S-Duality in \( \tau \)-Cohomology Theories

By

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Introduction

The concept of S-duality was introduced in Spanier [12] and generalized to the equivariant case by Wirthmüller [16]. \( \tau \)-cohomology theories [3] are \( G \)-cohomology theories for \( G = \mathbb{Z}/2\mathbb{Z} \) with their own sign convention. In the present work we translate S-duality into a form suitable for \( \tau \)-cohomology with respect to the sign convention, and discuss the duality between \( \tau \)-cohomology and homology.

Notation and terminology in [3] are used freely.

Section 1 is a preparatory section. The sign convention is described there. In Section 2 we observe the existence of the duality isomorphisms and S-duals. The main results of this section are Theorems 2.2 and 2.7. In Section 3 we see mainly the relations of slant products (which induces duality) with suspensions \( \sigma^{*,*}, \sigma_{*,*} \) and \( \sigma(*,*) \). In Section 4, using the results in Section 3, we discuss some properties of S-duality of stable \( \tau \)-maps.

In Section 5 we discuss the duality between \( \tau \)-cohomology and homology, and the representation of \( \tau \)-homology theories. The main results in this section are Theorems 5.2, 5.5 and Corollary 5.3. In Section 6 we see Atiyah-Poincaré type duality in \( \tau \)-cohomology.

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§ 1. \( \tau \)-Cohomology Theories

The main reference of this section is Araki-Murayama [3].

We work mainly on the category \( \mathcal{SF}_\tau \) of \( \tau \)-spaces (=spaces with invo-
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solutions) with base points. By τ-spaces, τ-maps and τ-homotopies we mean
τ-spaces with base points, equivariant maps preserving base points and
equivariant homotopies relative to the base points respectively, for simplicity.
Involutions are denoted by τ in most place. By [ , ]τ we denote the set of
τ-homotopy classes.

Let ψ: Sμτ→ Sμτ be the forgetful functor to forget involutions and
ϕ: Sμτ→ Sμτ be the fixed-point functor to restrict to fixed points. So ϕX
is the set of fixed points of a τ-space X and ϕf: ϕX→ϕY is the restriction of a
τ-map f: X→Y to the fixed-point sets. The forgetful functor ψ induces the
morphism ψ*: [X, Y]τ→[X, Y] and the fixed-point functor ϕ induces the
morphism φ*: [X, Y]τ→[ϕX, ϕY].

Let R^p,q be the euclidian space with the involution such that τ(x_1,..., x_p,
x_{p+1},..., x_{p+q})=(-x_1,..., -x_p, x_{p+1},..., x_{p+q}). Let B^p,q and S^p,q be the unit
ball and unit sphere in R^p,q. Let Σ^p,q=B^p,q/S^p,q. Σ^p,q is identified with the
one-point compactification of R^p,q. We identify Σ^r,0 ∩ Σ^p,q=Σ^r+p,q, Σ^p,q ∩ Σ^0,q
=Σ^p,q∗ by the standard τ-homeomorphisms.

Let X and Y be τ-spaces. We endow the set [Σ^p,qX, Y]τ with track ad-
dition along a fixed coordinate, where Σ^p,qX=Σ^p,q ∩ X and the involution on
Σ^p,qX is induced by the diagonal action on Σ^p,q × X. Then [Σ^p,qX, Y]τ is a
group for q≥1 and abelian for q≥2. Let J be the involution on Σ^1,0. After
the identification Σ^1,0 ∩ Σ^p-1,q=Σ^p,q we have an involutive τ-map J ∧ 1: Σ^p,q
→Σ^p,q. Thus we have an induced involution
ρ=(J ∧ 1)*: [Σ^p,qX, Y]τ→[Σ^p,qX, Y]τ
for p≥1. Clearly ψ*ρ=−1 and ϕ*ρ=1. Putting
A=Z[ρ]/(1−ρ²),

[Σ^p,qX, Y]τ is a A-module for p≥1 and q≥2. A is identified with the Burnside
ring A(Z/2Z) of Z/2Z [3], Section 2, and A=[Σ^p,q, Σ^p,q]τ for p≥1 and q≥1,
[3], Theorem 12.5.

