j=0}^{q-1} f_j(x) x^{j'/q} \mapsto \tilde{f}(z, x_2, x') = \sum_{j=0}^{q-1} f_j(x', x_2, x') z^j.$$

Moreover $f \in \mathcal{N}_{q,K}^l$ if and only if $\tilde{f}$ vanishes on $\tilde{S}$ up to order $l$.

**Proposition 1.** ([D]) Proposition 6) The characteristic Cauchy problem

$$D_2 g = f \in \mathcal{N}_{q,K}^l$$

admits a unique solution $g \in \mathcal{N}_{q,K}^{l+1}$. Moreover, if we have

$$|\tilde{f}(z, x_2, x')| \leq M(|z| + |x_2 - z|)^m$$

for some positive constant $M$ and a non-negative integer $m$, then

$$|\tilde{g}(z, x_2, x')| \leq \frac{M}{m+1} (|z| + |x_2 - z|)^{m+1}.$$

**Proof.** The equation $D_2 \tilde{g} = f$ is equivalent to $D_2 \tilde{f} = \tilde{f}$, and the initial surface $S$ is transformed into $\tilde{S}$. Since $\tilde{S}$ is noncharacteristic, we can find a unique holomorphic solution $\tilde{g}$. The estimate is obtained by an elementary integral representation. $\square$

This proposition suggests that $\mathcal{N}_{q,K}$ and its variants are more suitable classes for the study of characteristic Cauchy problems than that of holomorphic functions.

**Definition 3.** We can define

$$D_2^{-1} : \mathcal{N}_{q,K}^l \rightarrow \mathcal{N}_{q,K}^{l+1}$$

by using the proposition above. It is a right inverse of
RAMIFIED CHARACTERISTIC CAUCHY PROBLEM

\[ D_2 : \mathcal{N}_{q,K}^{l+1} \rightarrow \mathcal{N}_{q,K}^l \]

but it is not a left inverse.

Remark. If \( u \) is an element of \( \mathcal{N}_{q,K} \) and \( f \) is holomorphic near \( x = 0 \), then we can define \( D_2^{-l}(f(x)u(x)) \), \( l \in \mathbb{N}_0 \). It is the unique solution of the Cauchy problem

\[
\begin{cases}
D_2 w(x) = f(x)u(x) \\
w(x) \in \mathcal{N}_{q,K}^l.
\end{cases}
\]

On the other hand, \( D_2^{-l} \circ f(x) \) belongs to the symbol class \( \mathfrak{S}_K \) in \([D]\), and \((D_2^{-l} \circ f(x)) u(x) \in \mathcal{N}_{q,K}^l\) is defined in \([D]\). Dunau puts integration on the right:

\[
D_2^{-l} f(x) = f(x)D_2^{-l} + \sum_{j=l+1}^{\infty} f_j(x)D_2^{-j}
\]

for some \( f_j(x) \). He sets

\[
(D_2^{-l} \circ f(x)) u(x) = f(x)D_2^{-l} u(x) + \sum_{j=l+1}^{\infty} f_j(x)D_2^{-j} u(x).
\]

It satisfies the same equation as above and we see that

\[
D_2^{-l}(f(x)u(x)) = (D_2^{-l} \circ f(x)) u(x).
\]

So it makes no difference whether integration comes on the left or on the right.

Now we are ready to define \( \tilde{P}(x, D)^{-1}w(x) \in \tilde{\mathcal{N}}_{q,K} \), where \( \tilde{P} \) is as in the second section and \( w(x) \in \mathcal{N}_{q,K} \).

\( \tilde{P}^{-1} \) has the expression

\[
\tilde{P}^{-1} = \sum_{k=0}^{\infty} \sum_{|\beta| = -k} (-D)^{\beta} b_\beta(x) \in \mathfrak{B}(\|x\| \ll 1, \xi_1 \neq 0, \xi_2 \neq 0), \text{ ord } \tilde{P}^{-1} \leq 0.
\]

The partial sum consisting of the terms corresponding to \( \beta_1 \leq 0 \) belongs to Dunau’s class \( \mathfrak{S}_K \) and its action on \( \mathcal{N}_{q,K} \) is defined in \([D]\). Therefore, in order to define the action of \( \tilde{P}^{-1} \), we may assume without loss of generality that \( b_\beta \equiv 0 \) if \( \beta_1 \leq 0 \). This means that \( \beta_2 < 0 \) in the sum.
We set

\[ \widetilde{P}^{-1} (x, D) \varphi (x) = \sum_{k=0}^{\infty} \sum_{|\beta|=-k} (-D)^{\beta} b_\beta (x) \varphi (x). \]

