The Squaring Operations in the Eilenberg-Moore Spectral Sequence and the Classifying Space of an Associative $H$-Space, I

By
Mamoru MIMURA* and Masamitsu MORI**

§ 0. Introduction

Let $G$ be a compact, connected, simple Lie group. Let $p$ be a prime. Consider \{\(G; p\)\} the set of all compact, associative $H$-spaces $X$ such that $H^*(X; \mathbb{Z}_p) \cong H^*(G; \mathbb{Z}_p)$ as Hopf algebras over the Steenrod algebra $\mathcal{A}_p$. (Remark that we do not require the existence of any map between $X$ and $G$ inducing the isomorphism.) As is well known, $X$ has the classifying space $BX$ (see for example [8]).

The Eilenberg-Moore spectral sequence for $X$

\[(0, 1) \quad E_2(X) = \text{Cotor}_A(\mathbb{Z}_p, \mathbb{Z}_p) \Rightarrow H^*(BX; \mathbb{Z}_p),\]

where $A = H^*(X; \mathbb{Z}_p)$, is a machinery to calculate $H^*(BX; \mathbb{Z}_p)$. When $H^*(G; \mathbb{Z})$ has no $p$-torsion, it is quite easy to obtain $H^*(BX; \mathbb{Z}_p)$. In fact, $\text{Cotor}_A(\mathbb{Z}_p, \mathbb{Z}_p)$ is a polynomial algebra and the Eilenberg-Moore spectral sequence collapses. But when $H^*(G; \mathbb{Z})$ has $p$-torsion, it is, in general, difficult to obtain the structure of $H^*(BX; \mathbb{Z}_p)$.

Let $E_j$ be the compact, 1-connected, simple, exceptional Lie group of rank $j$ ($j=6, 7$). Recently, Kono-Mimura [6] and Kono-Mimura-Shimada [7] have determined the module structure of $H^*(BE_j; \mathbb{Z}_2)$ ($j=6, 7$). Their method was to calculate algebraically $\text{Cotor}_A(\mathbb{Z}_2, \mathbb{Z}_2)$ and then to show the collapsing of the spectral sequence $(0, 1)$ for $E_j$ by making use of the properties of $E_j$ as Lie groups.

The aim of this paper is to give a proof of the collapsing of the
spectral sequence (0.1) independently of the properties as Lie groups, namely, to show the collapsing of the spectral sequence (0.1) for $X_j$ of $\{E_j; 2\}$ ($j=6, 7$). Our method is to make use of the relationship between the differentials and the two kinds of the squaring operations in the spectral sequence, which was obtained by W. Singer [12].

We denote by $E_0H^*(BX; \mathbb{Z}_2)$ the bigraded, associated algebra of $H^*(BX; \mathbb{Z}_2)$ with respect to the filtration $F^pH^*(BX; \mathbb{Z}_2)$ in the sense of Eilenberg-Moore, that is,

$$E^p_H^*(BX; \mathbb{Z}_2) = F^pH^p+q(BX; \mathbb{Z}_2)/F^{p+1}H^p+q(BX; \mathbb{Z}_2).$$

We shall use the convention to identify the elements in $E_0H^*(BX; \mathbb{Z}_2)$ with those in $H^*(BX; \mathbb{Z}_2)$, since $E_0H^*(BX; \mathbb{Z}_2) \cong H^*(BX; \mathbb{Z}_2)$ as modules.

Our results are stated as follows.

**Theorem A.** For any $X_j \in \{E_j; 2\}$,

$$E_0H^*(BX_j; \mathbb{Z}_2) \cong \mathbb{Z}_2[y_4, y_5, y_7, y_{10}, y_{18}, y_{32}, y_{48}]/R,$$

as an algebra, where $R$ is the ideal generated by (3.7).

**Theorem B.** (i) In $H^*(BX_6; \mathbb{Z}_2)$ the following relations hold mod decomposables.

$$Sq^y y_4 = y_6, \quad Sq^y y_5 = y_7, \quad Sq^y y_6 = y_{10}, \quad Sq^y y_{18} = y_{19},$$

$$Sq^y y_{18} = y_{32}, \quad Sq^y y_{32} = y_{48}.$$  

(ii) $H^*(BX_6; \mathbb{Z}_2)$ is generated by $y_4$ and $y_{32}$ over $\mathbb{Z}_2$.

**Theorem C.** For any $X_7 \in \{E_7; 2\}$,

$$E_0H^*(BX_7; \mathbb{Z}_2) \cong \mathbb{Z}_2[y_4, y_5, y_7, y_{10}, y_{18}, y_{19}, y_{34}, y_{48}, y_{64}, y_{67}, y_{96}, y_{112}]/R,$$

as an algebra, where $R$ is the ideal generated by (3.9) and (3.10).

**Theorem D.** (i) In $H^*(BX_7; \mathbb{Z}_2)$ the following relations hold mod decomposables.
(4) \[ \begin{align*}
Sq^2 y_4 &= y_6, \\
Sq^1 y_6 &= y_7, \\
Sq^1 y_8 &= y_{10}, \\
Sq^2 y_9 &= y_11.
\end{align*} \]
\[ \begin{align*}
Sq^1 y_{10} &= y_{13}, \\
Sq^2 y_{13} &= y_{19}, \\
Sq^1 y_{14} &= y_{18}, \\
Sq^1 y_{18} &= y_{20}, \\
Sq^2 y_{20} &= y_{29}, \\
Sq^1 y_{29} &= y_{31}, \\
Sq^2 y_{31} &= y_{44}.
\end{align*} \]

(ii) \( H^*(BX; \mathbb{Z}) \) is generated by \( y_4 \) and \( y_{64} \) over \( \mathbb{A}_2 \).

Needless to say, Theorems A, B, C, D give the module structure of \( H^*(BE; \mathbb{Z}) \) \((j=6,7)\) over \( \mathbb{A}_2 \). These are simpler proof than those of [6] and [7].

**Remark.** Let \( G_2 \) and \( F_4 \) be the compact, 1-connected, simple exceptional Lie groups of rank 2 and 4 respectively. Let \( X_2 \in \{G_2; 2\} \) and \( X_4 \in \{F_4; 2\} \). The structure of \( H^*(BX; \mathbb{Z}) \) \((i=2,4)\) over \( \mathbb{A}_2 \) is obtained more easily by our argument. We leave them to the reader.

The paper is organized as follows. In § 1 we recollect the Singer's results on the two kinds of squaring operations in the Eilenberg-Moore spectral sequence. In § 2 we review that these operations coincide with those defined algebraically on \( \text{Cotor}_{\mathbb{A}}(\mathbb{Z}_2, \mathbb{Z}_2) \) through the isomorphism \( E_2 \cong \text{Cotor}_{\mathbb{A}}(\mathbb{Z}_2, \mathbb{Z}_2) \). In § 3 we calculate squaring operations on \( \text{Cotor}_{\mathbb{A}}(\mathbb{Z}_2, \mathbb{Z}_2) \) for \( A=H^*(X_2; \mathbb{Z}) \) and \( H^*(X_4; \mathbb{Z}) \). § 4 and § 5 show that the Eilenberg-Moore spectral sequences for \( X_6 \) and \( X_7 \) collapse and this leads us to our results. The final section, § 6, will be used to prove a lemma which is used in § 5.