A τ-complex is a G-complex for G=Z/2Z, generated by τ, [3, 6]. Let W^τ and
F^τ be the full subcategories of Sμτ in which the objects are τ-spaces
having τ-homotopy types of τ-complexes and finite τ-complexes, respectively.
Let CW^τ and CF^τ be the full subcategories of W^τ and F^τ with τ-complexes
and finite τ-complexes as objects, respectively. The base points of τ-complexes
are vertices as usual.
A reduced \( \tau \)-cohomology theory on the category \( \mathcal{W}_\sigma \) or on \( \mathcal{F}_\sigma \) is a system

\[
\check{h}^{*,*} = \{ \check{h}^{p,q}; (p, q) \in \mathbb{Z} \times \mathbb{Z} = \text{RO}(\mathbb{Z}/2\mathbb{Z}) \}
\]

of \( \Lambda \)-module-valued contravariant functors \( \check{h}^{p,q} \) satisfying the following four axioms (A1)–(A4).

A) Each \( \check{h}^{p,q} \) is a \( \tau \)-homotopy functor satisfying Wedge axiom and Mayer-Vietoris axiom on \( \mathcal{W}_\sigma \) or \( \mathcal{F}_\sigma \).

A2) Two kinds of suspension isomorphisms

\[
\check{\sigma} = \sigma^{1,0}: \check{h}^{p,q}(X) \cong \check{h}^{p+1,q}(\Sigma^{1,0}X)
\]

and

\[
\sigma = \sigma^{0,1}: \check{h}^{p,q}(X) \cong \check{h}^{p,q+1}(\Sigma^{0,1}X)
\]

are defined as natural isomorphisms of \( \Lambda \)-module-valued functors.

A3) The following diagram

\[
\begin{array}{ccc}
\check{h}^{p,q}(X) & \xrightarrow{\check{\sigma}} & \check{h}^{p,q+1}(\Sigma^{0,1}X) \\
\downarrow{\sigma} & & \downarrow{\sigma^{p+1,q+1}(\Sigma^{1,0}X)} \\
\check{h}^{p+1,q}(\Sigma^{1,0}X) & \xrightarrow{\check{\sigma}} & \check{h}^{p+1,q+1}(\Sigma^{0,1}X)
\end{array}
\]

is commutative for any \( X \), where \( T: \Sigma^{1,0} \Sigma^{1,0} \to \Sigma^{1,0} \Sigma^{0,1} \) is the \( \tau \)-map switching factors.

A4) Let \( J \) be the involution of \( \Sigma^{1,0} \), then

\[
(J \wedge 1)^* = \rho \text{ times: } \check{h}^{p,q}(\Sigma^{1,0}X) \to \check{h}^{p,q}(\Sigma^{1,0}X).
\]

Axioms A3) and A4) relate the ring \( \Lambda \) to sign conventions. Iterated suspension isomorphisms \( \sigma^{s,t}: \check{h}^{p,q}(X) \cong \check{h}^{p+s,q+t}(\Sigma^{s,t}X) \) are defined as the composite \( \sigma^{s,t} = \check{\sigma}^s \circ \sigma^t \) after the canonical identification \( \Sigma^{1,0} \wedge \cdots \wedge \Sigma^{1,0} \wedge \Sigma^{0,1} \wedge \cdots \wedge \Sigma^{0,1} = \Sigma^{s,t} \). We also use the notation \( \sigma^{-s,-t} = (\sigma^{s,t})^{-1} \) for inverses of suspensions.

The associated unreduced \( \tau \)-cohomology theory \( h^{*,*} = \{ h^{p,q}; (p, q) \in \mathbb{Z} \times \mathbb{Z} \} \) is defined as usual by \( h^{p,q}(X, A) = \check{h}^{p,q}(X/A) \) and \( h^{p,q}(X) = \check{h}^{p,q}(X_+) \), where \( X_+ = X \cup \{ pt \} \).

Reduced \( \tau \)-homology theories are defined in the obvious way and denoted by \( \tilde{h}^{*,*} \). Suspensions in reduced \( \tau \)-homology theories are denoted by \( \sigma_{s,t} \).