We are going to prove that it defines an element of \( \mathcal{N}_{q, \mathbb{K}} \). Put \( x^{1/q} = z, \)
\( \tilde{w} (z, x_2, x') = \varphi (z', x_2, x'), \) and \( \tilde{b}_\beta (z, x_2, x') = b_\beta (z', x_2, x'). \) Then

\[ (\widetilde{P}^{-1} \varphi) (x) = \sum_{k=0}^{\infty} \sum_{|\beta|=-k} \left( \frac{1}{q z^{d-1} D_z} \right)^{\beta_1} D_\beta^2 D^\beta' \cdot (-1)^{|\beta|} \tilde{b}_\beta (z, x_2, x') \tilde{w} (z, x_2, x'). \]

(3) in the second section implies that in a neighborhood \( X \) of \( (z, x_2, x') = 0, \) we have

\[ \left| (-1)^{|\beta|} \tilde{b}_\beta \tilde{w} \right| \leq C^{k+1}_{\delta_2, \delta'} \cdot \frac{k!}{[\frac{k}{m}]!} \delta_2^{-\delta_2} \delta'^{-|\delta'|} \sup_X |\tilde{w}|, \quad |\beta| = -k. \]

In a smaller neighborhood, there exists a positive constant \( r' > 0 \) such that

\[ \left| D^\beta' \cdot (-1)^{|\beta|} \tilde{b}_\beta \tilde{w} \right| \leq \beta'! r'^{-|\beta'|} C^{k+1}_{\delta, \delta'} \cdot \frac{k!}{[\frac{k}{m}]!} \delta_2^{-\delta_2} \delta'^{-|\delta'|} \sup_X |\tilde{w}|. \]

Then, we employ Proposition 1 repeatedly, first for \( m = 0, \) next for \( m = 1 \) and so on. We obtain

\[ \left| D_\beta^2 D'^\beta \cdot (-1)^{|\beta|} \tilde{b}_\beta \tilde{w} \right| \leq \frac{\lambda^{-\delta_2}}{(-\beta_2)!} \beta'! r'^{-|\delta'|} C^{k+1}_{\delta, \delta'} \cdot \frac{k!}{[\frac{k}{m}]!} \delta_2^{-\delta_2} \delta'^{-|\delta'|} \sup_X |\tilde{w}| \]

in \( \{|z| < \lambda/3, |x_2| < \lambda/3, \ldots, |x_m| < \lambda/3\}. \)

By using Lemma 6, we see that

(4)

\[ \left| \left( \frac{1}{q z^{d-1} D_z} \right)^{\beta_1} D_\beta^2 D'^\beta \cdot (-1)^{|\beta|} \tilde{b}_\beta \tilde{w} \right| \]

\[ \leq \left( \frac{\lambda^{-\delta_2}}{(-\beta_2)!} \beta'! r'^{-|\delta'|} C^{k+1}_{\delta, \delta'} \cdot \frac{k!}{[\frac{k}{m}]!} \delta_2^{-\delta_2} \delta'^{-|\delta'|} \sup_X |\tilde{w}| \right)^{\beta_1} \]

\[ \times \frac{\lambda^{-\delta_2}}{(-\beta_2)!} \beta'! r'^{-|\delta'|} C^{k+1}_{\delta, \delta'} \cdot \frac{k!}{[\frac{k}{m}]!} \delta_2^{-\delta_2} \delta'^{-|\delta'|} \sup_X |\tilde{w}| \]
in \( \{0 < |z| < r < \lambda/3, \, |x_2| < \lambda/3, \ldots, \, |x_n| < \lambda/3\} \).

There exists a constant \( C_z > 1 \) depending continuously on \( |z|, \, 0 < |z| < r \), such that \( \left\{ q |z|^q (|z| / r)^{q-1} (1 - |z| / r)^{-1} \right\} \leq C_z \). We have

\[
\frac{1}{q |z|^q (|z| / r)^{q-1} (1 - |z| / r)^{-1}} \leq C_z^{\beta_1} \leq C_z^{\beta_1 + |\beta'| + k} = C_z^{-\beta_2}.
\]

Moreover, if we take \( \delta' > 0 \) so small that \( r' \delta' < 1 \), then

\[
(r' \delta')^{-|\beta'|} \leq (r' \delta')^{-|\beta'| - \beta_1 - k} = (r' \delta')^{\delta_2}.
\]

In addition, it is easy to see that

\[
\frac{\beta_1 |\beta'| k!}{(-\beta_2)} \leq 1
\]

because \( \beta_1 + |\beta'| + k = -\beta_2 \). Combining (4) with these three inequalities, we obtain

\[
\left| \left( \frac{1}{q z^{q-1}} D_z \right)^{\beta_1} D_z^{|\beta'|} D' \beta' \cdot (-1)^{|\beta|} \overline{b}_{\beta} \overline{w} \right| \leq \frac{|\beta| q}{(1 - |z| / r)} \left[ \frac{k}{m} \right] ! \sup_X |\overline{w}|^r C_z^{k+1} \delta_{\delta_2, \delta'} \sup_X |\overline{w}|.
\]

