Some Inequalities for Generalized Commutators

By

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Abstract

Let $A$, $B$ be linear operators on a Banach space $\mathcal{H}$ with spectra in the set $S = \mathbb{C}\setminus(0, \infty)$, for which

$$
\|t(A + t)^{-1}\| \leq M, \quad \|t(B + t)^{-1}\| \leq N \quad (t > 0).
$$

Then for a certain class of holomorphic functions $f$ preserving $S$ and $f(0) = \lim_{s \to 0^+} f(s) = 0$, one has

$$
\|f(A)X - Xf(B)\| \leq a f(b \|AX - XB\|) \quad \text{for all } X \in B(\mathcal{H}), \quad \|X\| \leq 1,
$$

where $a = 2(M + N)$, $b = MN/(M + N)$. At that, if $\alpha \in \mathbb{C}$, $0 < \Re \alpha \leq 1$, then for all $X \in B(\mathcal{H})$

$$
\|A^\alpha X - XB^\alpha\| \leq C(M, N, \alpha) \|X\|^{1 - \Re \alpha} \|AX - XB\|^{\Re \alpha}
$$

where

$$
C(M, N, \alpha) = \frac{\sin \pi \alpha (MN)^{\Re \alpha} (M + N)^{1 - \Re \alpha}}{\pi \Re \alpha (1 - \Re \alpha)}.
$$

Also

$$
\|\exp(-tA^{1/2})X - X\exp(-tB^{1/2})\| \leq C(M, N) t^{2/3} \|X\|^{2/3} \|AX - XB\|^{1/3},
$$

where

$$
C(M, N) = 3 \pi^{-1} (MN)^{1/3} (M + N)^{2/3}, \quad t \geq 0, \quad X \in B(\mathcal{H}).
$$

§ 1. Introduction

Recently F. Kittaneh and H. Kosaki [4] obtained the interesting inequalities:

Let $A$, $B$ be two positive operators on a Hilbert space $\mathcal{H}$ and $f$ – an operator monotone function on $(0, \infty)$. Then

(1.1) If $\lim_{s \to 0^+} f(s) = 0$, one has

$$
\|f(A) - f(B)\| \leq f(\|A - B\|),
$$

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(1.2) If $A \geq a \geq 0$ and $B \geq b \geq 0$, one has for all $X \in B(\mathcal{H})$
$$\|f(A)X - Xf(B)\|_p \leq C(a, b)\|AX - XB\|_p, \, 1 \leq p \leq \infty,$$
where
$$C(a, b) = \begin{cases} \frac{f(a) - f(b)}{a - b} & a \neq b \\ f'(a) & a = b \end{cases}.$$ 

In particular, when $f(s) = s^\alpha, \, 0 < \alpha \leq 1,$ and $A \geq c > 0, \, B \geq c > 0,$ (1.2) turns into

$$\|A^{\alpha}X - XB^{\alpha}\|_p \leq c^{\alpha - 1}\|AX - XB\|_p \quad (X \in B(\mathcal{H})).$$

Unfortunately, when $c = 0$, or $a = b = 0$ and $f'(a) = \infty$ in (1.2), it is impossible to estimate $\|f(A)X - Xf(B)\|_p$ in terms of $\|AX - XB\|_p$ there even for $p = \infty$.

Such an estimate is sometimes important. For instance, in another development—for use in $C^*$-algebra theory, W. Arveson proved the following result (see [1], the Lemma on p. 332)

(1.4) Let $\mathcal{U}$ be a $C^*$-algebra, $f$ a continuous function on $[0, 1]$ (if $\mathcal{U}$ has no unit, one assumes $f(0) = 0$) and let $\varepsilon > 0$. There exists $\delta > 0$ such that any time when $a, x$ are in the unit ball of $\mathcal{U}$ and $a \neq 0$, one has
$$\|ax - xa\| < \delta \quad \text{implies} \quad \|f(a)x - xf(a)\| < \varepsilon.$$ 

For $f(s) = s^\alpha, \, 0 < \alpha < 1,$ a more precise estimate was found in the paper [7] (Lemma 2.1):

(1.5) If $a, x$ are elements in a $C^*$-algebra and $a \geq 0$, then for any $\alpha, \, 0 < \alpha < 1$ one has
$$\|a^\alpha x - xa^\alpha\| \leq (1 - \alpha)^{\alpha - 1}\|x\|^{1 - \alpha}\|ax - xa\|^\alpha.$$ 

As mentioned in the remarks on p. 4 of [7], U. Haagerup has reduced the constant $(1 - \alpha)^{\alpha - 1}$ in (1.5) to $\sin^\alpha(\pi\alpha(1 - \alpha)^{-1})$. We have come to this result independently and we present the refined inequality here in a general setting—see below (2.6).