§ 1. Squaring Operations in the Eilenberg-Moore Spectral Sequence

Let \( S_*(T) \) denote the normalized singular \( \mathbb{Z}_2 \)-chain complex of a space \( T \) with all vertices at the base point. Put \( S^*(T) = \text{Hom}(S_*(T), \mathbb{Z}_2) \).

Let \( X \) be a connected, associative \( H \)-space and \( BX \) the classifying space of \( X \) [8]. A special case of the dual statement to Théorème 3.1 of Moore [9] states that there is an isomorphism

\[ \begin{align*}
H^*(BX; \mathbb{Z}_2) &\cong \text{Cotor}_{S^*(X)}(\mathbb{Z}_2, \mathbb{Z}_2) \quad (\text{or Ext}_{S^*(X)}(\mathbb{Z}_2, \mathbb{Z}_2)).
\end{align*} \]

Let \( K \) denote the coalgebra \( S^*(X) \). Let \( \overline{C}(K) \) denote the cobar con-
struction of $K$, in which $\overline{C}^r(K) = K \otimes \cdots \otimes K$ ($s$-times) with $K = \sum_{i \in \delta} K^i$. Then $\overline{C}(K)$ is a double complex with the external differential induced from the coalgebra structure of $K$ and the internal differential induced from the differential in $K$. Let $\text{Tot} \overline{C}(K)$ denote the total complex of $\overline{C}(K)$. Then $\text{Cotor}_K(Z_2, Z_2)$ is, by definition, the cohomology of $\text{Tot} \overline{C}(K)$. The total complex $\text{Tot} \overline{C}(K)$ has a filtration such that

$$F^r \text{Tot}^n \overline{C}(K) = \sum_{p+q=r} \overline{C}^{p,q}(K),$$

where the first index $p$ is the external degree and the second one $q$ is the internal degree. This gives rise to a spectral sequence $\{E_r\}$ such that

$$E_2 \cong \text{Cotor}_{H^\cdot(X; Z_2)}(Z_2, Z_2) \Rightarrow H^\cdot(BX; Z_2).$$

We call the spectral sequence (1.2) the Eilenberg-Moore spectral sequence for $X$.

**Remark.** This is dual to the spectral sequence which is constructed in [9].

Now we recollect the Singer's results [12] for our purpose. Singer shows that products and squaring operations are defined in $\text{Cotor}_{s\cdot X}(Z_2, Z_2)$ as well as in $H^\cdot(BX; Z_2)$ and the isomorphism (1.1) preserves them (Proposition 1.1 of [12 I], Proposition 7.1 of [12 II]). This enables us to introduce products and squaring operations in the Eilenberg-Moore spectral sequence.

**Proposition 1.1** (Propositions 1.2, 1.3, 1.5 of [12 I]). In the Eilenberg-Moore spectral sequence $\{E_r\}$ for an associative $H$-space $X$ the following properties hold:

1. Each $E_r (r \geq 2)$ is a differential algebra and products on $E_2$ determine those on $E_r$ $(r \geq 2)$.
2. There are squaring operations

$$Sq^k : E_{r}^{p,q} \to E_{r}^{p+k,q} \quad (0 \leq k \leq q),$$

$$Sq^k : E_{r}^{p,q} \to E_{r}^{p+k-q,2q} \quad (k \geq q),$$
and the squaring operations on $E_2$ determine those on $E_r$ ($r \geq 2$).

(3) Let $\rho : F^p H^{*, q}(BX; \mathbb{Z}_p) \rightarrow E_{\rho, p}^{*, q}$ be the natural projection. For $u \in F^p H^{*, q}(BX; \mathbb{Z}_p)$ and $v \in F^r H^{*, q}(BX; \mathbb{Z}_p)$, we have

i) $uv \in F^{p+r} H^{*, q}(BX; \mathbb{Z}_p)$ and $\rho(uv) = \rho(u) \rho(v)$,

ii) if $0 \leq k \leq q$, then $Sq^k u \in F^p H^{*, q}(BX; \mathbb{Z}_p)$ and $\rho Sq^k u = Sq^k \rho u$,

iii) if $q \leq k$, then $Sq^k u \in F^{p+q-k} H^{*, q}(BX; \mathbb{Z}_p)$ and $\rho Sq^k u = Sq^k \rho u$.

The operation $Sq^k : E_{r, p}^{*, q} \rightarrow E_{r, p+q+k}^{*, q}$ will be called a vertical squaring operation and $Sq^k : E_{r, p}^{*, q} \rightarrow E_{r, p+q+k}^{*, q, 2q}$ a diagonal squaring operation.

**Proposition 1.2** (Proposition 1.4 of [12]). Let $u \in E_{r, p}^{*, q}$ ($r \geq 2$).

i) If $k \leq q - r + 1$, then $d_r Sq^k u = Sq^k d_r u$ in $E_r$.

ii) If $q - r + 1 \leq k \leq q$, then $Sq^k u$ survives to $E_{t, p}^{*, q, q}$, where $t = 2r + k - q - 1$, $Sq^k d_r u$ survives to $E_{t, p}^{*, q+2r-2}$ and $d_t [Sq^k u] = [Sq^k d_r u]$.

iii) If $q \leq k$, then $Sq^k u$ survives to $E_{t, p}^{*, q+k, q}$, where $t = 2r - 1$, $Sq^k d_r u$ survives to $E_{t, p}^{*, q+k-q, q-2r+1}$ and $d_t [Sq^k u] = [Sq^k d_r u]$.

Remark. We sometimes regard the vertical operation $Sq^k : E_{r, p}^{*, q} \rightarrow E_{r, p+q+k}^{*, q}$ as zero if $k > q$. In this sense the differentials commute with vertical operations, i.e., $d_r Sq^k u = Sq^k d_r u$ in $E_r$ for every $k \geq 0$ and $r \geq 2$.

§ 2. Squaring Operations on the $E_r$-Term

Let $X$ be an associative $H$-space. Put $A = H^*(X; \mathbb{Z}_2)$.

**Proposition 2.1** (Theorem 2.2 of [10]).

$E_2 \cong \text{Cotor}_A(\mathbb{Z}_2, \mathbb{Z}_2)$ as algebras.

We recall the two kinds of squaring operations on $\text{Cotor}_A(\mathbb{Z}_2, \mathbb{Z}_2)$.