For fixed \( k \) and \( \beta_2 \), we have

\[
\# \{ (\beta_1, \beta') ; \beta_1 > 0, \beta' \in \mathbb{N}^{n-2}, \beta_1 + \beta_2 + |\beta'| = -k \} \leq 2^{n-2-k-\beta_2}.
\]

Hence,

\[
\left| \sum_{k \geq 0} \sum_{|\beta| = -k} \left( \frac{1}{q z^{q-1}} D_z \right)^{\beta_1} D_z^{|\beta'|} D' \beta' \cdot (-1)^{|\beta|} \overline{b}_{\beta} \overline{w} \right| \leq \frac{|\beta| q}{(1 - |z| / r)} \sum_{k \geq 0} \frac{C_z^{k+1}}{[\frac{k}{m}] !} \sum_{\beta_1 + |\beta'| = -k - \beta_2} \sum_{\beta_2 = -\infty} (-2 \cdot C_z^{\delta_2})^{-\beta_2} \sup_X |\overline{w}|^r \leq 2^{n-2-k-\beta_2}.
\]
The right hand side converges on every compact set of \{0 < |z| \ll 1, |x_2| \ll 1, \ldots, |x_n| \ll 1\} if we take a sufficiently small \( \delta > 0 \) in accordance with the compact set.

Summing up, we have finally proved that

\[
(\tilde{P}^{-1}w)(x) \in \tilde{N}_{q,K}.
\]

Moreover, if \( w \in \mathcal{N}_{q,K}^{A_1+A_2} \), then it is easy to see that

\[
(\tilde{P}^{-1}w)(x) \in \tilde{N}_{q,K}^{A_1+A_2}.
\]

\section{Proof of Theorem 1}

First, remark that

\[
D_2^{A_2} : \mathcal{N}_{q,K}^{A_1+A_2} \rightarrow \mathcal{N}_{q,K}^{A_1}.
\]

Hence

\[
D_1^{A_1} \mathcal{N}_{q,K}^{A_1} = D_1^{A_1} D_2^{A_2} \mathcal{N}_{q,K}^{A_1+A_2}.
\]

Let us solve \( Pu = D_1^{A_1} D_2^{A_2} w, w \in \mathcal{N}_{q,K}^{A_1+A_2} \). The solution \( u \) is given by \( u = \tilde{P}^{-1}w \in \tilde{N}_{q,K}^{A_1+A_2} \). In fact,

\[
P u = P(\tilde{P}^{-1}w) = D_1^{A_1} D_2^{A_2} w
\]

holds.

The uniqueness is a consequence of Cauchy-Kowalevski theorem, which we apply at noncharacteristic points.

\section{Hamada's Example}

Hamada ([H]) gave the following example.

\[
\begin{cases}
(D_2^2 - D_1) u(x) = 0 \\
u|_s = \gamma_1 x_2, \ D_1 u|_s = \gamma_2 x_2
\end{cases}
\]
where

\[ S = (x_1 = x_2^2), \quad \gamma_1 = \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(m - \frac{3}{2})}{(2m)!}, \quad \gamma_2 = \sum_{m=0}^{\infty} (-1)^{m+1} \frac{\Gamma(m - \frac{1}{2})}{(2m)!}. \]

The solution \( u(x) \) is given by

\[ u(x) = \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(m - \frac{3}{2})}{(2m)!} x_1^{\frac{3}{2}-m} x_2^{2m}. \]

It is ramified and has an essential singularity. Let us interpret this phenomenon from our viewpoint. First we reduce the problem to the following one.

\[
\begin{cases}
(D_2^2 - D_1) u(x) = v(x), & v \in \mathcal{O}_{x=0} \text{ is given}, \\
|u| = 0, & D_1 |u| = 0.
\end{cases}
\]

By using

\[
(D_2^2 - D_1)^{-1} = (1 - D_1 D_2^{-2})^{-1} D_2^{-2} = \sum_{j=0}^{\infty} (D_1 D_2^{-2})^j D_2^{-2} = \sum_{j=0}^{\infty} D_1^j D_2^{-2j-2},
\]

we can express the solution by

\[ u(x) = \sum_{j=0}^{\infty} D_1^j D_2^{-2j-2} v(x). \]

Put \( z = x_1^{1/2} \). Then we obtain

\[ u(x^2, x_2, x') = \sum_{j=0}^{\infty} \left( \frac{1}{2z} D_z \right)^j D_z^{-2j-2} v(x^2, x_2, x'). \]

Ramification is caused by \( D_z^{-2j-2} \). The essential singularity appears because of the factor \( \left( \frac{1}{2z} D_z \right)^j \).
References