The aim of these notes is to describe a method of obtaining inequalities for generalized commutators and to illustrate it by some examples, thus complementing the results of Kittaneh-Kosaki. The inequalities here are stated for Banach space operators, although Banach algebra elements could be used also. The operator framework keeps in line with the notations in Kittaneh-Kosaki's paper and makes it possible to consider unbounded operators as well.
§ 2. Inequalities

We consider operator monotone functions $f$ on $[0, \infty)$ of the form

\begin{align}
(2.1) \quad f(z) &= kz + \int_0^\infty \frac{z}{z + t} d\mu(t), \quad k \geq 0, \\
f(0) &= \lim_{s \to 0^+} f(s) = 0, \quad \int_1^\infty \frac{d\mu(t)}{t} < \infty, \quad \mu(0) = 0,
\end{align}

and for the monotone increasing function $\mu$ we assume also that $\mu'$ exists and is positive in $(0, \infty)$. We also consider a pair of two bounded linear operators $A, B$ on a complex Banach space $\mathcal{H}$ satisfying

\begin{align}
(2.2) \quad \text{Sp}(A), \text{Sp}(B) \subset \mathbb{C} \setminus (-\infty, 0), \\
\|t(A + t)^{-1}\| \leq M, \quad \|t(B + t)^{-1}\| \leq N \quad \text{for all} \ t > 0.
\end{align}

For such functions $f$ and operators $A, B$ one can define

\begin{align}
\tag{2.3}
\|f(A)X - Xf(B)\| &= \|kAX - XB\| + \int_0^s t(X(B + t)^{-1} - (A + t)^{-1}X) d\mu(t) \\
&\quad + \int_s^\infty t(A + t)^{-1}(AX - XB)(B + t)^{-1} d\mu(t) \\
&\leq kc + (M + N)\mu(s) + cMN \int_s^\infty \frac{d\mu(t)}{t}.
\end{align}

This last expression takes its minimum for $s \geq 0$ in $s = \lambda = cMN/(M + N)$, as its derivative with respect to $s$ for $s > 0$ is

\begin{align}
\mu'(s)(M + N - cMNs^{-1}).
\end{align}

Let $a = 2(M + N)$ and let $MN \geq 1/2$ or $k = 0$. One easily checks that
This way we have proved the theorem:

**Theorem 2.1.** Let $A, B$ be two operators on the Banach space $\mathcal{H}$ satisfying (2.2). Then for every function $f$ of the form (2.1) and every $X \in B(\mathcal{H})$, $\|X\| \leq 1$:

$$\|f(A)X - Xf(B)\| \leq a\|AX - XB\|$$

where $a = 2(M + N)$, $b = MN/(M + N)$ and $MN \geq 1/2$ or $k = 0$.

Note that when $MN \geq 1$ and

$$\int_0^\infty d\mu(t)/t < \infty,$$ i.e. $f'(0)$ is finite, it follows from (2.3) for $s = 0$:

$$\|f(A)X - Xf(B)\| \leq f'(0)MN\|AX - XB\|$$

for all $X \in B(\mathcal{H})$ in accordance with (1.2).

We shall present now one variety of (2.4) which is of particular interest. Starting from the representation

$$z^s = \frac{\sin z\pi}{\pi} \int_0^\infty \frac{z}{z + t} t^{s-1} dt, \quad z \in \mathbb{C} \setminus (-\infty, 0), \quad 0 < \text{Re} z \leq 1$$

one can define the fractional powers

$$A^s = \frac{\sin z\pi}{\pi} \int_0^\infty (A + t)^{-1} A t^{s-1} dt$$

and in the same way $B^s$ (see [2]).

**Proposition 2.2.** For $A, B$ satisfying (2.2), $0 < \text{Re} z \leq 1$, and all $X \in B(\mathcal{H})$ one has

$$\|A^s X - XB^s\| \leq C(M, N, z)\|X\|^{1 - \text{Re} z}\|AX - XB\|^{\text{Re} z}$$

where $C(M, N, z) = \frac{|\sin z\pi|}{\pi |\text{Re} z(1 - \text{Re} z)|} (MN)^{\text{Re} z}(M + N)^{1 - \text{Re} z}$

**Proof.** Proceeding as in (2.3) we find for every $s > 0$ and $X \in B(\mathcal{H})$

$$\|A^s X - XB^s\|$$

$$\leq \frac{|\sin z\pi|}{\pi} \left( \|X\|(M + N) \int_0^s |t^{s-1}| dt + \|AX - XB\|MN \int_s^\infty |t^{s-2}| dt \right)$$
\[ |\sin x\pi| = \frac{1}{\pi} \left( \|X\|(M + N)_{s,Rez} + \|AX - XB\|MN_{s,Rez-1} \right) \quad (as \ |t'| \leq t^{Rez}). \]

Minimizing the right hand side for \( s > 0 \) we get (2.6).

**Proposition 2.3.** Let \( H \) be a Hilbert space, \( \| \cdot \|_p - the Schatten norm, 1 \leq p \leq \infty, f - a function as in (2.1) and \( A, B \in B(\mathcal{H}) - operators satisfying (2.2). Then one has

\[
\|f(A)X - Xf(B)\|_p \leq af(b \|AX - XB\|_p) \quad for all \ X \in B(\mathcal{H}), \|X\|_p \leq 1,
\]

where \( a = 2(M + N), b = MN/(M + N) \) and either \( MN \geq 1/2 \) or \( k = 0 \).