Let \( E = \{ E_n, e_n: \Sigma^{1,1}E_n \to E_{n+1} \} \) be a \( \tau \)-spectrum \( (E_n) \in \mathcal{W}_\sigma^* \), \( (p, q) \in \mathbb{Z} \times \mathbb{Z} \), and \( n > \max(-p, -q) \). For \( X \in \mathcal{W}_\sigma^* \) and \( Y \in \mathcal{W}_\sigma^* \) the \( \Lambda \)-homomorphism \( \bar{e}_n:: [\Sigma^{s+p,\sigma+q}X, E_n \wedge Y]^* \to [\Sigma^{s+p+1,\sigma+q+1}X, E_{n+1} \wedge Y]^* \) is defined as the composite.
Here, and henceforth, the \( \tau \)-homeomorphisms \( \Sigma^p \Sigma^q \approx \Sigma^{p+q} \) which are induced by the switching maps \( \Sigma^p \approx \Sigma^q \) are generally denoted by \( T \), for simplicity, \([3],[7]\), Section 7. Put

\[
[X, E \wedge Y]_{p,q} = \lim_{n} \left[ \Sigma^{n+p+q} X, E_n \wedge Y \right]
\]

for each \((p, q) \in \mathbb{Z} \times \mathbb{Z}\). Then \( \{ [\cdot, E \wedge Y]^{p,q} ; (p, q) \in \mathbb{Z} \times \mathbb{Z} \} \) is a reduced \( \tau \)-cohomology theory on \( \mathcal{F}^\tau \) for a fixed \( Y \in \mathcal{F}^\tau \) together with the suspension isomorphisms \( \sigma^* \), \([3],[7]\) and Theorem 7.7, and \( \{ [X, E \wedge -]_{p,q} ; (p, q) \in \mathbb{Z} \times \mathbb{Z} \} \) is a reduced \( \tau \)-homology theory on \( \mathcal{H}^\tau \) for a fixed \( X \in \mathcal{G}^\tau \) together with the suspension isomorphisms \( \sigma^* \), \([3],[13]\), Propositions 13.4 and 13.5.

Each \( \tau \)-cohomology theory \( h^{*,*} \) is associated with two (non-equivariant) cohomology theories: the one is the forgetful cohomology theory \( \psi h^* \) defined by \( \psi h^*(-) = h^{*,*}(S^{1,0} \times -) \), and the other is the fixed-point cohomology theory \( \phi h^* \) defined by \( \phi h^*(-) = \lim_{n} h^{*,*}(-) \). And the forgetful morphism \( \psi : \{ h^{p,q} \} \rightarrow \{ \psi h^{p,q} \} \) and the fixed-point morphism \( \phi : \{ h^{p,q} \} \rightarrow \{ \phi h^{p,q} \} \) are defined. These are a kind of natural transformations of cohomology theories. (Cf., \([3],[\S\S 2-3]\).)

Let \( E \) be a \( \tau \)-spectrum. Applying the forgetful and fixed-point functors to each term and map of \( E \), we obtain spectra \( \psi E \) and \( \phi E \) called the forgetful and fixed-point spectrum respectively. The cohomology theories \( h^*( ; \psi E) \) and \( h^*( ; \phi E) \) represented by \( \psi E \) and \( \phi E \) coincide with the forgetful cohomology theory \( \psi h^*( ; E) \) and the fixed-point cohomology theory \( \phi h^*( ; E) \) of \( h^{*,*}( ; E) \), respectively. The forgetful functor induces the homomorphisms \( \psi_* : [\Sigma^{n-p,-q} X, E_n]^* \rightarrow [\Sigma^{2n-p,-q} X, (\psi E)_{2n}] \) which form the map of the direct systems. Taking the direct limits, we get a homomorphism

\[
\psi_* : \tilde{E}^{p,q}(X) \rightarrow \psi \tilde{E}^{p+q}(X).
\]

This homomorphism coincides with the forgetful morphism \( \psi \) for \( \tilde{E}^{*,*} \), \([3],[7]\), (7.10). Also the fixed-point functor induces the homomorphism \( \phi_* : \tilde{E}^{p,q}(X) \)
→ϕ*E*(ϕX) which coincides with the fixed-point morphism ϕ for $E^{*, *}$, [3], (7.12).

