The Principle of Limiting Absorption for Second-order Differential Equations with Operator-valued Coefficients

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§0. Introduction

Let us consider differential operators of the form

\[ (0.1) \quad L = -\frac{d^2}{dr^2} + B(r) + C(r) \quad (0 < r < \infty), \]

where for each \( r \in (0, \infty) \) \( B(r) \) and \( C(r) \) are operators in a Hilbert space \( X \). \( L \) acts on \( X \)-valued functions on \( (0, \infty) \).

The purpose of the present paper is to justify the principle of limiting absorption for the equation

\[ (0.2) \quad (L - (\lambda + i\mu))u = f. \]

The essence of the above principle consists in the following: Let \( u_{\lambda+i\mu} \) be the solution of (0.2), where \( f \) is a given \( X \)-valued function on \( (0, \infty) \). Then a solution \( u_{\lambda} \) of the equation

\[ (0.3) \quad (L - \lambda)u = f \]

is given by \( u_{\lambda} = \lim_{\mu \to 0} u_{\lambda+i\mu} \). The meaning of the limit is to be determined suitably. For the literature of the principle of limiting absorption see, for example, Eidus [1].

Jäger [5] considers the differential operator \( L \) and gives, among others, the following result: Let \( B(r) \) be a non-negative self-adjoint
operator in $X$ and let $C(r)$ behave like $0(r^{-\frac{3}{2} - \varepsilon})$ ($\varepsilon > 0$) at infinity. Then with some other conditions imposed on $B(r)$ and $C(r)$ the principle of limiting absorption holds for equation (0.2) with boundary condition

\[(0.4) \quad u(0) = 0\]

and the "radiation condition"

\[(0.5) \quad \int_0^\infty |u'(r) - i\sqrt{z} u(r)|^2 dr < \infty \quad (z = \lambda + i\mu),\]

where $|\ |$ means the norm of $X$. He uses the above results to construct an eigenfunction expansion associated with $L$.

We shall extend J"ager's results to $L$ with $C(r)$ which behaves like $0(r^{-1 - \varepsilon})$ ($\varepsilon > 0$) at infinity. In our case the radiation condition (0.5) will be replaced by

\[(0.6) \quad \int_0^\infty (1 + r)^{-1 + \varepsilon} |u'(r) - i\sqrt{z} u(r)|^2 dr < \infty,\]

which is weaker than (0.5).

As an application we shall prove the principle of limiting absorption for the Schr"odinger operator $-\Delta + q(y)$ in $\mathbb{R}^n$ ($n \geq 3$) with $q(y) = 0(|y|^{-1 - \varepsilon})$ at infinity. In this case $X = L^2(S^{n-1})$ and

\[(0.7) \quad B(r) = \frac{1}{r^2} \left\{- A_n + \frac{(n-3)(n-1)}{4} \right\} \quad C(r) = q(r\omega) \times \omega = \frac{y}{r} \in S^{n-1},\]

where $S^{n-1}$ is $(n-1)$-sphere, and $A_n$ is the Laplace-Beltrami operator on $S^{n-1}$.

In §1 we state conditions imposed on $B(r)$ and $C(r)$ and prove some inequalities which will be used to obtain various a priori estimates for the solution of equation (0.2) in §3. §2 and §3 are devoted to showing the existence and uniqueness of the solution $u$ of the equation

\[(0.8) \quad (L - k^2) u = f \quad (\text{Im } k \geq 0)\]
which satisfies the boundary condition (0.4) and the radiation condition (0.6). Moreover we show that the solution $u$ continuously depends on $k$. Thus the principle of limiting absorption is justified. We discuss in § 4 the dependency on $C(r)$ of the solution of equation (0.8). In § 5 we apply these results to the Schrödinger operator in $\mathbb{R}^n$ ($n \geq 3$).

Using the results obtained in this paper we can develop a spectral and scattering theory for the differential operator $L$ with an application to Schrödinger operators $-\Delta + q(y)$ in $\mathbb{R}^n$, where $q(y) = 0(|y|^{-1-\varepsilon})$ at infinity. We shall discuss these elsewhere.\(^1\)

Recently we have been informed by Prof. T. Ikebe that the following very extensive results have been obtained by S. Agmon: Let

$$L = \sum_{|\alpha| \leq m} a_\alpha D^\alpha = L_0 + B$$

be an elliptic operator in $\mathbb{R}^n$ which has a unique self-adjoint extension in $L^2(\mathbb{R}^n)$, where $L_0 = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$ is an elliptic operator with constant coefficients, and $B = \sum_{|\alpha| \leq m} b_\alpha D^\alpha$ is a differential operator with $b(x) = 0(|x|^{-1-\varepsilon})$ as $|x| \to \infty$. Assume that $\lambda > 0$ does not belong to an exceptional set which is discrete in $(-\infty, \infty)$ and contains all the eigenvalues of $L$. Then the principle of limiting absorption holds good for $\lambda$, i.e., we have

$$v_{\lambda \pm i\mu} \to v_{\lambda \pm i0} \quad \text{as} \quad \mu \downarrow 0 \quad \text{in} \quad L_2(\mathbb{R}^n, (1 + |x|)^{-1-\epsilon} dx),$$

(0.10)

$$\int_{\mathbb{R}^n} (1 + |x|)^{-1-\varepsilon} |v_{\lambda \pm i\mu}(x)|^2 dx \leq C \int_{\mathbb{R}^n} (1 + |x|)^{1+\varepsilon} |f(x)|^2 dx,$$

where $v_{\lambda \pm i\mu} = (L - (\lambda \pm i\mu))^{-1} f$. In his method any radiation condition is unnecessary. These results are used to construct an eigenfunction expansion for $L$.

§ 1. Assumptions and Preliminary Lemmas

Let $X$ be a Hilbert space with the norm $| \cdot |$ and inner product $(\cdot, \cdot)$. For an open interval $J$ in $\mathbb{R}^2$ and $\beta \in \mathbb{R}$ we denote by $H^{\beta}(J, X)$ the

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1) See Y. Saito [7].

2) $\mathbb{R}$ is the set of all real numbers.
Hilbert space of all (equivalence classes of) \(X\)-valued function on \(J\) with the norm and inner product

\[
\begin{cases}
\|f\|_{\beta, J} = \left[ \int_{J} ((f, f))_{\beta, J} \right]^{\frac{1}{2}}, \\
((f, g))_{\beta, J} = \int_{J} (f(r), g(r))(1 + |r|)^{\beta} dr.
\end{cases}
\]

Let \(Y\) be a linear topological space, let \(m\) be a non-negative integer, and let \(J = (a_1, a_2)\) be an open interval in \(\mathbb{R}\). \(C^m(J, Y)\) denotes the set of all \(Y\)-valued functions on \(J\) having \(m\) strong continuous derivatives. We denote by \(\bar{C}^m(J, Y)\) the set of all \(Y\)-valued functions \(f(r)\) such that \(f \in C^m(J, Y)\) and \(\frac{d^j f}{dr^j}\) \((j = 0, 1, \ldots, m)\) can be extended to continuous functions on \(\bar{J}\). \(C^m_{\beta, a_i}(J, Y)\) \((i = 1, 2)\) denotes the set of all \(f \in \bar{C}^m(J, Y)\) satisfying \(f(r) = 0\) in some neighborhood of \(a_i\). We put \(C^m(J, Y) = C^m_{\beta, a_1}(J, Y) \cap C^m_{\beta, a_2}(J, Y)\). If \(Y = \mathbb{C}\), we omit \(\mathbb{C}\) as in \(C^m(J, \mathbb{C}) = C^m(J, \mathbb{R})\).

Let \(I = (0, \infty)\) and let \(B(r)\) and \(C(r)\) be operator-valued functions on \(I\). For local properties of \(B(r)\) and \(C(r)\) we make the following

**Assumption 1.1.** (a) For each \(r \in I\) \(B(r)\) is a non-negative, self-adjoint operator in \(X\) such that its domain \(\mathcal{D}(B(r)) = D^5\) does not depend on \(r\), and \(B(r)x \in C^0(I, X)\) for any \(x \in D^5\).

(b) Let \(x, y \in D\). Then \((B(r)x, y) \in C^2(I)\) and for any compact interval \(M \subset I\) there exists a constant \(c_1(M) > 0\) satisfying

\[
\left| \frac{d^j}{dr^j} (B(r)x, y) \right| \leq c_1(M) \left( |x| + |B^\frac{1}{2}(r)x| \left( |y| + |B^\frac{1}{2}(r)y| \right) \right),
\]

where \(r, s \in M\) and \(j = 1, 2\).

(c) For each \(r \in I\) \(C(r)\) is a symmetric operator in \(X\) with \(\mathcal{D}(C(r)) = D\) such that \(C(r)x \in C^1(I, X)\) for any \(x \in D\).

(d) Let \(M\) be a compact interval in \(I\). Then there exists a constant \(c_2(M) > 0\) such that

\[\text{3) } \bar{J}\text{ means the closure of } J.\]
\[\text{4) } \mathbb{C}\text{ is all complex numbers.}\]
\[\text{5) } \mathcal{D}(T)\text{ means the domain of } T.\]
(1.3) \[ \left| \frac{d}{dr} C(r) x \right| \leq c_2(M) \left( |x| + |B^1 r (r) x| \right), \]

holds for any \( x \in D \) and any \( r \in M \).

We introduce the norm \( \| \cdot \|_{B_r} \) and inner product \( ((\cdot, \cdot))_{B_r} \) by

\[
(1.4) \quad \|f\|_{B_r} = \sqrt{((f, f))_{B_r}}, \\
(1.5) \quad ((f, g))_{B_r} = ((f', g'), 0, r) + ((Bf, g))_{0, r} + ((f, g))_{0, r}. \]

We denote by \( C^2_{B_r} (J, X) (C^2_{a_1} (J, X), i = 1, 2) \) the linear space spanned by the set of all \( \varphi \in C^2 (J, X) \) having the form \( \varphi = \psi x \), where \( x \in D, \varphi \in C^2 (J) (\psi \in C^2_{a_1} (J), i = 1, 2) \) and \( \| \varphi \|_{B_r} < \infty \). We denote \( C^2_{a_1}^2 (J, X) \) \( \cap C^2_{a_2} (J, X) \) by \( C^2_{a_1} (J, X) \). We define Hilbert spaces \( H^1_{B_r} (J, X) \),

\( H^1_{a_1} (J, X) \) and \( H^1_{a_2} (J, X) \) \( (i = 1, 2) \), respectively, by the completion of \( C^2_{B_r} (J, X), C^2_{a_1} (J, X) \) and \( C^2_{a_2} (J, X) \) \( (i = 1, 2) \) in the norm \( \| \cdot \|_{B_r} \).

Let us denote by \( \text{loc} H^0 ((\hat{I}, X) \) the set of all \( X \)-valued functions \( f(r) \) on \( \hat{I} \) such that \( f \in H^0 ((0, b), X) \) for any \( b > 0 \). In a similar way \( \text{loc} H^1_{B_r} ((\hat{I}, X) \) and \( \text{loc} H^0_{a_1} ((\hat{I}, X) \) are also defined.

**Assumption 1.2**\(^{7)} (a) There exist constants \( \rho_1 > 0 \) and \( c_1 > 1 \) such that

\[
(1.6) \quad -\frac{d}{dr} (B(r) x, x) \geq \frac{c_1}{r} (B(r) x, x)
\]

holds for any \( x \in D \) and any \( r \geq \rho_1 \).

(b) For each finite \( b \in I \) the natural imbedding

\[
(1.7) \quad H^1_{a_1} ((0, b), X) \rightarrow H^0 ((0, b), X)
\]

is compact.

(c) There exists \( c_2 > 0 \) such that

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6) Here and in the sequel \( u' \) and \( u'' \) mean \( \frac{du}{dr} \) and \( \frac{d^2 u}{dr^2} \), respectively.

7) The conditions imposed on \( B(r) \) and \( C(r) \) are the same as in Jäger [5] except (c) of Assumption 1.2. Jäger [5] assumes that

\[ |C(r) x| \leq c_4 (1 + r)^{-\frac{3}{2} - l} (|x| + |B^i r (r) x|), \quad (r \in I, x \in D) \]

instead of (1.8).
\[(1.8) \quad |C(r) x| \leq c_2 (1+r)^{-1-\varepsilon} (|x| + |B^{\frac{1}{2}}(r) x|), \quad (r \in I, x \in D)\]

with some \(0 < \varepsilon < 1\).

For an open interval \(J \subset I\), \(\mathcal{U}(J)\) denotes the set of all linear, continuous functionals on \(H_0^1 B(J, X)\). \(\mathcal{U}(J)\) is a Banach space with the norm
\[(1.9) \quad \|\ell\|_J = \sup \{ |\langle \ell, \varphi \rangle| ; \varphi \in C_0^\infty B(J, X), \|\varphi\|_B = 1\}.