Let $\bar{C}(A)$ be the cobar construction of $A$. Let $\alpha = [x_1, \cdots, x_p] \in \bar{C}^{p, q}(A)$. Define an operation $Sq^k \cdot : \bar{C}^{p, q}(A) \rightarrow \bar{C}^{p, q+k}(A)$ by

$$Sq^k \cdot = \sum [Sq^k x_1, \cdots, Sq^k x_p], \ k_1 + \cdots + k_p = k.$$

Then $Sq^k \cdot$ commutes with the coboundary in $\bar{C}(A)$, since $A$ is the coalgebra over the Steenrod algebra. Hence this induces

$$Sq^k \cdot : \text{Cotor}_A^{p, q} \rightarrow \text{Cotor}_A^{p, q+k}.$$
Let $\overline{B}(A)$ be the bar construction of $A$, i.e.,
$$
\overline{B}^s(A) = H^*(X;\mathbb{Z}_2) \otimes \cdots \otimes H^*(X;\mathbb{Z}_2) \ (s\text{-times}).
$$

There is a map with external degree $i \geq 0$,
$$
\partial_i : \overline{B}(A) \rightarrow \overline{B}(A) \otimes \overline{B}(A),
$$
satisfying $d\partial_i + \partial_i d = \partial_{i-1} + T\partial_{i-1} \ (A_1 = 0)$. The cup-$i$-product
$$
\cup_i : \overline{C}^p(A) \otimes \overline{C}^q(A) \rightarrow \overline{C}^{p+q-i}(A)
$$
is defined by
$$
(\alpha \cup_i \beta)(c) = (\alpha \otimes \beta)\partial_i(c) \text{ for } \alpha \in \overline{C}^p(A), \ \beta \in \overline{C}^q(A),
$$
and satisfies
$$
\delta(\alpha \cup_i \beta) = \delta \alpha \cup_i \beta + \alpha \cup_i \delta \beta + \alpha \cup_i \beta + \beta \cup_i \alpha.
$$

Then an operation $Sq^p_D : \overline{C}^{p,q}(A) \rightarrow \overline{C}^{p+k,2q}(A)$ is defined by
$$
(2.2) \quad Sq^p_D(\alpha) = \alpha \cup_{p-k} \delta \alpha \cup_{p-k+1} \alpha \text{ for } \alpha \in \overline{C}^{p,q}(A).
$$

This commutes with the coboundary and induces
$$
Sq^p_D : \text{Cotor}^p_A \rightarrow \text{Cotor}^{p+k,2q}_A.
$$
The construction of $Sq^p_D$ is essentially due to [1]. The explicit formula for the cup-$i$-product may be found in [14]. Especially, we recall the formulae:
$$
[x_1 | \cdots | x_t] \cup_{0} [x_{t+1} | \cdots | x_{t+r}] = [x_1 | \cdots | x_{t+r}],
$$
$$
[x_1 | \cdots | x_t] \cup_{1} [x_{t+1} | \cdots | x_{t+r}] = \sum_{i=1}^{t} [x_1 | \cdots | x_{i-1} | x_{t}^{(i)} x_{t+1} | \cdots | x_{t+s-1}],
$$
$$
\cdots | x_{t}^{(r)} x_{t+s} | x_{t+1} | \cdots | x_{t+r},
$$
$$
[x_1 | \cdots | x_t] \cup_{s} [x_1 | \cdots | x_s] = [x_1 | \cdots | x_{s}],
$$
where $\psi^{(r-1)}(x) = \sum x^{(i)} \otimes \cdots \otimes x^{(r)}$, $\psi^{(r-1)} : A \rightarrow A \otimes \cdots \otimes A \ (r\text{-times})$, is the
$(r-1)$-iterated diagonal map.

**Proposition 2.2** (Propositions 7.2, 7.3 of [12 II]). Through the
isomorphism $E_2 \cong \text{Cotor}_A(Z_n, Z_n)$,

i) if $0 \leq k \leq q$, then the vertical squaring operation $Sq^k$ on $E_2$ coincides with $Sq^k \nu$ on $\text{Cotor}_A(Z_n, Z_n)$,

ii) if $q \leq k$, then the diagonal squaring operation $Sq^k$ on $E_2$ coincides with $Sq^{k-q} \nu$ on $\text{Cotor}_1(Z_n, Z_n)$.

**Corollary 2.3.** Let $\sum [x_1|\cdots|x_p] \in \overline{C}^{p, q}(A)$ and $\sum [x_{p+1}|\cdots|x_{p+r}] \in \overline{C}^{r, s}(A)$ represent $u \in E_2^{p, q}$ and $v \in E_2^{r, s}$ respectively. Then

i) $\sum [x_1|\cdots|x_p|x_{p+1}|\cdots|x_{p+r}] \in \overline{C}^{p+r, q+s}(A)$ represents $uv \in E_2^{p+r, q+s}$,

ii) if $0 \leq k \leq q$, then

$$\sum [Sq^k x_1|\cdots|Sq^k x_p] \in \overline{C}^{p+q+k}(A)$$

represents $Sq^k u \in E_2^{p+q+k}$.

iii) if $q \leq k$, then

$$\sum [x_1|\cdots|x_p] \cup \bigcup_{p-k \leq j \leq p} \{x_1|\cdots|x_p\} \in \overline{C}^{p+k-q, 2q}$$

represents $Sq^k u \in E_2^{p+k-q, 2q}$.

**Proof.** Immediate from Propositions 2.1, 2.2 and (2.1), (2.2). q.e.d.

Here we remark, for later use:

**Proposition 2.4.** As for the vertical squaring operation, the Cartan formula holds on $E_2$, i.e.,

$$Sq^k(uv) = \sum_{i+k} Sq^i u Sq^i v$$

for $u, v \in E_2$

and $E_r$ $(r \geq 2)$ inherits this formula.

**Proof.** We confirm this by Corollary 2.3, i), ii), and Proposition 1.1, (1), (2), though this may be proved by the standard argument. q.e.d.

Let $\phi$ be the diagonal map of $A = H^e(X; Z_2)$. Let $L$ be a quotient coalgebra of $\overline{A}$ over the Steenrod algebra $\mathcal{A}_2$ with projection $\theta: A \to L$. 
\[ \tilde{\theta} \] denotes the diagonal map of \( L \). Note that \( L \) is not equipped with unit. Construct the tensor algebra \( T(sL) \) with product \( \psi \), where \( s \) is the suspension, that is, the operation to make a copy with external degree added by one. Let \( I \) be the two-sided ideal generated by \( \psi \circ (s0 \otimes s0) \circ \phi(\text{Ker} \ 0) \). Let \( \overline{X} = T(sL)/I \). The differential \( \overline{d} \) on \( \overline{X} \) is induced so that \( \overline{d} = \psi \circ \tilde{\theta} \circ s^{-1} : sL \to T(sL) \) is derivative. Then \( \overline{d}(I) \subset I \) and \( \overline{d} \circ \overline{d} = 0 \), and this is well-defined. \( \overline{X} \) is the quotient of \( \overline{C}(A) \) as differential algebra with projection \( p: \overline{C}(A) \to \overline{X} \) such that \( p[x_1, \ldots, x_n] = s0x_1 \cdots s0x_n \) (see [11]). The (vertical) squaring operation on \( \overline{X} \) is defined by

\[ Sq^k_x = \sum_{k_1 + \cdots + k_n = k} sS_{Q}^k x_1 \cdots sS_{Q}^k x_n, \ x = sx_1 \cdots sx_n \in T(sL) \]

for \( k \geq 0 \).

**Proposition 2.5.** The projection \( p: \overline{C}(A) \to \overline{X} \) preserves the operation \( Sq^k_x \).

**Proof.** Immediate from Propositions 2.1, 2.2 and Corollary 2.3.

q.e.d.

**Corollary 2.6.** Assume that \( p: \overline{C}(A) \to \overline{X} \) induces an isomorphism on cohomology. Let \( \sum sx_1 \cdots sx_p \in \overline{X}^{p,q} \) represent \( u \in E^{p,q}_A \). Then if \( 0 \leq k \leq q \), the element

\[ \sum_{k_1 + \cdots + k_p = k} sS_{Q}^k x_1 \cdots sS_{Q}^k x_p \in \overline{X}^{p,q+k} \]

represents \( Sq^k u \in E^{p,q+k}_A \).