An example of τ-spectrum is the τ-spectrum of stable τ-homotopy

$$SR = \{ S^n, e_n = T: \Sigma_{\tau}^{1, 1} S^n.n \simeq \Sigma_{\tau}^{n+1, n+1} \}.$$  

In this case $ϕSR$ and $ψSR$ are both the sphere spectra.

**Proposition 1.1.** Let $X ∈ S^*$ and $Y ∈ \mathcal{F}_n$. Then, for each $(p, q) ∈ \mathbb{Z} × \mathbb{Z}$,

$$[X, SR \wedge Y]_{p,q} \cong [\Sigma^{n+p, n+q} X, S^n Y]$$

for large $n$.

This follows from [3], Proposition 13.12.

The cofibration sequence $S^* \to B^1 \to \Sigma^1 \to \Sigma^0 S^*$ induces exact sequences

$$\cdots \to \widetilde{h}^{p,q}(\Sigma X) \to \widetilde{h}^{p,q}(B^1 \wedge X) \to \widetilde{h}^{p,q}(S^0 \wedge X) \to \widetilde{h}^{p,q}(\Sigma X) \to \cdots$$

where the second row is called the forgetful exact sequence of $h^{*, *}$, [3], (5.1).

**Proposition 1.2.** Let $ϕ: h^{*, *} \to k^{*, *}$ be a natural transformation of reduced τ-cohomology theories of degree $(r, s)$. If

$$ϕ: h^{p,q}(X) \to k^{p+r, q+s}(X)$$

is isomorphic for a fixed X and each $(p, q) ∈ \mathbb{Z} × \mathbb{Z}$, then

$$ϕ: h^{p,q}(S^1 \wedge X) \to k^{p+r, q+s}(S^1 \wedge X)$$

and

$$ψϕ: ψk^{p+r, q+s}(X)$$

are isomorphic for any $(p, q) ∈ \mathbb{Z} × \mathbb{Z}$.

**Proof.** Compare the forgetful exact sequences of $h^{*, *}(X)$ and $k^{*, *}(X)$. Then 5-lemma implies the result.

Similarly we obtain the following

**Proposition 1.2’.** Let $Ψ: h_{*, *} \to k_{*, *}$ be a natural transformation of reduced τ-homology theories of degree $(r, s)$. If

$$Ψ: h_{p,q}(X) \to k_{p+r, q+s}(X)$$

is isomorphic for a fixed X and each $(p, q) ∈ \mathbb{Z} × \mathbb{Z}$, then
\[ \Phi : \tilde{h}_{p,q}(S^1_+ \wedge X) \to \tilde{h}_{p+r,q+s}(S^1_+ \wedge X) \]
is isomorphic for any \((p, q) \in \mathbb{Z} \times \mathbb{Z}.

Propositions 1.2 and 1.2' show that, if natural transformations \( \Phi : \tilde{h}^\ast \to \tilde{h}^\ast \) and \( \Psi : \tilde{h}^\ast \to \tilde{h}^\ast \) are isomorphic on \( \mathbb{Z}/\mathbb{Z} \), then \( \Phi : \tilde{h}^\ast(\Sigma^0, 0) \cong \tilde{h}^\ast(\Sigma^0, 0) \) and \( \Psi : \tilde{h}^\ast(\Sigma^0, 0) \cong \tilde{h}^\ast(\Sigma^0, 0) \). Then comparison theorem for \( \tau \)-(co-)homology theories has the following form. (Cf., [6], Chap. IV, 5 and [9], Comparison Theorem 2.14.)

**Theorem 1.3.** Let \( \tilde{h}^\ast \) and \( \tilde{k}^\ast \) be reduced \( \tau \)-cohomology theories on \( \mathcal{W} \) or on \( \mathcal{F} \), and \( \Phi : \tilde{h}^\ast \to \tilde{k}^\ast \) be a natural transformation of reduced \( \tau \)-cohomology theories of degree \((r, s)\). If

\[ \Phi : \tilde{h}^\ast(\Sigma^0, 0) \cong \tilde{k}^\ast(\Sigma^0, 0), \]

then

\[ \Phi : \tilde{h}^\ast(\Sigma^0, 0) \cong \tilde{k}^\ast(\Sigma^0, 0), \]

for any \( X \in \mathcal{W} \) or any \( X \in \mathcal{F} \).