For example, for \(g \in H^0(J, X)\) we define \(l[g] \in \mathcal{U}(J)\) by
\[(1.10) \quad \langle l[g], \varphi \rangle = ((g, \varphi))_{0, J} \quad (\varphi \in H_0^1 B(J, X)).\]

Then we can easily see
\[(1.11) \quad \|l[g]\|_J \leq \|g\|_{0, J}.

**Definition 1.3.** Let \(\ell \in \mathcal{U}(I), u \in H^{1, B}(I, X)\) and \(k \in \mathbb{C}^+\) be given, where
\[(1.12) \quad \mathbb{C}^+ = \{k \mid k \in \mathbb{C}, \text{Im} k \geq 0 \text{ and } \text{Re} k \neq 0\}. \quad \text{8)}

Then \(v \in \text{loc} H^{1, B}(I, X)\) is called a radiative function for \(\{L, k, l, u\}\), if the following three conditions hold:

(a) \(v - u \in \text{loc} H_1^1 B(I, X)\).

(b) \(v' - ikv \in H^{-1+\varepsilon}(I, X)\) (the “radiation condition”\(^9\))

(c) For all \(\varphi \in C_0^\infty B(I, X)\) we have
\[(1.13) \quad ((v, (L - k^2) \varphi))_{0, I} = \langle l, \varphi \rangle.

We shall give a lemma which will be used to prove the existence theorem of the radiative function.

**Lemma 1.4.\(^{10}\)** Let \(I_0 = (b, \infty), b > 0\). For each \(r \in I_0\) \(B(r)\) is assumed

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8) Im \(k\) and Re \(k\) mean the imaginary and real, respectively.

9) In Jäger [5] the radiation condition is defined by \(v' - ikv \in H^0(I, X)\).

to be a non-negative, self-adjoint operator in $X$ with $\mathcal{D}(B(r)) = D$ constant in $r$. Suppose that $(B(r)x, x) \in C^1(I_0)$ for any $x \in D$ and that we have

$$\frac{d}{dr} (B(r)x, x) \geq \frac{e_0}{r} (B(r)x, x) \quad (x \in D, r \geq b_0)$$

with constants $b_0 > b$ and $e_0 > 1$. Let $C(r), r \in I_0$, be a symmetric operator with $\mathcal{D}(C(r)) = D$. Let $v(r)$ be an $X$-valued function on $I_0$ which satisfies the following (i) $\sim$ (iii):

(i) \quad $v \in C^2(I_0, D)$, \quad $Bv, Cv \in C^0(I_0, X)$, and

(ii) \quad $v' - ikv \in H^{-1+\varepsilon}(I_0, X)$ and $v \in H^{-1-\varepsilon}(I_0, X)$.

(iii) \quad We have

$$|C(r)v(r)|^2 \leq e_1 r^{-2-2\varepsilon}(|v'(r)|^2 + |B(r)^{1/2}v(r)|^2) \quad (r \geq b)$$

with constants $e_1 > 0$, $0 < \varepsilon < 1$.

Then there exist constants $\delta_0 > 0$ and $r_0 \geq b_0 + 1$ which do not depend on $v(r)$ and $g(r)$ such that

$$\int_{r_0}^{\infty} r^{-1+\varepsilon} \{ |v'(r) - ikv(r)|^2 + (B(r)v(r), v(r)) \} dr$$

$$\leq \delta_0 \int_{r_0}^{\infty} (r^{-1-\varepsilon} |v(r)|^2 + r^{1+\varepsilon} |g(r)|^2) dr.$$ 

Moreover $\delta_0$ and $r_0$, as functions of $k$, are bounded on any bounded set in $C^+$. For the proof of this lemma we need the following lemma due to Jäger [5] (Lemma 4.1).

**Lemma 1.5.** Let $-\infty \leq a_2 < a_1 < b_1 < b_2 \leq \infty$ and put $I_i = (a_i, b_i)$,
Let $B(r)$ be a non-negative, self-adjoint operator in $X$ for each $r \in I_2$ with $\mathcal{D}(B(r)) = D$ constant in $r$. Let $C(r)$ be a symmetric operator in $X$ with $\mathcal{D}(C(r)) = D$ for each $r \in I_2$. Suppose that $v \in H^{1, B}(I_2, X)$ satisfies for any $\varphi \in C_0^2(I_2, X)$

\begin{equation}
\left( v , \left(-\frac{d^2}{dr^2} + B(\cdot) + C(\cdot) - k^2\right) \varphi \right)_{0, I_2} = \langle l, \varphi \rangle ,
\end{equation}

where $k \in \mathbb{C}^+$ and $l \in \mathcal{D}(I_2)$. Suppose, further, that $v$ satisfies

\begin{equation}
|C(r)v(r)| \leq c_2(|v'(r)| + |B^{\frac{1}{2}}(r)v(r)| + |v(r)|), \quad (r \in I_2)
\end{equation}

with a constant $c_2 = c_2(I_2) > 0$. Then there exists a constant $K = K(I_1, I_2, k) > 0$ such that

\begin{equation}
\|v\|_{B, I_2} \leq K(\|v\|_{0, I_2} + \|l\|_{I_2})
\end{equation}

holds. Further if we assume $v \in H^{1, B}_{0, a_1}(I_2, X)$ and $a_2 > -\infty$ then the conclusion is valid for $a_2 \leq a_1$.

**Proof of Lemma 1.4.** Take $r_0 \geq b_0 + 1$, where $b_0$ is given in (1.14). Let $\varphi \in C^1(I_0)$ such that $0 \leq \varphi \leq 1$, $\varphi'(r) \geq 0$, and

\begin{equation}
\varphi(r) = \begin{cases} 
0 & \text{for } r \in (b, r_0] , \\
1 & \text{for } r \in [r_0 + 1, \infty) .
\end{cases}
\end{equation}

Then we have for $r \geq r_0$

\begin{equation}
\frac{d}{dr} \left( r^\varepsilon \varphi(r) |v'(r) - ikv(r)|^2 \right) \\
= \varepsilon r^{1+\varepsilon} \varphi(r) |v'(r) - ikv(r)|^2 + r^\varepsilon \varphi'(r) |v'(r) - ikv(r)|^2 \\
+ 2r^\varepsilon \varphi(r) \Re (v''(r) - ikv'(r), v'(r) - ikv(r)) \\
\geq \varepsilon r^{1+\varepsilon} \varphi(r) |v' - ikv|^2 + 2r^\varepsilon \varphi(r) \Re (v'' - ikv', v' - ikv),
\end{equation}

since we have assumed that $\varphi'(r) \geq 0$. Noting that $\Im k \geq 0$ we have

\begin{equation}
\Re (v''(r) - ikv'(r), v'(r) - ikv(r))
\end{equation}
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\[ \text{Re}(v''(r) - B(r) v(r) + k^2 v(r), v'(r) - ikv(r)) \]
\[ + \text{Im}(k) \{ |v'(r) - ikv(r)|^2 + (B(r) v(r), v(r)) \} \]
\[ + \text{Re}(B(r) v(r), v'(r)) \]
\[ \geq \text{Re}(v''(r) - B(r) v(r) + k^2 v(r), v'(r) - ikv(r)) \]
\[ + \text{Re}(B(r) v(r), v'(r)). \]

We estimate \(2\varepsilon \text{Re}(v'' - Bv + k^2 v, v' - ikv)\) as follows:

\[(1.24) \quad 2\varepsilon \text{Re}(v''(r) - B(r) v(r) + k^2 v(r), v'(r) - ikv(r)) \]
\[ \geq -2\varepsilon |v''(r) - B(r) v(r) + k^2 v(r)| |v'(r) - ikv(r)| \]
\[ \geq -\frac{\varepsilon^2 \alpha^2}{\alpha} |v''(r) - B(r) v(r) + k^2 v(r)|^2 - \alpha \varepsilon^2 |v'(r) - ikv(r)|^2 \]
\[ (\alpha > 0, \beta + \eta = \varepsilon) \]
\[ \geq -\frac{\delta_1}{\alpha} \left[ r^{-2 - 2\varepsilon + 2\beta} \{ |v(r)|^2 + |v'(r) - ikv(r)|^2 + (B(r) v(r), v(r)) \} \right. \]
\[ \left. + r^{2\beta} |g(r)|^2 \right] - \alpha \varepsilon^2 |v'(r) - ikv(r)|^2, \]

since we have by (1.15) and (1.16)

\[(1.25) \quad |v''(r) - B(r) v(r) + k^2 v(r)|^2 \leq \delta_1 \left[ r^{-2 - 2\varepsilon} \{ |v(r)|^2 + |v'(r) - ikv(r)|^2 \right. \]
\[ \left. + (B(r) v(r), v(r)) \right] + |g(r)|^2 \]

with a constant \(\delta_1 = \delta_1(k) > 0\). We obtain from (1.14)

\[(1.26) \quad 2\text{Re}(B(r) v(r), v'(r)) = \frac{d}{dr}(B(r) v(r), v(r)) - \frac{d}{dr}(B(r) \chi, \chi) \bigg|_{x=e(r)} \]
\[ \geq \frac{d}{dr}(B(r) v(r), v(r)) + \frac{e\alpha}{r}(B(r) v(r), v(r)). \]

(1.22), (1.23), (1.24) and (1.26) are combined to give

\[(1.27) \quad \frac{d}{dr}(r^\varepsilon \psi(r) |v'(r) - ikv(r)|^2) \]
\[ \geq \phi(r) \left( e r^{-1 - \varepsilon} - \frac{\delta_1}{\alpha} r^{-2 - 2\varepsilon + 2\beta} - \alpha \varepsilon^2 \right) |v'(r) - ikv(r)|^2 \]
Putting \( \eta = \frac{1}{2} (-1 + \varepsilon) \), \( \beta = \frac{1}{2} (1 + \varepsilon) \) and \( \gamma = \frac{1}{2} \varepsilon \), we integrate (1.27) from \( r_0 \) to \( R (R \geq r_0 + 2) \) to obtain

\[
R^\varepsilon |v'(R) - i\varepsilon v(R)|^2
\]

\[
\geq \int_{r_0}^{R} \psi(r) \left( \frac{\varepsilon}{2} r^{-1+\varepsilon} - \frac{2\delta_1}{\varepsilon} r^{-1-\varepsilon} \right) |v'(r) - i\varepsilon v(r)|^2 \, dr
\]

\[
+ \int_{r_0}^{R} \psi(r) \left( (e_0 - \varepsilon) r^{-1+\varepsilon} - \frac{2\delta_1}{\varepsilon} r^{-1-\varepsilon} \right) (B(r) v(r), v(r)) \, dr
\]

\[
- \frac{2\delta_1}{\varepsilon} \int_{r_0}^{R} \psi(r) (r^{-1-\varepsilon} |v(r)|^2 + r^{1+\varepsilon} |g(r)|^2) \, dr
\]

\[- \int_{r_0}^{r_{0+1}} r^\varepsilon \psi'(r) (B(r) v(r), v(r)) \, dr,
\]

where we have made use of the estimate

\[
\int_{r_0}^{R} r^\varepsilon \psi(r) \frac{d}{dr} (B(r) v(r), v(r)) \, dr
\]

\[
= R^\varepsilon (B(R) v(R), v(R)) - \int_{r_0}^{R} \frac{d}{dr} (r^\varepsilon \psi(r)) \left( B(r) v(r), v(r) \right) \, dr
\]

\[
\geq - \int_{r_0}^{R} \varepsilon r^{-1+\varepsilon} \psi(r) (B(r) v(r), v(r)) \, dr
\]

\[- \int_{r_0}^{r_{0+1}} r^\varepsilon \psi'(r) (B(r) v(r), v(r)) \, dr.
\]

Now we take \( r_0 (\geq b_0 + 1) \) so large that we have with a constant \( \delta_2 > 0 \)

\[
\left\{ \begin{array}{l}
\frac{\varepsilon}{2} r^{-1+\varepsilon} - \frac{2\delta_1}{\varepsilon} r^{-1-\varepsilon} \geq \delta_2 r^{-1+\varepsilon} \\
(e_0 - \varepsilon) r^{-1+\varepsilon} - \frac{2\delta_1}{\varepsilon} r^{-1-\varepsilon} \geq \delta_2 r^{-1+\varepsilon}
\end{array} \right.
\]
for all \( r \geq r_0 \). On the other hand, using Lemma 1.5 with \( I_1 = (r_0, r_0 + 1) \) and \( I_2 = (r_0 - 1, r_0 + 2) \), we obtain the following estimate with constants \( K > 0 \) and \( \delta_3 > 0 \):

\[
(1.31) \quad \int_{r_0}^{r_0+1} r^\varepsilon \psi'(r) (B(r) v(r), v(r)) \, dr \\
\leq (r_0 + 1)^\varepsilon \left( \max_{r_0 \leq r \leq r_0 + 1} \psi'(r) \right) \int_{r_0}^{r_0+1} (B(r) v(r), v(r)) \, dr \\
\leq (r_0 + 1)^\varepsilon \left( \max_{r_0 \leq r \leq r_0 + 1} \psi'(r) \right) K \int_{r_0}^{r_0 + 2} (|v(r)|^2 + |g(r)|^2) \, dr \\
< \delta_3 \int_{r_0}^{r_0+1} (r^{-1+\varepsilon} |v(r)|^2 + r^{1+\varepsilon} |g(r)|^2) \, dr,
\]

where we used (1.11). It follows from (1.28), (1.30) and (1.31) that

\[
(1.32) \quad \delta_2 \int_{r_0+1}^R r^{-1+\varepsilon} \{|v'(r) - ikv(r)|^2 + (B(r) v(r), v(r))\} \, dr \\
\leq R^\varepsilon |v'(R) - ikv(R)|^2 \\
+ \left( \frac{2\delta_1}{\varepsilon} + \delta_3 \right) \int_{r_0+1}^R (r^{-1+\varepsilon} |v(r)|^2 + r^{1+\varepsilon} |g(r)|^2) \, dr.
\]

Since \( r^{-1+\varepsilon} |v'(r) - ikv(r)|^2 \) is integrable on \( I_0 \), we have

\[
(1.33) \quad R_j |v'(R_j) - ikv(R_j)|^2 \to 0, \quad j \to \infty
\]

for some sequence \( R_j \to \infty \). Thus we obtain (1.17) from (1.32). Q.E.D.