**Proof.** Immediate from Corollary 2.3 and Proposition 2.5.

q.e.d.

§ 3. Squaring Operations on \( \text{Cotor}_A \) for \( A = H^*(X_6; \mathbb{Z}_2) \) and \( H^*(X_7; \mathbb{Z}_2) \)

Let \( X_6 \in \{ E_6; 2 \} \) and \( X_7 \in \{ E_7; 2 \} \).

By definition and [2], we have

\[ H^*(X_6; \mathbb{Z}_2) = \mathbb{Z}[x_6]/(x_6^4) \otimes A(x_6, x_9, x_{12}, x_{17}, x_{22}), \]
The reduced diagonal map is given by Theorem 3.1 of \cite{6} and Theorem 1.8 of \cite{7}, namely,

\begin{align}
(3.1) & \quad \text{for } X_6, X_7, \quad \bar{\phi}(x_i) = 0 \quad (i = 3, 5, 9, 17), \\
(3.2) & \quad \text{for } X_6, \quad \bar{\phi}(x_{15}) = x_3 \otimes x_5^2, \\
& \quad \quad \quad \bar{\phi}(x_{23}) = x_{17} \otimes x_5^2, \\
(3.3) & \quad \text{for } X_7, \quad \bar{\phi}(x_{15}) = x_3 \otimes x_5^2 + x_9 \otimes x_9^2, \\
& \quad \quad \quad \bar{\phi}(x_{23}) = x_3 \otimes x_5^2 + x_{17} \otimes x_5^2, \\
& \quad \quad \quad \bar{\phi}(x_{27}) = x_3 \otimes x_5^2 + x_{17} \otimes x_5^2.
\end{align}

The squaring operations on the elements are given by \cite{2} and \cite{13}, namely,

\begin{align}
(3.4) & \quad \text{for } X_6, X_7, \quad x_5 = Sq^2 x_5, \quad x_9 = Sq^4 x_5, \quad x_{17} = Sq^8 x_9, \quad x_{23} = Sq^8 x_{19}, \\
(3.5) & \quad \text{for } X_7, \quad x_{27} = Sq^2 x_{23}, \\
(3.6) & \quad \text{for } X_6, X_7, \quad x_{17} = Sq^4 x_{15}.
\end{align}

**Proposition 3.1.** (i) Let $A = H^*(X_6; \mathbb{Z}_2)$. Then as an algebra,

$$E_2 \cong \text{Cotor}_A(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2[y_4, y_6, y_7, y_{10}, y_{18}, y_{32}, y_{48}, y_{82}, y_{34}] / R,$$

where the gradings of generators are given by

$$y_4 \in (1, 3), \quad y_6 \in (1, 5), \quad y_7 \in (1, 6), \quad y_{10} \in (1, 9),$$

$$y_{18} \in (1, 17), \quad y_{32} \in (2, 30), \quad y_{48} \in (2, 46), \quad y_{82} \in (2, 82), \quad y_{34} \in (2, 46),$$

\[(y \in (i, j) \text{ means } y \in E_2^{i,j}) \text{ and } R \text{ is the ideal generated by}

\begin{align}
(3.7) & \quad y_5 y_{10}, \quad y_7 y_{18}, \quad y_7 y_{32}, \quad y_7^2 + y_7^2 y_4, \quad y_4 y_8.
\end{align}

(ii) The following relations hold in $E_2$:

$$Sq^4 y_4 = y_6, \quad Sq^4 y_6 = y_7, \quad Sq^4 y_7 = y_{10}, \quad Sq^8 y_{10} = y_{18}, \quad Sq^{16} y_{18} = y_{48}.$$

**Proof.** The calculation of $\text{Cotor}_A(\mathbb{Z}_2, \mathbb{Z}_2)$ is purely algebraic, and hence (i) follows from Theorem 2.3 of \cite{6}. To determine the squaring operations, recall the outline of their calculation. Let $L = \{x_3, x_5, x_9, x_{17}, x_{15}, x_{23}\}$ and denote the corresponding elements by $sL = \{a_3, a_5, a_9, a_{17}, a_{15}, a_{23}\},$
\( a_{18}, b_{18}, b_{24} \). Then by (3.4) we have

\[
(3.8) \quad Sq^6 a_6 = a_6, \quad Sq^6 a_7 = a_7, \quad Sq^6 a_9 = a_9, \quad Sq^6 a_{10} = a_{10}, \quad Sq^6 b_{18} = b_{24}.
\]

Form a differential algebra \((\bar{X}, \bar{d})\) as in §2. Explicitly \(\bar{X}\) is isomorphic to the polynomial algebra \(Z[a_6, a_7, a_{10}, a_{18}, b_{18}, b_{24}]\). Then the projection \(p: \Omega(A) \to \bar{X}\) induces an isomorphism on cohomology, i.e., \(\text{Cotor}_4(Z_2, \bar{X}) \cong H(\bar{X}, \bar{d})\). Each \(y_i\) is represented in \(\bar{X}\) as follows:

\[
\begin{align*}
\gamma_1 &: a_i \quad (i = 4, 6, 7, 10, 18), \\
\gamma_{32} &: b_{18}, \\
\gamma_{48} &: b_{24}, \\
\gamma_{44} &: a_{10} b_{24} + a_{18} b_{18}.
\end{align*}
\]

Note that the squaring operations on \(y_i\) follow immediate from Corollary 2.6 and (3.8). q.e.d.

We next turn to \(X_7\).

**Proposition 3.2.** (i) Let \(A = H^*(X_7; Z_2)\). Then as an algebra,

\[
E_2 \cong \text{Cotor}_4(Z_2, Z_2) \cong Z[a_6, y_4, y_6, y_7, y_{10}, y_{11}, y_{15}, y_{19}, y_{24}, y_{25}, y_{69}, y_{96}, y_{98}, y_{112}] / R,
\]

where the gradings of generators are given by

\[
\begin{align*}
y_4 &\in (1, 3), \quad y_6 \in (1, 5), \quad y_7 \in (1, 6), \quad y_{10} \in (1, 9), \\
y_{11} &\in (1, 10), \quad y_{15} \in (1, 17), \quad y_{19} \in (1, 18), \quad y_{24} \in (2, 32), \\
y_{33} &\in (2, 33), \quad y_{69} \in (3, 63), \quad y_{97} \in (3, 64), \quad y_{96} \in (4, 60), \\
y_{98} &\in (4, 92), \quad y_{112} \in (4, 108),
\end{align*}
\]

and \(R\) is the ideal generated by

\[
(3.9) \quad y_6 y_{11} + y_{10} y_7, \quad y_6 y_{19} + y_{18} y_7, \quad y_{10} y_{19} + y_{18} y_{11},
\]

\[
\begin{align*}
y_7^2 + y_{18} y_6^2, \quad y_1 y_7^2, \quad y_1 y_6 + y_{18} y_8 + y_{19} y_7, \\
y_{10} y_{19} + y_{18} y_{35} + y_{19} y_{18}, \quad y_{10} y_{24} + y_{18} y_{25}, \quad y_{10} y_{18} + y_6 y_{19}, \\
y_{10} y_{25} + y_7 y_{19}, \quad y_{10} y_{19} + y_{18} y_{24} + y_{19} y_{25} + y_7 y_{19}, \\
y_{10} y_{19} + y_{18} y_{24} + y_{19} y_{25} + y_7 y_{19}, \quad y_{10} y_{19} + y_{18} y_{24} + y_{19} y_{25} + y_7 y_{19},
\end{align*}
\]
CLASSIFYING SPACE OF AN ASSOCIATIVE $H$-SPACE