**Theorem 1.3'.** Let \( \tilde{h}^\ast \) and \( \tilde{k}^\ast \) be reduced \( \tau \)-homology theories on \( \mathcal{W} \) or on \( \mathcal{F} \), and \( \Phi : \tilde{h}^\ast \to \tilde{k}^\ast \) be a natural transformation of reduced \( \tau \)-homology theories of degree \((r, s)\). If

\[ \Phi : \tilde{h}^\ast(\Sigma^0, 0) \cong \tilde{k}^\ast(\Sigma^0, 0), \]

then

\[ \Phi : \tilde{h}^\ast(\Sigma^0, 0) \cong \tilde{k}^\ast(\Sigma^0, 0), \]

for any \( X \in \mathcal{W} \) or any \( X \in \mathcal{F} \).

Next we state some isomorphisms of \( \tau \)-homotopy groups.

**Proposition 1.4.** Let \( X \) be a \( \tau \)-space such that i) \( X \) is \( m \)-connected and ii) \( \phi X \) is \( n \)-connected. Let \((K, L)\) be a pair of \( \tau \)-complexes such that \( \dim (K - L) \leq m + 1 \) and \( \dim (\phi K - \phi L) \leq n + 1 \). Then any \( \tau \)-map \( f : L \to X \) can be extended equivariantly on \( K \).

The proof is similar to [3], Proposition 11.1.

Let \( F(X, Y) \) be the base-point preserving function space from \( X \) to \( Y \). Then \( F(X, Y) \) is a \( \tau \)-space with \( \tau \)-action \( (\tau f)(x) = \tau f(tx), x \in X \).

**Proposition 1.5.** Let \( X \) be a locally compact \( \tau \)-complex and \( Y \) a \( \tau \)-space
Let \( r\phi : \phi F(X, Y) \to F(\phi X, \phi Y) \) be the map obtained by restriction to \( \phi X \). Assume that \( Y \) is \( m \)-connected. Then
\[
r\phi : \pi_j(\phi F(X, Y)) \to \pi_j(F(\phi X, \phi Y))
\]
is isomorphic if \( j \leq M \) and epimorphic if \( j \leq M + 1 \), where
\[
M = \begin{cases} 
  m - \dim (X - \phi X) & \text{if } X \neq \phi X \\
  \infty & \text{if } X = \phi X.
\end{cases}
\]

**Proof.** As \( X \) is locally compact, we have
\[
\pi_j(\phi F(X, Y)) \cong [\Sigma^0, X, Y]^*.
\]
(cf., [6], Chap. III). Let \( i : \phi X \to X \) be the inclusion. Consider
\[
(1 \wedge i)^* : [\Sigma^0, X, Y]^* \to [\Sigma^0(\phi X), Y]^*.
\]
Then, applying Proposition 1.4 to the pair \((\Sigma^0, X, \Sigma^0(\phi X))\) for surjectivity and to the pair \( (\Sigma^0, X \times I, \Sigma^0(\phi X) \times \{0, 1\} \cup \Sigma^0(\phi X) \times I) \) for injectivity, we get the proof.

### §2. S-Duality in the Stable \( \tau \)-Homotopy Theory

The \((p, q)\)-th stable \( \tau \)-homotopy group, \((p, q) \in \mathbb{Z} \times \mathbb{Z}\), is denoted by
\[
\{X, Y\}_{p, q} = [X, \mathbb{S} \mathcal{R} \wedge Y]_{p, q} = \lim_n [\Sigma^{n+p, n+q}X, \Sigma^{n, n}Y]^*.
\]
\(\{X, Y\}_{p, q}\) is also denoted by \(\{X, Y\}^{p, q}\). By Proposition 1.1
\[
\{X, Y\}_{p, q} \cong [\Sigma^{n+p, n+q}X, \Sigma^{n, n}Y]^*\]
for \(X \in \mathcal{C}_\mathcal{F}_s^\#\), \(Y \in \mathcal{N}_s^\#\) and large \(n\).