**Lemma 1.6.** Let us assume Assumption 1.1 and (a) and (c) of Assumption 1.2. Let \( k \in \mathbb{C}^+ \) and let \( v(r) \) be a radiative function for \( \{L, k, l^\varepsilon g\}, 0 \) with \( v \in \text{loc} H^{1+\varepsilon}(\bar{I}, X) \cap H^{-1-\varepsilon}(I, X) \) and \( g \in \text{loc} H^{1,B}(\bar{I}, X) \) \( \cap H^{1+\varepsilon}(I, X) \). Then there exists a constant \( \delta > 0 \) such that

\[
(1.34) \quad ||v' - ikv||_{-1+\varepsilon} + ||B^1 v||_{-1+\varepsilon} \leq \delta ||v||_{-1-\varepsilon} + ||g||_{1-\varepsilon},\quad \text{(12)}
\]

where \( \delta \) depends only on \( k \) and is bounded on any bounded set in \( \mathbb{C}^+ \).

---

12) Here and in the sequel we put \( \| \|_{s,t} = \| \|_s \) and \( \| \|_{s,t} = \| \|_B \) for the sake of simplicity.
Proof. It follows from Assumption 1.1 that we can apply the regularity theorem of Jäger [5] (Satz 3.1, p. 76) to see that \( v \in C^2(I, D) \), \( Bv \), \( Cv \in C^0(I, X) \), and \( v \) satisfies (1.15) for all \( r \in I \). From Lemma 1.4 we obtain

\[
(1.35) \quad \int_{r_0+1}^{r_1+1} (1+r)^{-1+\varepsilon} \left\{ |v'(r) - ikv(r)|^2 + (B(r)v(r), v(r)) \right\} dr 
\leq K_1 \int_{r_0-1}^{r_1} (1+r)^{-1-\varepsilon} |v(r)|^2 + r^{1+\varepsilon} |g(r)|^2 dr
\]

with constants \( r_0 \geq \rho_1 + 1 \) and \( K_1 > 0 \). Since \( v \in \text{loc}\, H^{1, B}_0(\tilde{I}, X) \), we can use the last statement of Lemma 1.5 with \( I_1 = (0, r_0+1) \) and \( I_2 = (0, r_0+2) \) to obtain

\[
(1.36) \quad \int_{0}^{r_0+1} (1+r)^{-1+\varepsilon} \left\{ |v'(r) - ikv(r)|^2 + (B(r)v(r), v(r)) \right\} dr 
\leq K_2 \int_{0}^{r_0+2} (1+r)^{-1-\varepsilon} |v(r)|^2 + (1+r)^{1+\varepsilon} |g(r)|^2 dr
\]

with a constant \( K_2 > 0 \). (1.34) follows from (1.35) and (1.36). Q.E.D.

§2. The Uniqueness Theorem

We shall show the uniqueness of the radiative function using arguments due to Jäger [5].

Lemma 2.1. Let \( B(r) \) satisfy (a) and (b) of Assumption 1.1. Let \( C(r) \) be a symmetric operator in \( X \) for each \( r \in I \) such that \( C(r) \) satisfies (c) and (d) of Assumption 1.1. and

\[
(2.1) \quad |C(r)x| \leq c(|x| + |B^{1, B}_{1/2}(r)x|), \quad (x \in D, r \in I)
\]

with a constant \( c > 0 \). Let \( v \in \text{loc}\, H^{1, B}_0(\tilde{I}, X) \) satisfy

\[
(2.2) \quad (v, (L - \tilde{k}^2)\varphi)_0 = (g, \varphi)_0 \quad (\varphi \in B^{0, B}_0(I, X))
\]

with \( g \in \text{loc}\, H^{1, B}_0(\tilde{I}, X) \) and \( k \in \mathbb{C}^+ \), where

\[
(2.3) \quad L = -\frac{d^2}{dr^2} + B(r) + C(r).
\]
Then we have for all \( r \in I, \)
\[
(2.4) \quad |v'(r) - ikv(r)|^2 = |v'(r) + (\text{Im} k) v(r)|^2 + (\text{Re} k)^2 |v(r)|^2 + 4(\text{Re} k)^2 (\text{Im} k) |v||_{0,(r)}^2 + 2(\text{Re} k) \text{Im}((g, v))_{0,(r)}.
\]

**Proof.** As we have seen in the proof of Lemma 1.6, it follows from the regularity theorem of Jäger [5] (p. 76) that

\[
(2.5) \quad \begin{cases}
\{v \in C^2(I, D) \text{ and } Bv, Cv \in C^0(I, X) \\
(L - k^2)v(r) = g(r) \quad (r \in I).
\end{cases}
\]

On the other hand we obtain from the fact that \( v \in \text{loc} H^{1, B}(I, X) \)
\[
(2.6) \quad \begin{cases}
v \in C^0(I, X) \\
v(0) = 0.
\end{cases}
\]

From (2.5) and (2.6) we see that
\[
(2.7) \quad \int_0^r (g(t), \varphi(t)) dt = \int_0^r ((L - k^2)v(t), \varphi(t)) dt = \int_0^r \{(v'(t), \varphi'(t)) + ((B(t) + C(t) - k^2)v(t), \varphi(t))\} dt - (v'(r), \varphi(r))
\]
holds for any \( \varphi \in C^2_{0, B}(I, X). \) Since \( v \in \text{loc} H^{1, B}(I, X), \) for any \( r > 0 \) there is a sequence \( \{\varphi_n\} \) in \( C^2_{0, B}(I, X) \) such that
\[
(2.8) \quad \begin{cases}
||\varphi_n - v||_{B,(0, r+1)} \rightarrow 0, \\
\varphi_n(t) \rightarrow v(t) \quad \text{in } X \quad (t \in [0, r+1])
\end{cases}
\]
as \( n \rightarrow \infty. \) Replacing \( \varphi \) by \( \varphi_n \) in (2.7) and letting \( n \rightarrow \infty, \) we have

13) Note that \( H^{1, B}(I, X) \) is continuously imbedded in \( C^0(I, X). \) See Jäger [5], p. 69.
Hence we obtain

\begin{equation}
\text{Im}(v'(r), v(r)) = -\text{Im}((g, v))_{0,0,r} - 2(\text{Re} k)(\text{Im} k\|v\|_0^2, 0, r).
\end{equation}

Using (2.10), we calculate $|v'(r) - ikv(r)|^2$ as follows:

\begin{equation}
|v'(r) - ikv(r)|^2 = |v'(r) + (\text{Im} k)v(r)|^2 + (\text{Re} k)^2|v(r)|^2 - 2(\text{Re} k)\text{Im}(v'(r), v(r))
\end{equation}

\begin{equation}
= |v'(r) + (\text{Im} k)v(r)|^2 + (\text{Re} k)^2|v(r)|^2 + 4(\text{Re} k)^2(\text{Im} k\|v\|_0^2, 0, r) + 2(\text{Re} k)\text{Im}((g, v))_{0,0,r}.
\end{equation}

Q.E.D.

**Theorem 2.2.** Let us assume Assumption 1.1 and (a) and (c) of Assumption 1.2. Let $l \in \mathcal{B}(I)$, $k \in \mathbb{C}^+$ and $u \in H^{1, b}(I, X)$ be given. Then the radiative function for $\{L, k, l, u\}$ is unique.

**Proof.** Let $v$ be a radiative function for $\{L, k, 0, 0\}$, where $k \in \mathbb{C}^+$. What we want to show is that $v$ is identically zero.

We start with the relation

\begin{equation}
|v'(r) - ikv(r)|^2 = |v'(r) + (\text{Im} k)v(r)|^2 + (\text{Re} k)^2|v(r)|^2 + 4(\text{Re} k)^2(\text{Im} k\|v\|_0^2, 0, r),
\end{equation}

which follows from Lemma 2.1.

If $\text{Im} k > 0$, then we obtain from (2.12) and the fact that $v' - ikv \\
\in H^{-1+\epsilon}(I, X)$

\begin{equation}
0 \leq \|v\|_{0,0,r}^2 \leq \frac{1}{4(\text{Re} k)^2(\text{Im} k)} |v'(r_j) - ikv(r_j)|^2 \to 0, \quad j \to \infty,
\end{equation}

\begin{equation}
\int_0^r (g(t), v(t)) dt = \int_0^r \{ |v'(t)|^2 + ((B(t) + C(t) - k^2)v(t), v(t)) \} dt - (v'(r), v(r)).
\end{equation}
for some sequence \( r_j \to \infty \). Hence we have \( \|v\|_0^2 = 0 \), i.e., \( v \equiv 0 \).

Next let us assume that \( \text{Im} k = 0 \). Then we have from (2.12) and the radiation condition \( v' - ikv \in H^{-1+\epsilon}(I, X) \)

\[
\lim_{r \to \infty} ( |v'(r)|^2 + k^2 |v(r)|^2 ) = \lim_{r \to \infty} |v'(r) - ikv(r)|^2 = 0.
\]

By the regularity theorem of Jäger [5] (p. 76) and (1.8) we have

\[
\left\{ \begin{array}{l}
v \in C^2(I, D) \\
|v''(r) - B(r)v(r) + k^2 v(r)|^2 = |C(r)v(r)|^2 \\
\leq 2c_2^2 (1+r)^{-2-2\epsilon} \{ |v(r)|^2 + (B(r)v(r), v(r)) \}
\end{array} \right. \quad (r \in I),
\]

where \( c_2 > 0 \) is given in (1.8). (2.14) and (2.15) enable us to apply Hilfssatz 1 of Jäger [3] (p. 66) on the growth property of solutions of the equation \( (L - k^2)v = 0 \) to show that the carrier of \( v \) is compact in \( I \). Hence, using Satz 3 of Jäger [4] (p. 32), a unique continuation theorem for solutions of the equation \( (L - k^2)v = 0 \), we see that \( v \equiv 0 \) on \( I \).

Q.E.D.

§ 3. The Existence Theorems

This section is devoted to showing the existence of the radiative function \( v \) for \( \{L, k, I, u\} \), where \( k \in \mathcal{C}, \ u \in H^{1,B}(I, X) \), and \( l \) belongs to a subspace \( \mathcal{U}_{1+\epsilon}(I) \) of \( \mathcal{U}(I) \). We shall first prove a priori estimates for radiative functions \( v \) for \( \{L, k, l, 0\}, \ k \in \mathcal{C}^+ \) and \( l \in \mathcal{U}_{1+\epsilon}(I) \) (Lemma 3.1 and Lemma 3.4). This corresponds to Satz 5.3 of Jäger [5]. But it seems that we have to modify its proof in order to obtain the a priori estimates needed in our case. Lemma 3.2 is necessary for this modification. Next we shall prove the existence theorems using our a priori estimates (Theorem 3.7 and Theorem 3.8). At the same time we shall see that the radiative function \( v \) for \( \{L, k, l, u\} \) depends continuously on \( k, l \) and \( u \).
Lemma 3.1. Let us assume Assumptions 1.1 and 1.2. Let $K$ be a compact set in $C^+$. Let $k \in K$ and $g \in H^{1+\epsilon}(I, X) \cap \text{loc} H^{1, B}(I, X)$. Let $v$ be a radiative function for $\{L, k, l \in g, 0\}$ such that

\begin{equation}
\tag{3.1}
v \in \text{loc} H^{1, B}(I, X) \cap H^{-1-\epsilon}(I, X).
\end{equation}

Then we have

\begin{equation}
\tag{3.2}
\|v\|_{1-\epsilon} + \|v' - ikv\|_{1+\epsilon} + \|B^2 v\|_{1+\epsilon} \leq \delta_1 \|g\|_{1+\epsilon}
\end{equation}

with a constant $\delta_1 > 0$, where $\delta_1$ depends only on $K$ and $L$.