(3.10) \[ y_1 y_6 + y_6 y_7 + y_7 y_8, \quad y_1 y_9 + y_3 y_8, \quad y_1 y_9 + y_1 y_7, \quad y_1 y_9 + y_9 y_7, \]
\[ y_3 y_9 + y_1 y_8 + y_1 y_9, \quad y_3 y_9 + y_1 y_8 + y_9 y_7, \quad y_3 y_9 + y_1 y_8 + y_9 y_7, \]
\[ y_3 y_9 + y_1 y_8 + y_9 y_7, \quad y_3 y_9 + y_1 y_8 + y_9 y_7, \quad y_3 y_9 + y_1 y_8 + y_9 y_7. \]

(ii) The vertical squaring operations in $E_2$ are given by
\[ Sq^1 y_1 = y_6, \quad Sq^1 y_6 = y_7, \quad Sq^1 y_7 = y_9, \quad Sq^1 y_9 = y_{10}, \quad Sq^1 y_{10} = y_{11}, \]
\[ Sq^1 y_{10} = y_{11}, \quad Sq^1 y_{13} = y_{19}, \quad Sq^1 y_{19} = y_{20}, \quad Sq^1 y_{20} = y_{21}, \quad Sq^1 y_{21} = y_{22}. \]

Proof. The calculation of $Cotor_A(Z_2, Z_2)$ is the same as that given by [7], although the relations (3.10) are dropped there. Recall the outline of their calculation. Let $L = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{17}, x_{18}, x_{19}, x_{20}, x_{21}, x_{22}, x_{23}, x_{24}, x_{25}, x_{26}, x_{27}, x_{28} \}$ and denote the corresponding elements by $sL = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, a_{16}, a_{17}, a_{18}, a_{19}, a_{20}, a_{21}, a_{22}, a_{23}, a_{24}, a_{25}, a_{26}, a_{27}, a_{28} \}$. Then by (3.4) and (3.5) we have
\[ Sq^1 a_4 = a_6, \quad Sq^1 a_6 = a_7, \quad Sq^1 a_7 = a_9, \quad Sq^1 a_9 = a_{10}, \quad Sq^1 a_{10} = a_{11}, \]
\[ Sq^1 a_{10} = a_{11}, \quad Sq^1 a_{11} = a_{12}, \quad Sq^1 a_{12} = a_{13}, \quad Sq^1 a_{13} = a_{14}, \quad Sq^1 a_{14} = a_{15}, \quad Sq^1 a_{15} = a_{16}, \quad Sq^1 a_{16} = a_{17}, \quad Sq^1 a_{17} = a_{18}, \quad Sq^1 a_{18} = a_{19}, \quad Sq^1 a_{19} = a_{20}, \quad Sq^1 a_{20} = a_{21}, \quad Sq^1 a_{21} = a_{22}. \]

Form a differential algebra $(\overline{X}, d)$ as in § 2. Explicitly
\[ \overline{X} \cong Z_2 \{a_i, b_j\} / I, \quad i = 4, 6, 7, 10, 11, 18, 19, \quad j = 16, 24, 28, \]
where $I$ is the ideal generated by all possible $[a_m, a_n] \bmod{[b_p, b_q]}$ and by $[a_i, b_j]$ except $(i, j) = (6, 16), (10, 16), (6, 24), (10, 28)$ and $[a_6, b_{16}] + a_{11}, [a_{10}, b_{16}] + a_{13}, [a_6, b_{18}] + a_{13}, [a_{10}, b_{28}] + a_{19}$. Then the projection $\overline{p}: \overline{C}(A) \rightarrow \overline{X}$ induces an isomorphism on cohomology, i.e., $Cotor_A(Z_2, Z_2) \cong H(\overline{X}, d)$. Each $y_i$ is represented in $\overline{X}$ as follows.
\[ y_i: \quad a_i \quad (i = 4, 6, 7, 10, 11, 18, 19) \]
\[ y_{14}: \quad a_{16} b_{16} + a_{10} b_{24} + a_{13} b_{28}, \quad y_{35}: \quad a_{19} b_{16} + a_{11} b_{24} + a_{7} b_{28}, \]
\[ y_{46}: \quad a_{16} b_{28}^2 + a_{10} b_{24}^2 + a_{13} b_{28}, \quad y_{67}: \quad a_{11} b_{24}^2 + a_{10} b_{24}, \]
\[ y_{ij}: \quad b_j^4 \quad (j = 16, 24, 28). \]

Remark that the representative of $y_{10}$ in [7] is incorrect. Now the squaring operations on $y_i$ follow immediate from Corollary 2.6 and (3.11).

q.e.d.
Proposition 3.3. $Sq^1 y_{64} = y_{67}$ in $\text{Cotor}_A(Z_2, Z_2)$, and hence $Sq^2 y_{64} = y_{67}$ in $E_2$.

Proof. Let $C$ be a representative of $y_{64}$ in the cobar construction $\overline{C}(A)$. The explicit form of $C$ is given by

$$C = [x_1 x_1] + [x_3 x_3] + [x_5 x_5] + [x_7 x_7] + [x_9 x_9] + [x_1 x_1 x_3] + [x_3 x_3 x_3] + [x_5 x_5 x_5] + [x_7 x_7 x_7].$$

Then $Sq^1 y_{64}$ is represented by $C \cup C$. By using the explicit formula for the cup-1-product, we have

$$C \cup_C C = [x_5 x_5 x_3] + [x_7 x_7 x_3] + [x_9 x_9 x_3] + r,$$

where $r \in \text{Ker}(p: \overline{C}(A) \to \overline{X})$. Hence

$$p(C \cup_C C) = a_{11} b_{18}^2 + a_{13} b_{18}^2.$$

Therefore $C \cup_C C$ represents $y_{67}$, and we have $Sq^1 y_{64} = y_{67}$ in $\text{Cotor}_A(Z_2, Z_2)$.

The latter relation $Sq^2 y_{64} = y_{67}$ in $E_2$ follows from Proposition 2.2.

q.e.d.

For later use we note

Lemma 3.4. $Sq^1 y_{64} = Sq^2 y_{64} = Sq^4 y_{64} = 0$ and $Sq^3 y_{64} = y_{18}$ in $E_2$.

Proof. Since $Sq^1 b_{16} = Sq^3 b_{16} = 0$ for dimensional reasons and $Sq^2 b_{16} = a_{18}$ in $\overline{X}$ by (3.6), and since $y_{64}$ is represented by $b_{18}^2$, the lemma follows from Corollary 2.6.

q.e.d.

§ 4. Collapsing of the Spectral Sequence for $X_6$

Let $X_6 \in \{E_2; 2\}$ and put $A = H^*(X_6; Z_2)$. Consider the Eilenberg-Moore spectral sequence for $X_6$:

$$(4.1) \quad E_2 \cong \text{Cotor}_A(Z_2, Z_2) \Rightarrow H^*(BX_6; Z_2),$$

where the $E_2$-term is given by Proposition 3.1.
Theorem 4.1. The Eilenberg-Moore spectral sequence (4.1) for $X_{e}$ collapses.