Let \( \mathcal{C}_\mathcal{F}_s^\# \) be the full subcategory of \( \mathcal{C}_\mathcal{F}_s^\# \) with objects \(X\) such that \(X\) and \(\phi X\) are path-connected. For \(X, X' \in \mathcal{C}_\mathcal{F}_s^\#\), a \(\tau\)-map
\[
u : X \wedge X' \to \Sigma^{r_1, s_1} \ldots \Sigma^{r_k, s_k}, \quad r = r_1 + \cdots + r_k, \quad s = s_1 + \cdots + s_k
\]
is called a \((r, s)\)-duality \(\tau\)-map or \(R\)-duality \(\tau\)-map, \(R = (r, s)\)) if \(\phi u : \phi X \wedge \phi X' \to \Sigma^s\) and \(\phi u : \phi X \wedge \phi X' \to \Sigma^s\) are duality maps in the sense of Spanier [12], page 360, and then \(X'\) is called an \((r, s)\)-dual by means of \(u\). For \(X, X' \in \mathcal{C}_\mathcal{F}_s^\#\), \(X'\) is called an \((\text{equivariant}) S\)-dual of \(X\) if some (iterated) suspension of \(X'\) is \((r, s)\)-dual of some (iterated) suspension of \(X\) for some \((r, s)\). If \(u : X \wedge X' \to \Sigma^{r, s}\) be an \((r, s)\)-duality \(\tau\)-map, then the \(\tau\)-map \(\tilde{u} : X' \wedge X \to \Sigma^{r, s}\) defined by \(\tilde{u}(x', x) = u(x, x')\), \(x \in X, x' \in X'\), is also an \((r, s)\)-duality \(\tau\)-map, [12], Lemma.
(5.4). For a $\tau$-map $u: X \wedge X' \to \Sigma^{r,s}$ and pairs $P = (p, q), P' = (p', q')$ of non-negative integers we define

$$u_{P, P'}: \Sigma^{p,q}X \wedge \Sigma^{p', q'}X' \to \Sigma^{p,q} \Sigma^{p', q'} \Sigma^{r,s}$$

to be the composite

$$\Sigma^{p,q}X \wedge \Sigma^{p', q'}X' \xrightarrow{1 \wedge u \wedge 1} \Sigma^{p,q} \Sigma^{p', q'} \Sigma^{r,s} \wedge \Sigma^{r,s}.$$ 

If $u$ is an $(r, s)$-duality $\tau$-map, then $u_{P, P'}$ is a $(P + P' + R)$-duality $\tau$-map.

Here, and henceforth, $T'$ denote switching maps in general.

For a $\tau$-map $u: X \wedge X' \to \Sigma^{r,s}$ and $P = (p, q) \in \mathbb{Z} \times \mathbb{Z}$ we define

$$\Gamma_{u}^P: \{Y, X\}_{p,q} \to \{Y \wedge X', \Sigma^{r,s}\}_{p,q}$$

as follows: if $f: \Sigma^{p+q}Y \to \Sigma^{r}Z$ represents an element $\{f\} \in \{Y, X\}_{p,q}$, then $\Gamma_{u}^P\{f\}$ is represented by the composite

$$\Sigma^{p+q}Y \wedge X' \xrightarrow{f \wedge 1} \Sigma^{r}Z \wedge X' \xrightarrow{N' \wedge O} \Sigma^{r,s},$$

where $N = (n, n)$ and $O = (0, 0)$. Then $\Gamma_{u}^P$ is a well-defined $\Lambda$-homomorphism and coincides with the slant product $\{u\}$, see Section 3. $\{\Gamma_{u}^P\}_{p\mathbb{Z} \times \mathbb{Z}}$ is a natural transformation of $\tau$-cohomology theories with respect to $Y$.

**Theorem 2.1.** Let $u: X \wedge X' \to \Sigma^{r,s}$ be an $(r, s)$-duality $\tau$-map. Then

$$\Gamma_{u}^P: \{Y, X\}_{p,q} \to \{Y \wedge X', \Sigma^{r,s}\}_{p,q}$$

is a $\Lambda$-isomorphism for any $Y \in \mathcal{F}_{r,s}$ and any $P = (p, q) \in \mathbb{Z} \times \mathbb{Z}$.