To prove this lemma we prepare

Lemma 3.2. Let $K, g, k$ and $v$ be as in Lemma 3.1. Then there exists a positive number $\alpha_0$ such that

\begin{equation}
\tag{3.3}
\int_{\rho}^{\infty} (1+r)^{-1-\epsilon} |v(r)|^2 dr \leq \alpha_0 (\|v\|_{1-\epsilon}^2 + \|g\|_{1+\epsilon}^2) \rho^{-\epsilon}, \quad (\rho \geq 1),
\end{equation}

where $\alpha_0$ depends only on $K$ and $L$.

Proof. From Lemma 2.1 we obtain

\begin{equation}
\tag{3.4}
(\text{Re} k)^2 |v(r)|^2 + 2(\text{Re} k) \text{Im} ((g, v))_0,0,r) \leq |v'(r) - ikv(r)|^2,
\end{equation}

whence we have

\begin{equation}
\tag{3.5}
|v(r)|^2 \leq \frac{1}{(\text{Re} k)^2} |v'(r) - ikv(r)|^2 + \frac{2}{|\text{Re} k|} \int_0^r |g(t)| |v(t)| dt
\end{equation}

\begin{align*}
\leq & \frac{1}{(\text{Re} k)^2} |v'(r) - ikv(r)|^2 \\
& + \frac{2}{|\text{Re} k|} \left\{ \int_0^r (1+t)^{1+\epsilon} |g(t)|^2 dt \right\}^{1/2} \left\{ \int_0^r (1+t)^{-1-\epsilon} |v(t)|^2 dt \right\}^{1/2} \\
\leq & \frac{1}{(\text{Re} k)^2} |v'(r) - ikv(r)|^2 + \frac{2}{|\text{Re} k|} \|g\|_{1+\epsilon} \|v\|_{1-\epsilon}.
\end{align*}

Multiplying both sides of (3.5) by $r^{-1-\epsilon}$ and integrating from $\rho$ to $\infty$, we have
(3.6) \[
\int_\rho r^{-1-\varepsilon} |v(r)|^2 dr \leq \frac{1}{(\text{Re} \, k)^2} \int_\rho r^{-1-\varepsilon} |v'(r) - ikv(r)|^2 dr
\]
\[
+ \frac{2}{\varepsilon |\text{Re} \, k|} \|g\|_{1+\varepsilon}\|v\|_{-1-\varepsilon} \rho^{-\varepsilon}
\]
\[
\leq \frac{1}{(\text{Re} \, k)^2} \rho^{-2\varepsilon} \int_\rho r^{-1+\varepsilon} |v'(r) - ikv(r)|^2 dr
\]
\[
+ \frac{2}{\varepsilon |\text{Re} \, k|} \|g\|_{1+\varepsilon}\|v\|_{-1-\varepsilon} \rho^{-\varepsilon}.
\]

(3.3) follows from (3.6) and Lemma 1.6. Q.E.D.

Proof of Lemma 3.1. It follows from Lemma 1.6 that it is enough to show

(3.7) \[\|v\|_{-1-\varepsilon} \leq \alpha \|g\|_{1+\varepsilon}\]

with a constant \(\alpha > 0\) depending only on \(K\) and \(L\). Let us assume that (3.7) is false. Then for each positive integer \(n\) we can find \(k_n \in K\), \(h_n \in \text{loc} \, H^{1,2}(\tilde{I}, X)\), and radiative functions \(u_n\) for \(\{L; k_n, l[\bar{h}_n], 0\}\) such that

(3.8) \[\|u_n\|_{-1-\varepsilon} > n \|h_n\|_{1+\varepsilon}\]

Since we see \(\|u_n\|_{-1-\varepsilon} > 0\) from (3.8), we obtain radiative functions \(v_n = \frac{u_n}{\|u_n\|_{-1-\varepsilon}}\) for \(\{L; k_n, l[\bar{g}_n], 0\}\), \(g_n = \frac{h_n}{\|u_n\|_{-1-\varepsilon}}\), with

(3.9) \[
\begin{cases} 
\|v_n\|_{-1-\varepsilon} = 1, \\
\|g_n\|_{1+\varepsilon} < \frac{1}{n}.
\end{cases}
\]

Let \(\{k_{nm}\}\) be a subsequence of \(\{k_n\}\) satisfying

(3.10) \[k_{nm} \to k_0, \quad m \to \infty\]

with \(k_0 \in K\). Without loss of generality we can assume

(3.11) \[k_n \to k_0, \quad n \to \infty.\]

In view of (3.9) we have for any \(R \in I\)
Therefore it follows from Lemma 1.5 that

\[
\sup_n \|v_n\|_{0,(0,R+1)} < \infty, \\
\sup_n \|g_n\|_{0,(0,R+1)} < \infty.
\] (3.12)

Therefore it follows from Lemma 1.5 that

\[
\sup_n \|v_n\|_{B,(0,R)} < \infty
\] (3.13)

for all \( R > 0 \). Since for all \( 0 < R < \infty \) the imbedding \( H^{1,B}_0((0, R), X) \to H^0((0, R), X) \) is compact by (b) of Assumption 1.2, we obtain a subsequence of \( \{v_n\} \) which is a Cauchy sequence in \( H^0((0, R), X) \) for all \( R \in I \). Without loss of generality we can assume that \( \{v_n\} \) itself is a Cauchy sequence in \( H^0((0, R), X) \) for all \( R \in I \). The sequence \( \{v_n\} \) is a Cauchy sequence in \( H^{1,B}_0((0, R), X) \) for all \( R \in I \), too. In fact for each pair \((n, m)\) \( v_n - v_m \) is the radiative function for \( \{I, k, l[g_{nm}], 0\} \), where

\[
g_{nm} = g_n - g_m - (k_0^2 - k_n^2)v_n + (k_0^2 - k_m^2)v_m
\] (3.14)

and \( k_0 \) is given as in (3.11). From (3.9) and (3.11) we obtain \( g_{nm} \to 0, n, m \to \infty \) in \( H^0((0, R+1), X) \) for any \( R > 0 \). Hence, noting that \( \{v_n\} \) is a Cauchy sequence in \( H^0((0, R+1), X) \), we can apply Lemma 1.5 to show

\[
\|v_n - v_m\|_{B,(0,R)} \lesssim \beta(\|v_n - v_m\|_{0,(0,R+1)} + \|g_{nm}\|_{0,(0,R+1)})
\] (3.15)

\[
\to 0, \quad n, m \to \infty,
\]

where \( \beta > 0 \) depends only on \( R, k \) and \( L \). Therefore there exists \( v \in \text{loc} \, H^{1,B}_0(I, X) \) satisfying

\[
v_n \to v, \quad n \to \infty
\] (3.16)

both in \( H^{1,B}_0((0, R), X) \) and in \( H^0((0, R), X) \) for any \( R \in I \).

Letting \( n \to \infty \) in the relation

\[
((v_n, (L - \bar{k}_n^2)\varphi))_0 = ((g_n, \varphi))_0, \quad (\varphi \in C^{1,B}_0(I, X))
\] (3.17)

we obtain from (3.16), (3.9) and (3.11)
(3.18) \((v, (L-\kappa_0^2)\varphi))_0=0\).

Using (3.9), (3.16) and Lemma 1.6 we estimate \(\|v' - ik_0 v\|_{-1-\varepsilon, (0,R)}\) as follows:

\[
(3.19) \quad \|v' - ik_0 v\|_{-1-\varepsilon, (0,R)} = \lim_{n \to \infty} \|v'_n - ik_0 v_n\|_{-1-\varepsilon, (0,R)}
\]

\[
\leq \sup_n \|v'_n - ik_0 v_n\|_{-1-\varepsilon}
\]

\[
\leq \delta \sup_n \{\|v_n\|_{-1-\varepsilon} + \|g_n\|_{1+\varepsilon}\}
\]

\[
\leq \delta \sup_n \left(1 + \frac{1}{n}\right) \leq 2\delta,
\]

where \(\delta > 0\) is as in Lemma 1.6. Since the last member of (3.19) does not depend on \(n\) and \(R\), we have \(v' - ik_0 v \in H^{-1+\varepsilon}(I, X)\), i.e., \(v\) satisfies the radiation condition. Thus \(v\) is a radiative function for \(\{L, k_0, 0, 0, R\}\), and hence \(v = 0\) by Theorem 2.2.

From Lemma 3.2 we obtain for \(\rho \geq 1\)

\[
(3.20) \quad \lim_{n \to \infty} \|v_n\|_{-1-\varepsilon}^2 \leq \lim_{n \to \infty} \|v_n\|_{-1-\varepsilon, (0,\rho)}^2 + \sup_n \|v_n\|_{-1-\varepsilon, (\rho, \infty)}^2
\]

\[
\leq \|v\|_{-1-\varepsilon, (0,\rho)}^2 + \alpha_0 \rho^{-\varepsilon} \sup_n \{\|v_n\|_{-1-\varepsilon} + \|g_n\|_{1+\varepsilon}\}
\]

\[
= 0(\rho^{-\varepsilon}),
\]

where we have noted (3.9) and the fact \(v = 0\). Since \(\rho \geq 1\) is arbitrary, we obtain \(\lim_{n \to \infty} \|v_n\|_{-1-\varepsilon} = 0\), which contradicts the assumption that \(\|v_n\|_{-1-\varepsilon} = 1\).

Q.E.D.

Now we introduce a subspace of \(\mathcal{V}(I)\).

**Definition 3.3.** Let \(\mathcal{V}_{1+\varepsilon}(I)\) be the set of all \(l \in \mathcal{V}(I)\) such that

\[
(3.21) \quad \|l\|_{1+\varepsilon} = \sup \{|< l, (1+r)^{1+\varepsilon}\varphi>|, \varphi \in C_0^{\infty}(I, X), \|\varphi\|_B = 1\} < \infty.
\]

\(\mathcal{V}_{1+\varepsilon}(I)\) is a Banach space with the norm \(\|\|_{1+\varepsilon}\).

It is easy to see that we have

\[
(3.22) \quad \|l\| \leq \alpha_0 \|l\|_{1+\varepsilon} \quad (l \in \mathcal{V}_{1+\varepsilon}(I))
\]
with a constant $a_0 > 0$.

We shall show that the inequality (3.2) also holds for the radiative function for $\{L, k, l, 0\}$, where $l \in \mathcal{U}_{1+\delta}(I)$.

**Lemma 3.4.** Let us assume Assumptions 1.1 and 1.2. Let $K$ be as in Lemma 3.1. Let $k \in K$ and $l \in \mathcal{U}_{1+\delta}(I)$. Let $v$ be a radiative function for $\{I, k, l, 0\}$ such that $v \in H^{-1-\delta}(I, X) \cap \text{loc} H^{1-\delta}_{0}B(I, X)$.

Then there exists a constant $\delta_2 > 0$ such that

$$
||v||_{1-\delta} + ||v' - ikv||_{1-\delta} + ||B^{1/2}v||_{1+\delta} \leq \delta_2 ||l||_{1+\delta},
$$

where $\delta_2$ depends only on $K$ and $L$.

To prove this lemma we need

**Lemma 3.5.** Let $B(r)$ satisfy (a) of Assumption 1.1 and let $C(r)$ $(r \in I)$ be a symmetric operator in $X$ with the domain $\mathcal{D}(C(r)) = D$ such that

$$
|C(r)x| \leq c(|x| + |B^{1/2}(r)x|) \quad (r \in I, x \in D)
$$

with a constant $c > 0$. Let $k_0 \in C^+$ and $\text{Im} k_0 > 0$. Let $l \in \mathcal{U}(I)$. Then the equation

$$
(u, (L - k_0^2)\varphi)_0 = <l, \varphi> \quad (\varphi \in C_0^{1/2}B(I, X))
$$

has a unique solution $u$ in $H^{1/2}_{0}B(I, X)$ with the estimate

$$
||u||_{B} \leq \beta_1 ||l||,
$$

where $\beta_1 = \beta_1(k_0) > 0$ is a constant. Further, if $l \in \mathcal{U}_{1-\delta}(I)$, then we have $u \in H^{1+\delta}(I, X)$ and

$$
||u||_{1+\delta} \leq \beta_2 ||l||_{1+\delta}
$$

with a constant $\beta_2 = \beta_2(k_0) > 0$.