Also

\[
\|A^*X - XB^*\|_p \leq C(M, N, a)\|X\|^{1-Rez}_p \|AX - XB\|_p^{Rez}
\]

for all \( X \in B(\mathcal{H}), \|X\|_p < \infty, 0 < Rez \leq 1 \) and \( C(M, N, a) \) as in (2.6).

**Proof.** We repeat the proofs of (2.4) and (2.6), using \( \| \cdot \|_p \) instead of \( \| \cdot \| \) and estimating on the right hand sides of the inequalities in the following way:

\[
\|X(B + t)^{-1} - (A + t)^{-1}X\|_p \leq (M + N)\|X\|_p t^{-1},
\]

\[
\|(A + t)^{-1}(AX - XB)(B + t)^{-1}\|_p \leq MN\|AX - XB\|_p t^{-2}.
\]

**Remarks.** The technique used in the above proofs—dividing the integral in two parts on \([0, s]\) and \([s, \infty)\) estimated in different ways and then minimizing for \( s > 0 \)—is not new. It has been used, for instance, by Matsaev and Palant [6] for obtaining the inequality

\[
\|A^* - B^*\| \leq C(M, N, a)\|A - B\|_p\]

\( (0 < a \leq 1, C(M, N, a) as in (2.6)) \)

for operators \( A, B \) essentially as in (2.2).

This method has been used also for proving moment type inequalities

\[
\|A^*\| \leq C(M, a)\|X\|^{1-a}_p \|Ax\|_p
\]

\( (0 < a \leq 1, C(M, a) = C(M, 1, a)) \)

for \( A \) as in (2.2)—see [5], [8].

Note that Matsaev-Palant's inequality (2.9) follows immediately from (2.6) by putting there \( X = 1 = identity operator. In the same way we obtain from (2.4) the immediate corollary:

**Corollary 2.4.** Let \( f \) be as in (2.1) and let \( A, B \) satisfy (2.2). Then

\[
\|f(A) - f(B)\| \leq af(b \|A - B\|)
\]
(a, b as in (2.4)), which is a Banach space variety of (1.1).

In this connection see also Theorem 3.4 in [4]. A natural modification of the proof turns it into a theorem for generalized commutators.

§ 3. Further Inequalities

We want to point out that many functions of the form

\[ f(z) = \int_{0}^{\infty} \frac{z}{z + t} g(t) dt, \text{ or } f(z) = \int_{0}^{\infty} \frac{h(t)}{z + t} dt, \quad z \in \mathbb{C}\setminus(-\infty, 0), \]

with explicitly given \( g, h \) can successfully be used for obtaining moment type inequalities for generalized commutators via the above method. To illustrate this we shall present the following example:

\[ \exp(-tz^{1/2}) = \frac{1}{\pi} \int_{0}^{\infty} \sin(t\sqrt{\lambda}) \frac{d\lambda}{z + \lambda} \quad (t \geq 0) \text{ (see [2], [10])}. \]

It is well-known that if \( A \) is a closed linear operator satisfying (2.2), its square root \( A^{1/2} \) (defined as in Section 2) is the generator of a (holomorphic) one-parameter operator semigroup which can be determined by the formula

\[ \exp(-tA^{1/2}) = \frac{1}{\pi} \int_{0}^{\infty} (A + \lambda)^{-1} \sin(t\sqrt{\lambda}) d\lambda, \quad t \geq 0, \]

(3.1)

(see [2], Section 5).

Proposition 3.1. Let \( A, B \) be closed (possibly unbounded) linear operators on the Banach space \( \mathcal{H} \) satisfying (2.2). Then for any \( X \in B(\mathcal{H}) \) for which \( AX - XB \) is bounded, we have

\[ \| \exp(-tA^{1/2})X - X\exp(-tB^{1/2}) \| \leq C(M, N)t^{2/3}\|X\|^{2/3}\|AX - XB\|^{1/3}, \]

\[ C(M, N) = 3\pi^{-1}(MN)^{1/3}(M + N)^{2/3}, \quad t \geq 0. \]

Proof. For all \( s > 0 \):

\[ \pi\| \exp(-tA^{1/2})X - X\exp(-tB^{1/2}) \|
\]

\[ \leq t \int_{0}^{s} \left| \frac{\sin(t\sqrt{\lambda})}{t\sqrt{\lambda}} \right| (A + \lambda)^{-1} \|X - (B + \lambda)^{-1}\| d\lambda
\]

\[ + \int_{s}^{\infty} \left| \sin(t\sqrt{\lambda}) \right| (A + \lambda)^{-1}(AX - XB)(B + \lambda)^{-1}\| d\lambda
\]

\[ \leq 2t\|X\|(M + N)\sqrt{s} + MN\|AX - XB\|s^{-1}. \]

Minimizing this for \( s > 0 \) we come to (3.2).
References