This will follow from the following lemmas.

Lemma 4.2. The element $y_{i}$ survives, and hence so do $y_{b}$, $y_{r}$, $y_{10}$, $y_{18}$.

Proof. For dimensional reasons $y_{1}$ survives and so do the other elements by Propositions 1.2 and 3.1. q.e.d.

We need the following facts.

(4.2) i) $y_{18}^{2} \neq 0$ in $H^{*}(BX_{e}; Z_{2})$.

ii) $y_{1} \neq 0$ in $E_{1}$.

iii) $y_{i}y_{b}y_{r} \neq 0$, $y_{b}y_{b}y_{b} \neq 0$ in $E_{1}$.

iv) $y_{i}y_{i} \neq 0$ in $E_{1}$.

Proof is clear for dimensional reasons.

Lemma 4.3. The element $y_{34}$ survives.

Proof. Denote $F^{p} = F^{p}H^{*}(BX_{e}; Z_{2})$. First note that $Sq^{18}y_{18} = y_{18}^{2} \neq 0$ in $H^{*}(BX_{e}; Z_{2})$ by (4.2). Remark that $y_{18}^{2} \in F^{3}$. By Adem relation

$$y_{18}^{2} = Sq^{18}y_{18} = Sq^{1}Sq^{18}y_{18} + Sq^{17}Sq^{17}y_{18}.$$  

For dimensional reasons $Sq^{18}y_{18} \in F^{3}$, and hence $Sq^{17}Sq^{18}y_{18} \in F^{3}$ by Proposition 1.1. Now assume that $y_{34}$ does not survive, then $Sq^{34}y_{18} \in F^{3}$, and hence $Sq^{3}Sq^{16}y_{18} \in F^{3}$. This is a contradiction to $y_{18}^{2} \in F^{3}$. Thus $y_{34}$ survives and furthermore we must have

$$Sq^{18}y_{34} \equiv y_{34} \text{ mod } F^{3}.$$  

This completes the proof. q.e.d.

In the above proof we have shown...
Proposition 4.4. \( Sq^* y_{18} \equiv y_{34} \mod \text{decomposables in } H^*(BX_6; \mathbb{Z}_2) \).

Lemma 4.5. \( Sq^* y_{32} = Sq^* y_{32} = 0 \) in \( E_2 \).

Proof. Recall that \( y_{32} \) is represented by \( b_{16}^2 \) in \( X \) (See the proof of Proposition 3.1) and \( Sq^2 b_{16} = 0 \) for dimensional reasons. Hence \( Sq^2 y_{32} \) is represented by

\[
Sq^2 b_{16} = Sq^2 b_{16} \Rightarrow Sq^2 b_{16} = 0
\]

by Corollary 2.6.

Therefore we have \( Sq^2 y_{32} = 0 \) in \( E_2 \). It is easier to see \( Sq^1 y_{32} = 0 \).

q.e.d.

Lemma 4.6. The element \( y_{32} \) survives, and so does \( y_{48} \).

Proof. We first show that \( y_{32} \in E_3^{*, 28} \) is a permanent cocycle. Consider \( d_r : E_r^{2, 28} \to E_r^{2+r, 21-r} \) (\( r \geq 2 \)). For dimensional reasons the possible elements to be killed by \( y_{32} \) are

\[
\begin{align*}
E_2^{4, 29} & \ni y_4^4 y_{18} = 0, y_6 y_{10} = 0, \\
E_3^{6, 28} & \ni y_6 y_4 y_{14} = 0, y_8 y_{12}, \\
E_4^{8, 27} & \ni y_4 y_4 y_{10} = 0, y_4 y_8 y_{12}, y_6 y_{12}, \\
E_5^{10, 28} & \ni y_6 y_6 y_{12}.
\end{align*}
\]

Put \( d_4(y_{32}) = ay_4^4 y_{14}^4 \) with \( a \in \mathbb{Z}_2 \). Applying \( Sq^4 \), we have

\[
0 = d_4(Sq^4 y_{32}) = Sq^4 d_4(y_{32}) = ay_4^4
\]

by Propositions 1.2, 3.1 and Lemma 4.5. Then since \( y_4^4 \neq 0 \) by (4.2) we have \( a = 0 \). Next put \( d_4(y_{32}) = ay_4^4 y_{14} + by_8 y_{12} \) with \( a, b \in \mathbb{Z}_2 \). Applying \( Sq^4 \), we have \( 0 = d_4(Sq^4 y_{32}) = Sq^4 d_4(y_{32}) = ay_4^4 y_{14}^4 \). Since \( y_4^4 y_{14}^4 \neq 0 \) by (4.2), we have \( a = 0 \). Then applying \( Sq^8 \) to \( d_4(y_{32}) = by_8 y_{12} \), we have

\[
0 = d_4(Sq^8 y_{32}) = Sq^8 d_4(y_{32}) = by_8 y_{12}^4.
\]

Since \( y_8 y_{12}^4 \neq 0 \) by (4.2), we have \( b = 0 \). Thus \( d_4(y_{32}) = 0 \). Finally put \( d_5(y_{32}) = ay_4^4 y_{14} y_{12} \) with \( a \in \mathbb{Z}_2 \). Applying \( Sq^4 \),

\[
0 = d_5(Sq^4(y_{32})) = Sq^4 d_5(y_{32}) = ay_4^4 y_{14} y_{12}.
\]

Since \( y_4^4 y_{14} y_{12} \neq 0 \) by (4.2), we have \( a = 0 \) and \( d_5(y_{32}) = 0 \). Thus we have
shown that \( y_{32} \) is a permanent cocycle. Since \( y_{32} \) is not killed for dimensional reasons, we conclude that \( y_{32} \) survives, and hence \( y_{48} = Sq^n y_{32} \) survives by Proposition 1.2. q.e.d.

Now Theorem 4.1 follows from Lemmas 4.2, 4.3 and 4.6.

Theorems A and B follow immediately from Propositions 3.1 and 4.4 and Theorem 4.1.

§ 5. Collapsing of the Spectral Sequence for \( X_7 \)

Let \( X_7 \subset \{E_7, 2\} \) and put \( A = H^*(X_7; Z) \). Consider the Eilenberg-Moore spectral sequence for \( K_7 \):

\[
E_2 \cong \text{Cotor}_A(Z_2, Z_2) \Rightarrow H^*(BX_7; Z),
\]

where the \( E_2 \)-term is given by Proposition 3.2.

Theorem 5.1. The Eilenberg-Moore spectral sequence for \( X_7 \) collapses.

This will follow from the following lemmas.

Lemma 5.2. The element \( y_4 \) survives, and so do \( y_6, y_7, y_{10}, y_{11}, y_{18}, y_{19} \).

Proof. The element \( y_4 \) survives for dimensional reasons, and so do the other elements by Propositions 1.2 and 3.2. q.e.d.

Lemma 5.3. The element \( y_{34} \) survives and so does \( y_{35} \).

Proof is quite similar to that of Lemma 4.3, though the existence of the element of degree 19 may make a proof a little bit complicated.