**Proof.** Let us define a bilinear form $\mathcal{A}_{L-k_0^2}[-,\cdot]$ on $H^{1/2}_{0}B(I, X)$

14) Cf. Jäger [5], Lemma 2.3 (p. 75) and the proof of Satz 5.3 (p. 86).
Then we shall show

\begin{equation}
(3.29) \quad d_j \| w \|_B^2 \geq |A_{L-k_0^2}[w, w]| \geq d_2 \| w \|_B^2 \quad (w \in H_0^{1,B}(I, X)),
\end{equation}

where \( d_j = d_j(k_0) > 0 \) (\( j = 1, 2 \)) are constants. Since \( C_0^{2,B}(I, X) \) is dense in \( H_0^{1,B}(I, X) \), and we have by integration by parts

\begin{equation}
(3.30) \quad A_{L-k_0^2}[\varphi_1, \varphi_2] = \langle (\varphi_1, (L-k_0^2)\varphi_2) \rangle_0 \quad (\varphi_1, \varphi_2 \in C_0^{2}(I, X)),
\end{equation}

it is sufficient to show (3.29) that we show

\begin{equation}
(3.31) \quad \begin{cases}
   d_1 \| \varphi \|_B^2 \geq |\langle (\varphi, (L-k_0^2)\varphi) \rangle_0|,
   \\
   d_2 \| \varphi \|_B^2 \leq |\langle (\varphi, (L-k_0^2)\varphi) \rangle_0|,
\end{cases} \quad (\varphi \in C_0^{2,B}(I, X)).
\end{equation}

Let us prove (3.31). From (3.24) we see that

\begin{equation}
(3.32) \quad \| C\varphi \|_B^2 \leq \int_I c^2 |\varphi(r) - B^2(r)\varphi(r(r))|^2 dr \leq 2c \| \varphi \|^2_B,
\end{equation}

whence follows for all \( \varphi \in C_0^{2,B}(I, X) \)

\begin{equation}
(3.33) \quad |\langle (\varphi, (L-k_0^2)\varphi) \rangle_0| \leq \| \varphi \|_B^2 + \sqrt{2c} \| \varphi \|_B \| \varphi \|_0 + |k_0|^2 \| \varphi \|_B^2 \leq (1 + \sqrt{2c} + |k_0|^2) \| \varphi \|_B^2.
\end{equation}

Thus we have shown the first inequality of (3.31) with \( d_1 = 1 + \sqrt{2c} + |k_0|^2 \). On the other hand we have

\begin{equation}
(3.34) \quad |\langle (\varphi, (L-k_0^2)\varphi) \rangle_0|^2 = \| \varphi \|_B^4 + \langle ((C-\lambda-1)\varphi, \varphi) \rangle_0^2 + \mu^2 \| \varphi \|_6^4,
\end{equation}

where \( \lambda = \text{Re}k_0^2 \) and \( \mu = \text{Im}k_0^2 \neq 0 \). Hence, using (3.32) again, we have

\begin{equation}
(3.35) \quad |\langle (\varphi, (L-k_0^2)\varphi) \rangle_0|^2 \geq \| \varphi \|_B^2 - 2\| \varphi \|_B^2 \langle ((C-\lambda-1)\varphi, \varphi) \rangle_0 | + \langle ((C-\lambda-1)\varphi, \varphi) \rangle_0^2 + \mu^2 \| \varphi \|_6^4 \\
\geq (1 - \alpha) \| \varphi \|_B^2 - \left( \frac{1}{\alpha} - 1 \right) \langle ((C-\lambda-1)\varphi, \varphi) \rangle_0^2 + \mu^2 \| \varphi \|_6^4
\end{equation}
with $\alpha > 0$. Take $0 < \alpha < 1$ in (3.35). Then, noting that we obtain from (3.32)

\begin{align*}
((C - \lambda - 1) \varphi, \varphi)_0 & \leq \| (C - \lambda - 1) \varphi \|_B \| \varphi \|_B \\
& \leq 2 \{ 2 \gamma_0^2 + (| \lambda | + 1)^2 \} \| \varphi \|_B \| \varphi \|_B \\
& \leq \{ 2 \gamma_0^2 + (| \lambda | + 1)^2 \} \left( \beta \| \varphi \|_B^2 + \frac{1}{\beta} \| \varphi \|_B \right) \quad (\beta > 0),
\end{align*}

we arrive at

\begin{align*}
|((\varphi, (L - \overline{k}^2) \varphi))_0|^2 & \geq \left\{ 1 - \alpha - \frac{1 - \alpha}{\alpha} \beta c_1 \right\} \| \varphi \|_B^2 \\
& + \left\{ \mu^2 - \frac{1 - \alpha}{\alpha \beta} c_1 \right\} \| \varphi \|_B^2 \quad (0 < \alpha < 1, \beta > 0),
\end{align*}

where we put $c_1 = 2 \gamma_0^2 + (| \lambda | + 1)^2$. Putting $\beta = \frac{\alpha}{2c_1}$ and taking $1 - \alpha > 0$ small enough, we obtain from (3.37)

\begin{align*}
|((\varphi, (L - \overline{k}^2) \varphi))_0|^2 & \geq \frac{1}{2} (1 - \alpha) \| \varphi \|_B^2,
\end{align*}

whence follows the second inequality of (3.31) with $d_2 = \sqrt{\frac{1 - \alpha}{2}}$.

Since (3.29) has been justified, we can make use of the Lax-Milgram theorem\(^{15}\) to show that there exists a unique solution of $u$ in $H_0^{1, B}(I, X)$ of the equation

\begin{align*}
B_{L - \overline{k}^2} [u, w] = < l, w > \quad (w \in H_0^{1, B}(I, X))
\end{align*}

for $l \in \mathcal{U}(I)$. Since $B_{L - \overline{k}^2} [u, \varphi] = ((u, (L - \overline{k}^2) \varphi))_0$ for $\varphi \in C_0^{1, B}(I, X)$, it follows from (3.38) that $u$ is a unique solution of (3.25). (3.29) and (3.39) are combined to give

\begin{align*}
\| u \|_B^2 & \leq \frac{1}{d_2} \| B_{L - \overline{k}^2} [u, u] \| = \frac{1}{d_2} \| < l, u > \| \leq \frac{1}{d_2} \| l \| \| u \|_B,
\end{align*}

which implies (3.26) with $\beta_1 = \frac{1}{\sqrt{d_2}}$.

---

\(^{15}\) See, for example, Yosida [6], p. 92.
Next let us show (3.27). Let \( \phi \in C^1(\mathbb{R}) \) such that \( 0 \leq \phi(r) \leq 1 \), \( 0 \leq |\phi'(r)| \leq 1 \) and

\[
\psi(r) = \begin{cases} 
0 & \text{for } r \geq 2, \\
1 & \text{for } r \leq \frac{1}{2}.
\end{cases}
\]

(3.41)

For each \( m = 1, 2, \ldots \) we define

\[
\phi_m(r) = (1 + r)^{\frac{1+\varepsilon}{2}} \phi\left(\frac{r}{m}\right),
\]

(3.42)

\[ u_m = \psi_m u, \]

where \( u \) is the solution of the equation (3.38). Then we obtain from (3.39) and (3.28)

(3.43)

\[
\mathcal{B}_{L-k_1^2}[u_m, u_m] = ||u_m||_B^2 + ((C(r) - k_0^2) u_m, u_m)_0
\]

\[
= ((\phi_m u', u_m')_0 + (B^2 u, B^2 \phi_m u_m)_0
\]

\[
+ ((C(r) - k_0^2) u, \psi_m u_m)_0
\]

\[
= \mathcal{B}_{L-k_0^2}[u, \psi_m u_m] + ((\phi_m' u, u_m')_0 - ((\phi_m' u', u_m))_0
\]

\[
= \langle L, \phi_m u_m \rangle + ((\phi_m' u, u_m')_0 - ((\phi_m' u', u_m))_0.
\]

It follows from (3.43) and (3.29)

(3.44)

\[
||u_m||_B^2 \leq \frac{1}{d_2} ||\mathcal{B}_{L-k_1^2}[u_m, u_m]| |
\]

\[
\leq \frac{1}{d_2} \left\{ \langle L, (1 + r)^{\frac{1+\varepsilon}{2}} \phi\left(\frac{r}{m}\right) u_m \rangle
\right\}
\]

\[
+ ||\phi_m' u||_0 ||u_m||_0 + ||\phi_m' u'||_0 ||u_m||_0
\]

\[
\leq \frac{1}{d_2} \left\{ ||L||_1 + \varepsilon ||\phi\left(\frac{r}{m}\right) u_m ||_B + (||\phi_m' u||_0 + ||\phi_m' u'||_0 ||u_m||_0
\right\}
\]

\[
\leq \frac{2}{d_2} \left\{ \frac{1+\varepsilon}{2} ||u||_B \right\} ||u_m||_B,
\]

where we note \( |\phi\left(\frac{r}{m}\right)| \leq 1 \), \( \left| \frac{1}{m} \phi'\left(\frac{r}{m}\right) \right| \leq 1 \) and
for any \( r \in I \) and for any \( m = 1, 2, \ldots \). Taking account of (3.26) and (3.22), we obtain from (3.44)

\[
\|u_m\|_B \leq \frac{2}{d_2} \left\{ \|l\|_{1+\varepsilon} + \left( \frac{1+\varepsilon}{2} + 3 \right) \beta_1 \|l\| \right\}
\]

\[
\leq \frac{2}{d_2} \left\{ 1 + \left( \frac{1+\varepsilon}{2} + 3 \right) \beta_1 a_0 \right\} \|l\|_{1+\varepsilon},
\]

which implies with \( \frac{2}{d_2} \left( 1 + \left( \frac{1+\varepsilon}{2} + 3 \right) \beta_1 a_0 \right) = \beta_2 \)

\[
\|\psi_m \|_{1+\varepsilon} = \|l\|_{1+\varepsilon} = \|u_m\|_0 = \|u_m\|_B \leq \beta_2 \|l\|_{1+\varepsilon}.
\]

Thus, letting \( m \to \infty \) in (3.47), we obtain (3.27). Q.E.D.

**Proof of Lemma 3.4.** Since \( l \in \mathcal{U}_{1+\varepsilon}(I) \), it follows from Lemma 3.5 that the equation

\[
((g, (L+i) \varphi))_0 = \langle l, \varphi \rangle \quad (\varphi \in C^{0,\beta}_{0:B}(I, X))
\]

has a solution \( g \in H^{1,\beta}_{0:B}(I, X) \cap H^{1+\varepsilon}(I, X) \). Put \( w = g - v \), where \( v \) is a radiative function for \( \{L, k, l, 0\} \), i.e., \( v \) satisfies

\[
((v, (L-k^2) \varphi))_0 = \langle l, \varphi \rangle \quad (\varphi \in C^{0,\beta}_{0:B}(I, X))
\]

and

\[
\|v' - ikv\|_{-1+\varepsilon} < \infty.
\]

From (3.48) and (3.49) we see that

\[
((w, (L-k^2) \varphi))_0 = (k^2 - i)((g, \varphi))_0.
\]

Noting \( g \in H^{1,\beta}_{0:B}(I, X) \) and (3.50), we have

\[
\|w' - ikw\|_{-1+\varepsilon} \leq \|v' - ikv\|_{-1+\varepsilon} + \|g' - ikg\|_{-1+\varepsilon}
\]

\[
\leq \|v' - ikv\|_{-1+\varepsilon} + \|g'\|_0 + |k| \|g\|_0 < \infty.
\]
Hence $w$ is a radiative function for $\{L, k, (k^2-i)l[g]\}$. We make use of Lemma 3.1 to obtain

$$\|w\|_{-1-\varepsilon} + \|w' - ikw\|_{-1+\varepsilon} + \|B^{1/2}w\|_{-1+\varepsilon} \lesssim \delta_1(1 + \|k\|)\|g\|_{1-\varepsilon},$$

where $\delta_1 = \delta_1(K)$ is given in (3.2). It is implied by (3.26) and (3.22) that

$$\|g\|_{-1-\varepsilon} + \|g' - ikg\|_{-1+\varepsilon} + \|B^{1/2}g\|_{-1+\varepsilon}$$

$$\lesssim (1 + |k|)\|g\|_0 + \|g''\|_0 + \|B^{1/2}g\|_0$$

$$\lesssim (3 + |k|)\|g\|_B \lesssim (3 + |k|)\|l\| \lesssim (3 + |k|)a_0\|l\|_{1-\varepsilon}.$$

Since $v = g + w$, (3.23) follows from (3.53), (3.54) and (3.27). Q.E.D.

**Lemma 3.6.** Let us assume Assumptions 1.1 and 1.2. Let $k_m \in \mathbb{C}^+$, $l_m \in \mathcal{U}_{1+\varepsilon}(I)$ for each $m = 1, 2, \ldots$. Let $v_m, m = 1, 2, \ldots$ be radiative functions for $\{L, k_m, l_m, 0\}$ such that

$$v_m \in H^{-1-\varepsilon}(I, X) \quad (m = 1, 2, \ldots).$$

Let us assume

$$\lim_{m \to \infty} k_m = k,$$

$$\lim_{m \to \infty} \|l - l_m\|_{1+\varepsilon} = 0$$

with $k \in \mathbb{C}^+$ and $l \in \mathcal{U}_{1+\varepsilon}(I)$. Then there exists the radiative function $v$ for $\{L, k, l, 0\}$ satisfying

$$v_m \to v$$

both in $H^{-1-\varepsilon}(I, X)$ and in $\text{loc}H^{1}_{1+\varepsilon}(I, X)$ as $m \to \infty$.