**Proof.** Consider the map $u_n: [\Sigma^{p+q}Y, \Sigma^{r,s}] \to [\Sigma^{p+q}Y \wedge X', \Sigma^{r,s}]$ in the definition of $\Gamma_{u}^P$, with $\lim_n u_n = \Gamma_{u}^P$. Define $\lambda_n: \Sigma^{r,s} \to F(X', \Sigma^{r,s})$ by $\lambda_n(x)(x') = u_n, x, x', x \in \Sigma^{r,s}, x' \in X'$. Then the following diagram is commutative:

$$\begin{array}{ccc}
[\Sigma^{p+q}Y, \Sigma^{r,s}] & \xrightarrow{u_n} & [\Sigma^{p+q}Y \wedge X', \Sigma^{r,s}] \\
\downarrow \lambda_n & & \downarrow \mu_n \\
[\Sigma^{p+q}Y, F(X', \Sigma^{r,s})] & & \\
\end{array}$$

where $\mu_n$ is the isomorphism induced by a $\tau$-homeomorphism

$$F(\Sigma^{p+q}Y \wedge X', \Sigma^{r,s}) \simeq F(\Sigma^{p+q}Y, F(X', \Sigma^{r,s}))$$

taking $f$ into $\tilde{f}$ defined by $\tilde{f}(y)(x') = f(y, x')$. Therefore $u_n$ is isomorphic if and only if $\lambda_{n*}$ is isomorphic.

We show that $\lambda_{n*}$ is isomorphic for large $n$. Define $v_n: \Sigma^{n}(\phi X) \to F(\phi X')$,
$\Sigma^{n+s}$ by $v_n(x)(x') = \phi u_{N,O}(x, x')$. Let $r_\phi: \phi F(X', \Sigma^{n,s}) \to F(\phi X', \Sigma^{n+s})$ be the map obtained by restriction to $\phi X'$. Then the following diagram is commutative:

$$
\pi_j(\Sigma^n(\phi X)) \xrightarrow{\phi_{n+1}} \pi_j(\phi F(X', \Sigma^{n,s}))
\xrightarrow{\psi_{n+1}} \pi_j(F(\phi X', \Sigma^{n+s})).
$$

By Proposition 1.5 $r_\phi$ is isomorphic if $j \leq (2n + r + s - 1 - \dim X')$. Recall that $\phi u_{N,O}$ is a duality map. By [12], (2.8) and the proof of Theorem (5.5), $v_{n+1}$ is isomorphic for $j < 2n - 2\dim X' - 1$. Recall that $\psi u_{N,O}$ is a duality map. Then $\psi_{n+1}: \pi_j(\Sigma^{2n}X) \to \pi_j(F(X', \Sigma^{2n+s}))$ is isomorphic for $j < 2(2n + r + s - \dim X')$ and large $n$. Then, by [3], Proposition 11.2 $\lambda_{n+1}$ is isomorphic for $n > 2 \dim X' + \dim Y + p + q + 2$. Thus $\lambda_{n+1}$ is isomorphic for large $n$.

The duality isomorphism $\Gamma^p_u: \{Y, X\}_{p,q} \to \{Y \wedge X', \Sigma^{r,s}\}_{p,q}$, $P = (p, q)$, induces the homomorphisms

$$
\psi(\Gamma^p_u): \{\psi Y, \psi X\}_{p+q} \to \{\psi Y \wedge \psi X', \Sigma^{r+s}\}_{p+q}
$$

and

$$
\phi(\Gamma^p_u): \{\phi Y, \phi X\}_{q} \to \{\phi Y \wedge \phi X', \Sigma^{q}\}_{q}
$$

which correspond to $\Gamma^{p+1}_{\phi u}$ and $\Gamma^{q}_{\phi u}$ respectively, where $\{\ , \\}_n$ denotes the (non-equivariant) stable homotopy group. By the definition of a duality $\tau$-map and [12], Lemma (5.8), we see that $\psi(\Gamma^p_u) = \Gamma^{p+q}_{\phi u}$ and $\phi(\Gamma^p_u) = \Gamma^{q}_{\phi u}$ are isomorphisms.