As an analogous result to Proposition 4.4 we can show

Proposition 5.4. \( Sq^n y_{18} \equiv y_{34} \mod \text{decomposables in } H^*(BX_7; Z) \).
Lemma 5.5. The element $y_{66}$ survives and so does $y_{67}$.

Proof. Denote $F^p = F^pH^*(BX_7; Z_2)$. By Proposition 3.3 the relation $Sq^{4k}y_{64} = y_{67}$ holds in $E_2$. Hence the element $y_{67}$ survives to $E_\infty$ by Proposition 1.2, and we obtain

$$y_{67} = Sq^{4}y_{64} = Sq^{4}Sq^{4}y_{64} \mod F^4$$

in $H^*(BX_7; Z_2)$. Assume that $y_{66}$ does not survive. Then $Sq^{4k}y_{64} \in F^i$ for dimensional reasons. So $y_{67} = 0 \mod F^4$, which is a contradiction to $y_{67} \in F^3$. Therefore $y_{66}$ survives and furthermore we must have

$$Sq^{4}y_{64} = y_{66} \mod F^4.$$

q.e.d.

In the proof we have obtained

Proposition 5.6. $Sq^{4k}y_{64} = y_{66} \mod$ decomposables in $H^*(BX_7; Z_2)$.

Lemma 5.7. $d_r(Sq^iy_{64}) = 0$ for $i = 1, 2, 4, 8$ and for all $r \geq 2$.

Proof. Immediate from Lemma 3.4. q.e.d.

Lemma 5.8. The element $y_{64}$ survives and hence so do $y_{66}$ and $y_{112}$.

(The proof will be given in §6.)

Now Theorem 5.1 follows from Lemmas 5.2, 5.3, 5.5 and 5.8. Theorems C and D follow from Propositions 3.2, 5.4 and 5.6 and Theorem 5.1.

§ 6. Proof of Lemma 5.8

The proof of Lemma 5.8 given here is quite analogous to that of Lemma 4.3, although it is much more complicated. To prove the lemma, it suffices to show that the element $y_{64}$ survives, since $y_{66} = Sq^{4}y_{64}$ and $y_{112} = Sq^{4}y_{64}$ by Proposition 3.2. For dimensional reasons $y_{64}$ is not killed.
and hence we need only to check that \( y_{64} \in E^*_{\mathfrak{g}} \) is a permanent cocycle.

Let \( S(n) \) be the set of monomials in \( E^*_{\mathfrak{g}} \).

\[(6.1) \quad y_{4}^{a}y_{6}^{b}y_{7}^{c}y_{10}^{d}y_{11}^{e}y_{12}^{f}y_{13}^{g}y_{34}^{h}y_{35}^{i} \]

with

\[(6.2. n) \quad 4a + 6b + 7c + 10d + 11e + 18f + 19g + 34h + 35i = n. \]

Note that the \( \mathbb{Z}_{2} \)-module generated by \( S(n) \) is closed under the vertical squaring operations. The set \( S(n) \) is ordered lexicographically from the right, for example, \( y_{4}y_{6}^{2}y_{10}^{2}y_{13} > y_{6}^{2}y_{10}^{2}y_{13} \) in \( S(65) \). Since there are relations

\[(6.3) \quad y_{4}y_{11} + y_{16}y_{7} = 0, \quad y_{6}y_{19} + y_{21}y_{7} = 0, \quad y_{10}y_{16} + y_{18}y_{11} = 0, \]

\[y_{16}^{a} + y_{19}y_{7} = 0, \quad y_{19}^{c} + y_{18}^{d} = 0, \quad y_{18}y_{24} + y_{16}y_{25} + y_{14}y_{11} = 0, \]

\[y_{11}y_{34} + y_{18}y_{35} + y_{17}y_{19} = 0, \quad y_{12}y_{34} + y_{14}y_{35} = 0, \]

the monomials of \( S(n) \) satisfying one of the following

\[(6.4) \quad i) \ c \geq 1, \ d \geq 1, \ ii) \ c \geq 1, \ f \geq 1, \ iii) \ e \geq 1, \ f \geq 1, \]

\[iv) \ e \geq 3, \ v) \ e \geq 1, \ g \geq 2, \ vi) \ g \geq 3, \ vii) \ e \geq 2, \ g \geq 1, \]

\[viii) \ c \geq 1, \ g \geq 2, \ ix) \ g \geq 1, \ h \geq 1, \]

can be reduced either to a trivial one or to a linear combination of the other monomials of higher order. A monomial is irreducible unless it satisfies one of the relations \( (6.3) \). Thus the set of the irreducible monomials of degree \( n \) forms a \( \mathbb{Z}_{2} \)-basis of \( S(n) \).

Remark that the first (possibly) non-trivial differential is

\[d_{r}: E^{4,60}_{r} \rightarrow E^{4+2,61}_{r-1}, \]

since the elements \( y_{i} \) \( (i = 4, 6, 7, 10, 11, 18, 19, 34, 35) \) are cocycles. So the following lemmas are clear for dimensional reasons.

**Lemma 6.1.** The irreducible monomials \( y_{4}^{a}y_{6}^{b}y_{7}^{c}y_{10}^{d}y_{11}^{e}y_{12}^{f}y_{13}^{g}y_{34}^{h}y_{35}^{i} \)

are non-trivial in \( E^{p,q}_{r} \) for the following cases:

1. \( p + q \leq 68 \) and \( p + q \neq 65 \), when \( a > 0 \),
2. \( p + q = 69 \) and \( 73 \), when \( a = 0 \),
Lemma 6.2. The non-negative integer solutions of the equation (6.2.65) and

\[(6.5) \quad a + b + c + d + e + f + g + 2h + 2i = 4 + r\]

except the cases (6.4) gives a basis \(\{m_{r,i}\}\) of \(E_r^{4+r.61-r}\).

Using this basis, each element of \(E_r^{4+r.61-r}\) is expressed as \(\sum k_i m_{r,i}\) with \(k_i \in \mathbb{Z}_r\). Explicitly we have

(6.6) i)  \[E_6^{4.55}: k_1 y_6^5 y_{10} y_{35} + k_2 y_6^5 y_1 y_{35} + k_3 y_6^5 y_{11} y_{35} + k_4 y_6 y_{10} y_{35} + k_5 y_6 y_{11} y_{35}\]

\[+ k_6 y_6 y_{10} y_{35} + k_7 y_6 y_{11} y_{35}\]

ii)  \[E_6^{4.58}: k_1 y_6^5 y_{10} y_{35} + k_2 y_6^5 y_1 y_{35} + k_3 y_6^5 y_{11} y_{35} + k_4 y_6^5 y_{35}\]

\[+ k_5 y_6^5 y_{10} y_{35} + k_6 y_6^5 y_{11} y_{35}\]

iii) \[E_6^{4.57}: k_1 y_6^5 y_{10} y_{35} + k_2 y_6^5 y_1 y_{35} + k_3 y_6^5 y_2 y_{35} + k_4 y_6^5 y_{10} y_{35}\]

\[+ k_5 y_6^5 y_{11} y_{35} + k_6 y_6^5 y_{10} y_{35} + k_7 y_6^5 y_{11} y_{35}\]

iv) \[E_6^{4.56}: k_1 y_6^5 y_6 y_{35} + k_2 y_6^5 y_{10} y_{35} + k_3 y_6^5 y_1 y_{35} + k_4 y_6^5 y_{11} y_{35} + k_5 y_6^5 y_{10} y_{35}\]

\[+ k_6 y_6^5 y_{11} y_{35} + k_7 y_6^5 y_{10} y_{35} + k_8 y_6^5 y_{11} y_{35}\]

v) \[E_6^{4.55}: k_1 y_6^5 y_{10} y_{35} + k_2 y_6^5 y_1 y_{35} + k_3 y_6^5 y_{11} y_{35} + k_4 y_6^5 y_{10} y_{35}\]

\[+ k_5 y_6^5 y_{11} y_{35} + k_6 y_6^5 y_{10} y_{35} + k_7 y_6^5 y_{11} y_{35}\]
The above elements are the candidates to be killed off by \( y_{64} \). That is,

\[(6.7) \quad d_r(y_{64}) = \sum_i k_im_{r,i} \text{ with } k_i \in \mathbb{Z}_2\]

for \( d_r: E_r^{4,0} \to E_r^{4+r,81-r} \) (\( r \geq 2 \)). We will show that all the coefficients \( k_i \) are zero in the following way.