**Proof.** As in the proof of Lemma 3.4 we put $v_m = g_m + w_m$, where $g_m \in H^{1/2}_0(B(I, X) \cap H^{1+\varepsilon}(I, X)$ is the solution of the equation

$$((g_m, (l + i)\varphi))_0 = \langle l_m, \varphi \rangle \quad (\varphi \in C^{2,0}_0(B(I, X),$$

and $w_m$ is the radiative function for $\{L, k_m, (k^2-m-i)l[g_m], 0\}$ for each $m = 1, 2, \ldots$. For each pair $(m, n)$ we have
and hence we obtain, using Lemma 3.5,

\[
\begin{align*}
|g_m - g_n|_B &\leq \beta_1(\sqrt{L}) \|l_m - l_n\| \\
|g_m - g_n|_{1+\varepsilon} &\leq \beta_2(\sqrt{L}) \|l_m - l_n\|_{1+\varepsilon} 
\end{align*}
\] (3.60)

as \(m, n \to \infty\). We put \(g = \lim_{m \to \infty} g_m\). Then \(g \in H^1_{0;B}(I, X) \cap H^{1+\varepsilon}(I, X)\), and \(g\) is the solution of equation (3.48).

Now we turn to the sequence \(\{w_m\}\). Since the sequence \(\{l_m\}\) is uniformly bounded in \(\mathfrak{U}_{1+\varepsilon}(I)\), it follows from Lemma 3.4 and Lemma 3.5 that the sequence \(\{v_m\}_{1-\varepsilon} + \|v_m - ik v_m\|_{1+\varepsilon} + \|B^2 v_m\|_{1+\varepsilon}\), \(\{\|g_m\|B\}\) and \(\{\|g_m\|_{1+\varepsilon}\}\) are also uniformly bounded. Therefore, noting that \(w_m = v_m - g_m\), we obtain the uniform estimate

\[
\|w_m\|_{1-\varepsilon} + \|w_m' - ik w_m\|_{1+\varepsilon} + \|B^2 w_m\|_{1+\varepsilon} \leq \alpha \quad (m = 1, 2, \ldots)
\] (3.61)

with a constant \(\alpha > 0\). From (3.61) we have

\[
\sup_{m=1,2,\ldots} \|w_m\|_{B, (0, R)} < \infty
\] (3.62)

for any \(R \in I\). Hence, proceeding as in the proof of Lemma 3.1, we obtain a subsequence \(\{w_{m_j}\}\) of \(\{w_m\}\) which converges to \(w\) in \(\text{loc} H^1_{0;B}(I, X)\).

On the other hand, using Lemma 3.2 and the uniform boundedness of \(\{\|g\|_{1+\varepsilon}\}\), we have uniformly with respect to \(m\)

\[
\int_0^\infty (1+r)^{-1-\varepsilon} |w_m(r)|^2 dr \leq \alpha_0(\|w_m\|_{1-\varepsilon}^2 + |k_m^2 - i|^2 \|g_m\|_{1+\varepsilon}) \rho^{-\varepsilon}
\]

\[
= 0 (\rho^{-\varepsilon}) \quad (\rho \to \infty),
\]

where we have noted that \(\{k_m\}\) is uniformly bounded and \(w_m\) is a radiative function for \(\{L, k, (k_m^2 - i)l[g_m], 0\}\). It is implied by (3.63) and the convergence of \(\{w_m\}\) in \(\text{loc} H^1_{0;B}(I, X)\) that \(w_{m_j}\) converges to \(w\) in \(H^{1-\varepsilon}(I, X)\). Therefore, taking note of (3.61) and \(k_m \to k, m \to \infty\), we see that \(w\) is a radiative function for \(\{L, k, (k^2 - i)l[g], 0\}\) and we have

\[
w_{m_j} \to w \quad (j \to \infty)
\] (3.64)
both in $\text{loc} H^1_0(B(I, X))$ and $H^{-1-\varepsilon}(I, X)$.

Finally put $v_{m_j} = g_{m_j} + w_{m_j}$. Then we obtain from (3.60) and (3.64)

\begin{equation}
(3.65) \quad v_{m_j} \to v \quad (j \to \infty)
\end{equation}

both in $\text{loc} H^1_0(B(I, X))$ and $H^{-1-\varepsilon}(I, X)$, where $v = g + w$ is a radiative function for $\{L, k, l, 0\}$. Since $v$ is unique by the uniqueness of the radiative function (Theorem 2.2), it follows from (3.65) that the original sequence $\{v_m\}$ itself converges to $v$ both in $H^{-1-\varepsilon}(I, X)$ and $\text{loc} H^1_0(B(I, X))$.

Q.E.D.

We can now prove the existence theorem of the radiative function for $\{L, k, l, 0\}$, where $k \in \mathbb{C}^+$, $l \in \mathscr{U}_{1+\varepsilon}(I)$.

**Theorem 3.7.** Let us assume Assumptions 1.1 and 1.2. Let $k \in \mathbb{C}^+$ and $l \in \mathscr{U}_{1+\varepsilon}(I)$. Then there exists a unique radiative function $v = v(\cdot, k, l)$ for $\{L, k, l, 0\}$ in $H^{-1-\varepsilon}(I, X)$. If $k$ belongs to a compact set $K$ in $\mathbb{C}^+$ then we have

\begin{equation}
(3.66) \quad \|v\|_{-1-\varepsilon} + \|v' - ikv\|_{-1+\varepsilon} + \|B^2 v\|_{-1+\varepsilon} \leq \delta_2 \|l\|_{1+\varepsilon}
\end{equation}

with a constant $\delta_2 > 0$, depending only on $K$. Denote by $\Sigma_0$ the mapping

\begin{equation}
(3.67) \quad \Sigma_0: \mathbb{C}^+ \times \mathscr{U}_{1+\varepsilon}(I) \ni (k, l) \\
\quad \quad \rightarrow v(\cdot, k, l) \in H^{-1-\varepsilon}(I, X) \cap \text{loc} H^1_0(B(I, X)).
\end{equation}

Then $\Sigma_0$ is continuous as a mapping from $\mathbb{C}^+ \times \mathscr{U}_{1+\varepsilon}(I)$ into $H^{-1-\varepsilon}(I, X)$ and is also continuous as a mapping from $\mathbb{C}^+ \times \mathscr{U}_{1+\varepsilon}(I)$ into $\text{loc} H^1_0(B(I, X))$.

**Proof.** First assume that $\text{Im} k > 0$. Then from Lemma 3.5 we obtain a unique radiative function $v(\cdot, k, l)$ for $\{L, k, l, 0\}$ such that $v \in H^{1+\varepsilon}(I, X) \cap H^1_0(B(I, X))$. Next assume that $\text{Im} k = 0$. Then, putting for $m = 1, 2, \ldots$

\begin{equation}
(3.68) \quad \begin{cases} 
k_m = k + \frac{i}{m} \\
v_m = v_m(\cdot, k_m, l),
\end{cases}
\end{equation}
we see from Lemma 3.6 that the radiative function \( v = v(\cdot, k, l) \) for \( \{L, k, l, 0\} \) is obtained as \( v = \lim_{m \to \infty} v_m \). The other statements follow from Lemma 3.4 and Lemma 3.6. Q.E.D.

Finally we prove the existence of the radiative function for \( \{L, k, l, u\} \).

Let \( v = v(\cdot, k, l, u) \) be a radiative function for \( \{L, k, l, u\} \), where \( k \in \mathbb{C}^+, l \in \mathcal{U}_{1+\varepsilon}(I) \) and \( u \in H^{1,B}(I, X) \). We define \( l_1 \in \mathcal{U}_{1+\varepsilon}(I) \) by

\[
< l_1, \varphi > = < l, \varphi > - ((\psi u, (L - k^2) \varphi))_0 \quad (\varphi \in C_0^{1,B}(I, X)),
\]

where \( \psi \in C^1(I), 0 \leq \psi \leq 1 \) and

\[
\psi(r) = \begin{cases} 
1 & (0 < r \leq 1), \\
0 & (r \geq 2).
\end{cases}
\]

Then it is easy to see that \( v_0 = v - \psi u \) is a radiative function for \( \{L, k, l, 0\} \). Thus we can reduce the equation with the boundary value \( v(0) = u(0) \) to the equation with the boundary value \( v_0(0) = 0 \). Therefore, noting that \( l_1 = l_1(u) \) is a \( \mathcal{U}_{1+\varepsilon}(I) \)-valued continuous function on \( H^{1,B}(I, X) \), we obtain from Theorem 3.7 the following

**Theorem 3.8.** Let us assume Assumptions 1.1 and 1.2. Let \( k \in \mathbb{C}^+, l \in \mathcal{U}_{1+\varepsilon}(I) \) and \( u \in H^{1,B}(I, X) \). Then there exists a unique radiative function \( v = v(\cdot, k, l, u) \) for \( \{L, k, l, u\} \) in \( H^{-1-\varepsilon}(I, X) \). Denote by \( \Sigma \) the mapping

\[
\Sigma: \mathbb{C}^+ \times \mathcal{U}_{1+\varepsilon}(I) \times H^{1,B}(I, X) \ni (k, l, u) \\
\rightarrow v(\cdot, k, l, u) \in H^{-1-\varepsilon}(I, X) \cap \text{loc} H^{1,B}(\bar{I}, X).
\]

Then \( \Sigma \) is continuous as a mapping from \( \mathbb{C}^+ \times \mathcal{U}_{1+\varepsilon}(I) \times H^{1,B}(I, X) \) into \( H^{-1-\varepsilon}(I, X) \) and is also continuous as a mapping from \( \mathbb{C}^+ \times \mathcal{U}_{1+\varepsilon}(I) \times H^{1,B}(I, X) \) into \( \text{loc} H^{1,B}(\bar{I}, X) \).

§ 4. The Dependency of Radiative Functions on \( C(\tau) \)

Let \( C_m(\tau), m = 1, 2, \ldots \), be a sequence of operator-valued functions on
Let $C(r)$ be as above. In this section we study the relations between radiative functions for $L$ and radiative functions for $L_m = -\frac{d^2}{dr^2} + B(r) + C_m(r)$ when $C_m(r) \to C(r)$ as $m \to \infty$.

**Assumption 4.1.** (a) For each $r \in I$ $C_m(r)$ is a symmetric operator in $X$ with $\mathcal{D}(C_m(r)) = D$ such that $C_m(r) x \in C^1(I, X)$ for any $x \in D$. Moreover for any compact interval $M$ in $I$ there exists a constant $c(m)(M) > 0$ such that

\begin{equation}
\frac{d}{dr} C_m(r) x \leq c(m)(M) \left( |x| + |B^2(r) x| \right)
\end{equation}

holds for any $x \in D$ and any $r \in M$.

(b) There exists a constant $c_0 > 0$ such that

\begin{equation}
|C_m(r) x| \leq c_0 (1 + r)^{-1-\varepsilon} \left( |x| + |B^2(r) x| \right) \quad (x \in D, r \in I)
\end{equation}

for any $m=1, 2, \ldots$, where $c_0$ does not depend on $m$, and $0 < \varepsilon < 1$ is as given in (1.8).

(c) We have

\begin{equation}
\lim_{m \to \infty} |C(r) x - C_m(r) x| = 0
\end{equation}

for any $x \in D$ and any $r \in I$.

Since $C_m$ is assumed to satisfy (a) and (b) of Assumption 4.1 for each $m=1, 2, \ldots, C_m(r)$ is so smooth and tends to zero at $r=\infty$ so rapidly that the results of §2 and §3 can be applied to the operator

\begin{equation}
L_m = -\frac{d^2}{dr^2} + B(r) + C_m(r),
\end{equation}

i.e., there exists a unique radiative function $v_m(r, k, l, u)$ for $\{L_m, k, l, u\}$, where $(k, l, u) \in C^+ \times \mathcal{U}_{1+\varepsilon}(I) \times H^{1,1}(I, X)$.

**Theorem 4.2.** Let $B(r)$ and $C(r)$ satisfy Assumptions 1.1 and 1.2. Let $C_m(r), m=1, 2, \ldots, $ satisfy Assumption 4.1. Let $K$ be a compact set such that $K \subset C^+$ and let $v_m = v_m(r, k_m, l_m), m=1, 2, \ldots,$ be the radiative
function for \( \{L_m, k_m, l_m, 0\} \), where \( k_m \in K \) and \( l_m \in \mathcal{U}_{1+\delta}(I) \). Then there exists a constant \( \delta_0 > 0 \) such that

\[
(4.5) \quad \|v_m\|_{-1-\delta} + \|v'_m - ik_m v_m\|_{-1+\delta} + \|B^2 v_m\|_{-1+\delta} \leq \delta_0 \|l_m\|_{1+\delta}.
\]

\( \delta_0 \) depends only on \( K \).