First we apply \( Sq^l \) on both sides of (6.7). Since \( Sq^l d_r y_{64} = d_r Sq^l y_{64} = 0 \) by Lemma 5.7, we have

\[\sum_i k_i Sq^l m_{r,i} = 0,\]

where \( Sq^l m_{r,i} \) is calculated by Proposition 3.2 and by the Cartan formula. Then the linear independency of \( \{ Sq^l m_{r,i} \} \) by Lemma 6.1 implies that \( k_i = 0 \). By this argument we get

**Lemma 6.3.** \( k_i \) is trivial for

\[
i = 1, 6, 7, 8, 9 \quad \text{in (6.6.1)},
\]

\[
i = 4, 6, 7, 8, 10, 11 \quad \text{in (6.6.ii)},
\]
\[ i = 6, 7, 9, 10, 11, 13, 14, 15 \quad \text{in (6. 6. iii)}, \]
\[ i = 4, 5, 6, 8, 10, 12, 13 \quad \text{in (6. 6. iv)}, \]
\[ i = 2, 4, 5, 7, 8, 9, 11, 13, 15 \quad \text{in (6. 6. v)}, \]
\[ i = 1, 3, 5, 6, 7, 9, 11, 13 \quad \text{in (6. 6. vi)}, \]
\[ i = 1, 3, 5, 6, 8 \quad \text{in (6. 6. vii)}, \]
\[ i = 1, 3, 5 \quad \text{in (6. 6. viii)}, \]
\[ i = 1, 3 \quad \text{in (6. 6. ix)}, \]
\[ i = 1 \quad \text{in (6. 6. x)}. \]

Then by applying \( \text{Sq}^i \) on both sides of (6. 7), we get by Lemma 5. 7
\[
\sum k_i \text{Sq}^i m_{r,i} = 0,
\]
where the summation runs over \( i \) not listed in Lemma 6. 3.

The linear independency of \( \{ \text{Sq}^i m_{r,i} \} \) by Lemma 6. 1 implies

**Lemma 6. 4.** \( k_i \) is trivial for
\[ i = 3, 4, 5 \quad \text{in (6. 6. i)}, \]
\[ i = 3, 5 \quad \text{in (6. 6. ii)}, \]
\[ i = 1, 3, 4, 5, 8, 12 \quad \text{in (6. 6. iii)}, \]
\[ i = 3, 11, 14 \quad \text{in (6. 6. iv)}, \]
\[ i = 1, 3, 6, 10, 12, 14 \quad \text{in (6. 6. v)}, \]
\[ i = 4, 12 \quad \text{in (6. 6. vi)}, \]
\[ i = 2, 4, 7 \quad \text{in (6. 6. vii)}, \]
\[ i = 4 \quad \text{in (6. 6. viii)}, \]
\[ i = 2 \quad \text{in (6. 6. ix)}. \]

**Corollary 6. 5.**

1. \( \nu_s^3 \nu_s^5 \nabla_{11} \nu_s \) and \( \nu_s^3 \nu_s^5 \nabla_{11} \nu_s \) are not trivial in \( E_{8.35} \).
2. \( \nu_s^3 \nu_s^5 \nabla_{11} \nu_s \) is not trivial in \( E_{8.35} \).
3. \( \nu_s^3 \nu_s^5 \nabla_{11} \nu_s \) is not trivial in \( E_{18.56} \).
4. \( \nu_s^3 \nu_s^5 \nabla_{19} \), \( \nu_s^3 \nu_s^5 \nabla_{20} \), \( \nu_s^3 \nu_s^5 \nabla_{21} \), \( \nu_s^3 \nu_s^5 \nabla_{22} \), \( \nu_s^3 \nu_s^5 \nabla_{23} \), \( \nu_s^3 \nu_s^5 \nabla_{24} \) are not trivial in \( E_{8.35} \).
Proof. (1) and (2): The elements $y_{8}^{i}y_{8}^{i}y_{11}y_{8}$ and $y_{8}^{i}y_{11}y_{8}$ are not $d_{r}$-images of $y_{8}$, since $k_{2} = k_{4} = 0$ in (6.6.i). So $y_{8}^{i}y_{8}^{i}y_{11}y_{8}$ and $y_{8}^{i}y_{11}y_{8}$ for $a = 3, 4$ are not trivial, since $d_{r} = 0$ in these degrees.
(3) follows from that $k_{1} = 0$ in (6.6.vii).
(4) follows from that $k_{1} = 0$ for $i = 1, 3, 4, 5, 12$ in (6.6.iii).

q.e.d.

Then by applying $Sq'$ on the both sides of (6.7) we get the following lemma by virtue of Lemma 6.1 and Corollary 6.5.

**Lemma 6.6.** $k_{i}$ is trivial for

<table>
<thead>
<tr>
<th>$i$</th>
<th>in (6.x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i = 2$</td>
<td>(6.6.i)</td>
</tr>
<tr>
<td>$i = 1, 2, 9$</td>
<td>(6.6.ii)</td>
</tr>
<tr>
<td>$i = 1, 2, 7, 9$</td>
<td>(6.6.iv)</td>
</tr>
<tr>
<td>$i = 2$</td>
<td>(6.6.viii)</td>
</tr>
</tbody>
</table>

**Corollary 6.7.** $y_{8}^{i}y_{8}^{i}y_{11}y_{8}$, $y_{8}^{i}y_{11}y_{8}$ and $y_{8}^{i}y_{8}^{i}y_{11}y_{8}$ are not trivial in $E_{r}^{11,82}$.

**Proof.** This follows from that $k_{1} - k_{2} - k_{4} = 0$ in (6.6.iv).

q.e.d.

Now we apply $Sq^{8}$ on the both sides of (6.7) and by Lemma 6.1 and Corollaries 6.5 and 6.7 we get

**Lemma 6.8.** $k_{i}$ is trivial for

<table>
<thead>
<tr>
<th>$i$</th>
<th>in (6.x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i = 2$</td>
<td>(6.6.iii)</td>
</tr>
<tr>
<td>$i = 2, 8, 10$</td>
<td>(6.6.vi)</td>
</tr>
</tbody>
</table>

Thus we have shown that all $k_{i}$ are trivial. This completes the proof of Lemma 5.8.
References