**Proof.** Denote by \( g_m \) the radiative function for \( \{L_m, \sqrt{i}, l_m, 0\} \). We see from Lemma 3.5 that \( g_m \in H^{1+\delta}(I, X) \) for each \( m=1, 2, \ldots \). We denote by \( w_m \) the radiative function for \( \{L_m, k_m, (k_m^2 - i)l[-g_m, 0]\} \). Obviously we have \( v_m = g_m + w_m \). Proceeding as in the proof of Lemma 3.5, from \( (4.2) \) we obtain uniformly for \( m=1, 2, \ldots \),

\[
(4.6) \quad \alpha \|\phi\|_B^2 \geq |(\phi, (L_m + i) \phi)_0| \geq \beta \|\phi\|_B^2 \quad (\phi \in C_0^0 B(I, X)),
\]

with constants \( \alpha, \beta > 0 \), whence follows that we obtain uniformly for \( m=1, 2, \ldots \)

\[
(4.7) \quad \begin{cases} \\
\|g_m\|_B \leq \gamma_0 \|l_m\|,
\\
\|g_m\|_{1+\epsilon} \leq \gamma_0 \|l_m\|_{1+\epsilon}
\end{cases}
\]

with a constant \( \gamma_0 > 0 \). Re-examining the proof of Lemma 1.6, we can see from \( (4.2) \) that we obtain uniformly for \( m=1, 2, \ldots \)

\[
(4.8) \quad \|w'_m - ikw_m\|_{-1+\delta} + \|B^2 w_m\|_{-1+\delta} \leq \gamma_1 (\|w_m\|_{-1-\delta} + \|g_m\|_{1+\delta}),
\]

with a constant \( \gamma_1 = \gamma_1(K) > 0 \). Finally, proceeding as in the proof of Lemma 3.1, we can show by reduction to absurdity that we have uniformly for \( m=1, 2, \ldots \)

\[
(4.9) \quad \|w_m\|_{-1-\delta} \leq \gamma_2 \|g_m\|_{1+\delta}
\]

with a positive constant \( \gamma_2 = \gamma_2(K) \). Thus we have \( (4.5) \) from \( (4.7), (4.8), (4.9) \) and \( (3.22) \) as follows:

\[
(4.10) \quad \begin{align*}
\|v_m\|_{-1-\delta} + \|v'_m - ik_m v_m\|_{-1+\delta} + \|B^2 v_m\|_{-1+\delta} \\
\leq \|w_m\|_{-1-\delta} + \|w'_m - ik_m w_m\|_{-1+\delta} + \|B^2 w_m\|_{-1+\delta} \\
+ \|g_m\|_0 + \|g'_m\|_0 + |k_m| \|g_m\|_0 + \|B^2 g_m\|_0
\end{align*}
\]
where we put $T = \sup_{m=1,2,...} |k_m|$, and $a_0$ is given as in (3.22). Q.E.D.

**Theorem 4.3.** Let $B(r)$ and $C(r)$ satisfy Assumptions 1.1 and 1.2. Let $C_m(r)$, $m=1, 2, ..., s$ satisfy Assumption 4.1.

(i) Let $k_m \in C^+$ and $l_m \in \mathcal{A}_{1+\epsilon}(I)$ such that

\[
\lim_{m \to \infty} k_m = k
\]

\[
\lim_{m \to \infty} \|l - l_m\|_{1+\epsilon} = 0
\]

with $k \in C^+$ and $l \in \mathcal{A}_{1+\epsilon}(I)$. Denote by $v_m(\cdot, k_m, l_m)$ the radiative function for $\{L_m, k_m, l_m, 0\}$ for each $m=1, 2, ...$. Then we have

\[
v_m(\cdot, k_m, l_m) \to v(\cdot, k, l)
\]

both in $H^{-1-\epsilon}(I, X)$ and in $\text{loc} H^{1, B}(I, X)$, where $v(\cdot, k, l)$ is the radiative function for $\{L, k, l, 0\}$.

(ii) Let $K$ be a compact set in $C^+$ and let $M$ be a compact metric space. For each $m=1, 2, ..., l_m(k, s)$ is assumed to be a $\mathcal{A}_{1+\epsilon}(I)$-valued, continuous function on $K \times M$ such that

\[
\lim_{m \to \infty} \|l(k, s) - l_m(k, s)\|_{1+\epsilon} = 0
\]

uniformly on $K \times M$ with a $\mathcal{A}_{1+\epsilon}(I)$-valued, continuous function $l(k, s)$ on $K \times M$. Denote by $v_m(\cdot, k, s)$ the radiative function for $\{L_m, k, l_m(k, s), 0\}$. Then we have

\[
\lim_{m \to \infty} v_m(\cdot, k, s) = v(\cdot, k, s)
\]

both in $H^{-1-\epsilon}(I, X)$ and in $\text{loc} H^{1, B}(I, X)$ uniformly on $K \times M$, where $v(\cdot, k, s)$ is the radiative function for $\{L, k, l(k, s), 0\}$.

**Proof.** First let us prove (i). Let $g_m$ be the radiative function for
\( \{L, \sqrt{t}, l_m, 0\} \) and let \( w_m \) be the radiative function for \( \{L, k, (k_m^2 - i) l \lfloor g_m \rfloor, 0\} \). Then we have \( v_m = g_m + w_m \). Similarly we have \( v = g + w \), where \( g \) is the radiative function for \( \{L, \sqrt{t}, l, 0\} \) and \( w \) is the radiative function for \( \{L, k, (k^2 - i) l \lfloor g \rfloor, 0\} \). It follows from Lemma 3.5 and the regularity theorem of Jäger [5] that \( g, g_m \in H^{1+\varepsilon}(I, X) \cap H_0^1(I, X) \cap C^2(I, D) \). We can show that

\[
\lim_{m \to \infty} \|(C - C_m) g\|_{1+\varepsilon} = 0.
\]

In fact we obtain from (4.3) and the fact that \( g(r) \in D \)

\[
\lim_{m \to \infty} |(C(r) - C_m(r)) g(r)| = 0 \quad (r \in I),
\]

and also obtain from (1.8) and (4.2)

\[
|\langle (C(r) - C_m(r)) g(r) \rangle|^2 \leq [(c_2 + c_0)(1 + r)^{-1-\varepsilon}(| g(r) | + | B_1^0 r g(r) |)]^2
\]

\[
\leq 2(c_2 + c_0)^2(1 + r)^{-2-2\varepsilon}(| g(r) |^2 + | B_1^0 r g(r) |^2)
\]

\[
\in L^1(I, (1 + r)^{1+\varepsilon} dr).
\]

(4.15) directly follows from (4.16) and (4.17). Noting that \( g - g_m \) satisfies the equation

\[
((g - g_m, (L + i) \varphi))_0 = \langle l - l_m, \varphi \rangle + (((C - C_m) g, \varphi))_0
\]

\[
(\varphi \in C_0^1 B(I, X)),
\]

We see from (4.7) and (4.15) that

\[
\| g - g_m \|_B \leq \eta_0 \{ |l - l_m| + \| (C - C_m) g \|_{1+\varepsilon} \} \to 0,
\]

\[
\| g - g_m \|_{1+\varepsilon} \leq \eta_0 \{ |l - l_m|_{1+\varepsilon} + \| (C - C_m) g \|_{1+\varepsilon} \} \to 0
\]

as \( m \to \infty \). Using (4.19) and Theorem 4.2, we can proceed as in the proof of Lemma 3.6 to show that the sequence \( w_m \) converges to \( w \) both in \( H^{-1-\varepsilon}(I, X) \) and in \( \text{loc} \, H_0^1(B(I, X)) \). Thus we have shown that \( v_m = g_m + w_m \) converges to \( v = g + w \) both in \( H^{-1-\varepsilon}(I, X) \) and in \( \text{loc} \, H_0^1(B(I, X)) \) which completes the proof of (i).
Next let us prove (ii). It follows from (i) that for each pair \((k, s)\in K\times M\) \(v_m(\cdot, k, s)\) converges to \(v(\cdot, k, s)\) both in \(H^{-1-\varepsilon}(I, X)\) and in \(\text{loc} H_0^{1-B}(I, X)\). Assume that the convergence of \(v_m\) in \(H^{-1-\varepsilon}(I, X)\) is not uniform on \(K\times M\). Then there exists \(\varepsilon_0>0\) and the set of positive integers \(\{m_j\}_{j=1}^\infty\) and \((k_j, s_j)\in K\times M\) such that \(m_j\to\infty\) as \(j\to\infty\) and
\[
\|v(\cdot, k_j, s_j) - v_m(\cdot, k_j, s_j)\|_{-1-\varepsilon} \geq \varepsilon_0. \tag{4.20}
\]
Since the set \(\{(k_j, s_j) | j=1, 2, \ldots\}\) has at least an accumulating point \((k_0, s_0)\in K\times M\), we can assume \(k_j\to k_0\) and \(s_j\to s_0\) without loss of generality. Then, using the continuity of \(l(k, s)\) and the uniform convergence of \(l_m(k, s)\), we obtain
\[
\|l(k_0, s_0) - l_m(k_j, s_j)\|_{1+\varepsilon} \leq \|l(k_0, s_0) - l(k_j, s_j)\|_{1+\varepsilon} + \|l(k_j, s_j) - l_m(k_j, s_j)\|_{1+\varepsilon} \to 0, \tag{4.21}
\]
\(j\to\infty\).

Therefore it follows from (i) that
\[
\|v(\cdot, k_0, s_0) - v_m(\cdot, k_j, s_j)\|_{-1-\varepsilon} \to 0, \quad j\to\infty. \tag{4.22}
\]
On the other hand we obtain from Lemma 3.6
\[
\|v(\cdot, k_0, s_0) - v(\cdot, k_j, s_j)\|_{-1-\varepsilon} \to 0, \quad j\to\infty. \tag{4.23}
\]
(4.22) and (4.23) are combined to give \(\|v(\cdot, k_j, s_j) - v_m(\cdot, k_j, s_j)\|_{-1-\varepsilon} \to 0, j\to\infty\), which contradicts (4.20). Hence \(v_m(\cdot, k, s)\) converges to \(v(\cdot, k, s)\) in \(H^{-1-\varepsilon}(I, X)\) uniformly for \((k, s)\in K\times M\). Similarly we can show that \(v_m(\cdot, k, s)\) converges to \(v(\cdot, k, s)\) in \(\text{loc} H_0^{1-B}(I, X)\) uniformly for \((k, s)\in K\times M\).

By an argument similar to the one used in obtaining Theorem 3.8 from Theorem 3.7, we can show the following

**Theorem 4.4.** Let \(B(r), C(r)\) and \(C_m(r)\), \(m=1, 2, \ldots\), be as in Theorem 4.3. Let \(u\in H^{1-B}(I, X)\).

(i) Let \(k_m\in C^+\) and \(l_m\in \mathcal{U}_{1+\varepsilon}(I)\) satisfy (4.11). Denote by \(v_m(\cdot, k_m, l_m, u)\) the radiative function for \(\{L_m, k_m, l_m, u\}\) for each \(m=1, 2, \ldots\).
Then we have $v_m(\cdot, k_m, l_m, u) \to v(\cdot, k, l, u)$, $m \to \infty$, both in $H^{-1-\varepsilon}(I, X)$ and in $\text{loc} \, H^{1,B}(\tilde{I}, X)$, where $v(\cdot, k, l, u)$ is the radiative function for \{L, k, l, u\}.

(ii) Let $k, M, l_m(k, s)$ and $l(k, s)$ be as in (ii) of Theorem 4.3. Let (4.13) be satisfied. Then we have

\begin{equation}
\lim_{m \to \infty} v_m(\cdot, k_m, s_m, u) = v(\cdot, k, s, u)
\end{equation}

both in $H^{-1-\varepsilon}(I, X)$ and in $\text{loc} \, H^{1,B}(\tilde{I}, X)$ uniformly on $K \times M$, where $v_m(\cdot, k_m, s_m, u)$ and $v(\cdot, k, s, u)$ are the radiative functions for \{L_m, k_m, l_m(k, s), u\} and \{L, k, l(k, s), u\}, respectively.

§5. The Schrödinger Operator in $\mathbb{R}^n$ ($n \geq 3$)

In this section we apply the results obtained in the preceding sections to the Schrödinger operator in $\mathbb{R}^n$ ($n \geq 3$).

Let $X = L^2(S^{n-1})$, $S^{n-1}$ being $(n-1)$-sphere. We define a unitary operator $U$ from $L^2(\mathbb{R}^n)$ onto $H^0(I, X)$ by

\begin{equation}
(UF)(r) = r^{n-1} F(r \omega) \quad (F(\gamma) \in L^2(\mathbb{R}^n)),
\end{equation}

where $r = |\gamma|$ and $\omega = \frac{\gamma}{r} \in S^{n-1}$.

Let us consider the Laplacian on $\mathbb{R}^n$

\begin{equation}
-\Delta F(\gamma) = - \sum_{i=1}^{n} \frac{\partial^2 F}{\partial y_i^2}.
\end{equation}

We denote by $H_0$ the restriction of $-\Delta$ to $C^\infty_0(\mathbb{R}^n)$, i.e.,

\begin{align}
\mathcal{D}(H_0) &= C^\infty_0(\mathbb{R}^n), \\
H_0 \Phi &= -\Delta \Phi.
\end{align}

As is well known, we have for $\Phi \in C^\infty_0(\mathbb{R}^n)$

\begin{equation}
UH_0 \Phi = L_0 U \Phi,
\end{equation}

16) $C^\infty_0(\mathbb{R}^n)$ is the set of all infinitely continuously differentiable functions on $\mathbb{R}^n$ with compact carrier.
where

\[
\begin{aligned}
L_0 &= -\frac{d^2}{dr^2} + B(r) \\
\mathcal{D}(B(r)) &= D = \mathcal{D}(\mathcal{A}_n), \\
B(r) &= \frac{1}{r^2} \left(-\mathcal{A}_n + \frac{(n-1)(n-3)}{4}\right),
\end{aligned}
\]

(5.5)

and \(\mathcal{A}_n\) is the Laplace-Beltrami operator on \(S^{n-1}\). As is well-known, \(-\mathcal{A}_n\) is a non-negative, self-adjoint operator in \(L^2(S^{n-1})\), and hence we can easily see that \(B(r)\) satisfies (a) and (b) of Assumptions 1.1 and 1.2.

We obtain from (5.4)

\[
\begin{aligned}
U\mathcal{D}_L^1(\mathbb{R}^n) &= H^1_0(I, X) \quad (F \in \mathcal{D}^1(\mathbb{R}^n)).
\end{aligned}
\]

(5.6)

Let \(\mathcal{V}(\mathbb{R}^n)\) be the set of all linear continuous functionals \(\alpha\) on \(\mathcal{D}_L^1(\mathbb{R}^n)\). \(\mathcal{V}(\mathbb{R}^n)\) is a Banach space with the norm

\[
|\alpha| = \sup \{ |\langle \alpha, F \rangle| ; F \in \mathcal{D}_L^1(\mathbb{R}), \|F\|_1 = 1\}. 
\]

(5.7)

Then a linear mapping \(\bar{U}\) from \(\mathcal{V}(\mathbb{R}^n)\) into \(\mathcal{D}(I)\) is defined by

\[
\langle \bar{U}\alpha, \varphi \rangle = \langle \alpha, U^{-1}\varphi \rangle \quad (\varphi \in H^1_0(I, X)).
\]

(5.8)

We have

\[
\begin{cases}
\bar{U}\mathcal{V}(\mathbb{R}^n) = \mathcal{V}(I), \\
|\alpha| = \|\bar{U}\alpha\|.
\end{cases}
\]

(5.9)

Denote by \(q(y)\) a real-valued function on \(\mathbb{R}^n\). \(q(y)\) is assumed to satisfy the following conditions:

\(\text{Q) } q(y) \text{ is continuously differentiable on } \mathbb{R}^n \text{ and behaves like }\)

\(O(|y|^{-1-\epsilon}) \quad (\epsilon > 0) \text{ at infinity, i.e., there exist constants } c > 0, \rho > 0 \text{ such that }\)

\(17) \quad \text{The Hilbert space } \mathcal{D}_L^1(\mathbb{R}^n) \text{ is defined as the completion of } C^\infty_0(\mathbb{R}^n) \text{ in the norm }\)

\[
|F|_{H^1} = \left\{ \int_{\mathbb{R}^n} \left\{ \sum_{i=1}^n \left| \frac{\partial F}{\partial y_i} \right|^2 + |F(y)|^2 \right\} dx \right\}^{1/2}
\]
\( |q(y)| \leq c |y|^{-1-\varepsilon} \quad (|y| \geq \rho) \)

with \( 0 < \varepsilon < 1 \).

Let us define \( C(r) \) by
\[
\begin{cases}
C(r) = q(r\omega) \times \\
\mathcal{D}(C(r)) = D.
\end{cases}
\]

It is easy to see that \( C(r) \) satisfies Assumptions 1.1 and 1.2.

Define a differential operator \( H \) by
\[
\begin{cases}
\mathcal{D}(H) = C^\omega_\omega(\mathbb{R}^n) \\
H\Phi = -\Delta \Phi + q(y) \Phi.
\end{cases}
\]

Then we have
\[
UH\Phi = LU\Phi \quad (\Phi \in C^\omega_\omega(\mathbb{R}^n)),
\]
where
\[
L = -\frac{d^2}{dt^2} + B(r) + C(r).
\]

Denote by \( \mathcal{V}_{1+\varepsilon}(\mathbb{R}^n) \) the set of all \( \alpha \in \mathcal{V}(\mathbb{R}^n) \) such that
\[
|\alpha|_{1+\varepsilon} = \sup \{ |<\alpha, (1+r)^{1+\varepsilon} F>|; F \in \mathcal{D}_{1+\varepsilon}(\mathbb{R}^n), \|F\|_{(1)} = 1 \} < \infty.
\]

We have \( \tilde{\mathcal{V}}_{1+\varepsilon}(\mathbb{R}^n) = \mathcal{V}_{1+\varepsilon}(I) \) and \( |\alpha|_{1+\varepsilon} = \|\tilde{\mathcal{V}}\alpha\|_{1+\varepsilon} \) for \( \alpha \in U_{1+\varepsilon}(\mathbb{R}^n) \).

We now give the definition of the radiative function for \( H \) as follows:

Let \( k \in \mathbb{C}^+ \) and \( \alpha \in \mathcal{V}(\mathbb{R}^n) \). Then \( F \in \text{loc} \mathcal{D}_{1+\varepsilon}(\mathbb{R}^n) \)

is called the radiative function for \( \{H, k, \alpha\} \), if \( F \) satisfies the following conditions:

1. For any \( \Phi \in C^\omega_\omega(\mathbb{R}^n) \) we have
\[
(F, (H-k^2)\Phi)_{L^2(\mathbb{R}^n)} = <\alpha, \Phi>.
\]

---

18) \( \text{loc} \mathcal{D}_{1+\varepsilon}(\mathbb{R}^n) \) is the set of all \( F(y) \) on \( \mathbb{R}^n \) such that \( \phi_n F \in \mathcal{D}_{1+\varepsilon}(\mathbb{R}^n) \) for any \( n = 1, 2, \ldots \), where \( \phi_n \in C^\omega_\omega(\mathbb{R}^n) \), \( 0 \leq \phi_n \leq 1 \) and
\[
\phi_n(x) = \begin{cases} 
1 & \text{for } |x| \leq n, \\
0 & \text{for } |x| \geq n+1.
\end{cases}
\]
(2) The "radiation condition"

\[
(5.17) \quad \int_{|y| = 1} (1 + |y|)^{-1+\varepsilon} \left| \frac{\partial F}{\partial y} - ikF(y) \right|^2 dy < \infty
\]

holds.

Let $F$ be the radiative function for $\{H, k, 0\}$, $k \in \mathbb{C}^+$. Then, putting $v = UF \in \text{loc} \, H^1_B(\bar{I}, X)$, we have

\[
(5.18) \quad ((v, (L - k^2) \varphi))_0 = 0 \quad (\varphi \in C_0^2(I, X)),
\]

and

\[
(5.19) \quad \left\| v - \frac{n - 1}{2r} v - ikv \right\|_{-1+\varepsilon, (1, \infty)} < \infty.
\]

Modifying slightly the proof of Lemma 2.1, we obtain

\[
(5.20) \quad \left\| v'(r) - \frac{n - 1}{2r} v(r) - ikv(r) \right\|^2
\]

\[
= \left| v' + \left( \text{Im} k - \frac{n - 1}{2r} \right) v(r) \right|^2 + (\text{Re} k)^2 |v(r)|^2
\]

\[
- 2(\text{Re} k) \text{Im} (v'(r), v(r))
\]

\[
= \left| v'(r) + \left( \text{Im} k - \frac{n - 1}{2r} \right) v(r) \right|^2 + (\text{Re} k)^2 |v(r)|^2
\]

\[
+ 4(\text{Re} k)^2 (\text{Im} k) \|v\|_{0, (0, r)}^2.
\]

If $\text{Im} k \neq 0$, then we see from (5.20)

\[
(5.21) \quad \lim_{r_j \to \infty} \|v\|_{0, (0, r_j)}^2
\]

\[
\leq \frac{1}{4(\text{Re} k)^2 (\text{Im} k)} \lim_{r_j \to \infty} \left| v'(r_j) - \frac{n - 1}{2r} v(r_j) - ikv(r_j) \right|^2 = 0
\]

along some sequence $\{r_j\}_{j=1}^\infty$, and hence $\|v\|_0 = 0$, i.e., $v \equiv 0$. If $\text{Im} k = 0$, then we obtain from (5.20)

\[
(5.22) \quad \left| v'(r) - \frac{n - 1}{2r} v(r) - ikv(r) \right|^2
\]

\[
\geq \left| v'(r) - \frac{n - 1}{2r} v(r) \right|^2 + (\text{Re} k)^2 |v(r)|^2
\]
\[ \frac{1}{2} |v'(r)|^2 + \left( k^2 - \frac{(n-1)^2}{4r^2} \right) |v(r)|^2 \]
\[ \geq \frac{1}{2} \left\{ |v'(r)|^2 + k^2 |v(r)|^2 \right\}, \quad \left( r \geq \frac{n-1}{\sqrt{2k}} \right), \]
whence follows \( \lim_{r \to \infty} (|v'(r)|^2 + k^2 |v(r)|^2) = 0 \). Therefore, proceeding as in the proof of Theorem 2.2, we have \( v \equiv 0 \). Thus the uniqueness of the radiative function for \( H \) has been proved.

Next let \( \alpha \in \mathcal{V}_{1,\varepsilon}(\mathbb{R}^n) \) and \( k \in \mathbb{C}^+ \). Since we have \( \tilde{U}\alpha \in \mathcal{V}_{1,\varepsilon}(I) \), it follows from Theorem 3.7 that there exists the radiative function \( v = v(\cdot, k, \tilde{U}\alpha) \) for \( \{L, k, \tilde{U}\alpha, 0\} \). Put

\[ (5.23) \quad F = U^{-1} v(\cdot, k, \tilde{U}\alpha). \]

Then \( F \in \text{loc} \mathcal{L}_2(\mathbb{R}^n) \) and it follows from (5.23) that

\[ (5.24) \quad \langle F, (H - \overline{k}^2) \Phi \rangle_{\mathcal{L}_2(\mathbb{R}^n)} = \langle (v, (L - \overline{k}^2) U\Phi) \rangle_0 = \langle \tilde{U}\alpha, U\Phi \rangle = \langle \alpha, \Phi \rangle \]
holds for any \( \Phi \in C_0^\infty(\mathbb{R}^n) \). Since \( v \in H^{-1-\varepsilon}(I, X) \) and \( 0 < \varepsilon < 1 \), we have \( \frac{n-1}{2r} \in H^{-1+\varepsilon}(1, \infty), X) \). This together with \( v' - ikv \in H^{-1+\varepsilon}(I, X) \) implies that \( v' - ikv - \frac{n-1}{2r} v \in H^{-1+\varepsilon}(1, \infty), X) \). Hence we obtain

\[ (5.25) \quad \int_{|y| \geq 1} \left( 1 + |y| \right)^{-1-\varepsilon} \left| \frac{\partial F}{\partial |y|} - ikF(y) \right|^2 dy < \infty. \]

Therefore it has been shown that \( F = U^{-1} v \) is the radiative function for \( \{H, k, \alpha\} \). It follows from \( v \in H^{-1-\varepsilon}(I, X) \) that \( F \in L^2(\mathbb{R}^n, (1 + |y|)^{-1-\varepsilon}) \). Thus we obtain

**Theorem 5.1.** Let \( n \) be an integer such that \( n \geq 3 \). Let \( q(y) \) satisfy the condition (Q). Then for given \( k \in \mathbb{C}^+ \) and \( \alpha \in \mathcal{V}(\mathbb{R}^n) \) the radiative function \( F(\cdot, k, \alpha) \) for \( \{H, k, \alpha\} \) is unique. For given \( k \in \mathbb{C}^+ \) and \( \alpha \in \mathcal{V}_{1,\varepsilon}(\mathbb{R}^n) \) there exists the radiative function \( F(\cdot, k, \alpha) \) for \( \{H, k, \alpha\} \) such that \( F(\cdot, k, \alpha) \in L^2(\mathbb{R}^n, (1 + |y|)^{-1-\varepsilon}) \). Denote by \( \sigma \) the mapping
(5.26) \[ \sigma : C^+ \times \nu_{1+\varepsilon}(\mathbb{R}^n) \ni (k, \alpha) \rightarrow F(\cdot, k, \alpha) \in L^2(\mathbb{R}^n, (1 + |y|)^{-1-\varepsilon} dy) \cap \text{loc } \mathcal{D}'(\mathbb{R}^n). \]

Then \( \sigma \) is continuous as a mapping from \( C^+ \times \nu_{1+\varepsilon}(\mathbb{R}^n) \) into \( L^2(\mathbb{R}^n, (1 + |y|)^{-1-\varepsilon} dy) \) and is also continuous as a mapping from \( C^+ \times \nu_{1+\varepsilon}(\mathbb{R}^n) \) into \( \text{loc } \mathcal{D}'(\mathbb{R}^n) \).

References