Adding the converse to the above results, we obtain the following

**Theorem 2.2.** Let $X, X' \in \mathcal{C} \mathcal{F}_0$ and $u: X \wedge X' \to \Sigma^{r,s}$ be a $\tau$-map. Then the following are equivalent:

1. $u$ is a duality $\tau$-map,
2. $\Gamma^p_u: \{\Sigma^{0,0}, X\}_{p,q} \cong \{X', \Sigma^{r,s}\}_{p,q}$ for every $P = (p, q) \in \mathbb{Z} \times \mathbb{Z},$
3. $\Gamma^p_u: \{Y, X\}_{p,q} \cong \{Y \wedge X', \Sigma^{r,s}\}_{p,q}$ for any $Y \in \mathcal{C} \mathcal{F}_0$ and $P = (p, q) \in \mathbb{Z} \times \mathbb{Z},$
4. $\Gamma^q_{\phi u}: \{\Sigma^{0}, \psi X\}_n \cong \{\psi X', \Sigma^{r+s}\}_n$ and
   $$
   \Gamma^q_{\phi u}: \{\Sigma^{0}, \phi X\}_n \cong \{\phi X', \Sigma^{r+s}\}_n
   $$
   for every $n \in \mathbb{Z},$
5. $\Gamma^q_{\phi u}: \{Y, \psi X\}_n \cong \{Y \wedge \psi X', \Sigma^{r+s}\}_n$ and
   $$
   \Gamma^q_{\phi u}: \{Y, \phi X\}_n \cong \{Y \wedge \phi X', \Sigma^{r+s}\}_n
   $$
   for any $Y \in \mathcal{C} \mathcal{F}_0$ and every $n \in \mathbb{Z},$

q.e.d.
where $\mathcal{C} \mathcal{F}_0$ denotes the category of finite pointed CW-complexes.

**Proof.** The implications (2)$\iff$(3) and (4)$\iff$(5) follow from comparison theorems (Theorem 1.3). The implications (1)$\implies$(2) and (1)$\implies$(3) are the result of Theorem 2.1.

Proof of (2)$\implies$(4). Since $\{\Gamma^0_p\}_{p \in \mathbb{Z} \times \mathbb{Z}}$ is a natural transformation of $\tau$-cohomology theories, Proposition 1.2 implies that $\Gamma^q_{\psi}$ is isomorphic for each $n \in \mathbb{Z}$. As the spectra $\psi \mathcal{SR} \wedge \psi X$ and $\psi \mathcal{SR} \wedge \Sigma^{r+s}$ are connective and $X'$ is finite, we see that $\phi: \{\Sigma^{0,0}, X\}_{p,q} \cong \{\Sigma^{0,0}, \phi X\}_{q}$ and $\phi: \{X', \Sigma^{r,s}\}_{p,q} \cong \{\phi X', \Sigma^{r,s}\}_{q}$ for large $p$ by [3], Proposition 5.4. Then (2) implies that $\Gamma^q_{\phi}$ is isomorphic for each $q \in \mathbb{Z}$ because of $\phi(\Gamma^p_\psi) = \Gamma^q_{\phi}$.

Proof of (4)$\implies$(1). By [12], Lemma (4.7) and Theorem (5.7) we see that $\psi u$ and $\phi u$ are duality maps. This shows that $u$ is a duality $\tau$-map. q.e.d.

**Remark 2.3.** The above theorems show that Wirthmüller's definition of a duality [16] is equivalent to our definition under Propositions 1.1 and 1.2, i.e., let $\{u\} \in \{X \wedge X', \Sigma^{0,0}\}^{r,s}$ be an $(r,s)$-duality in the sense of [16] for $X$, $X' \in \mathcal{C} \mathcal{F}_0$, then there exists a duality $\tau$-map $u$ showing that $X$ and $X'$ are $S$-duals of each other, and the converse follows from Theorem 2.1.

Next we show the existence of equivariant $S$-duality.

**Proposition 2.4.** Let $X$ and $X'$ be $\tau$-subcomplexes of $\Sigma^{r,s+1}$ such that $X$, $X' \in \mathcal{C} \mathcal{F}_0$, $X \cap X' = \emptyset$ and the inclusion $X' \rightarrow \Sigma^{r,s+1} - X$ (which may not preserve the base points) is a $\tau$-homotopy equivalence. Then there exists an $(r,s)$-duality $\tau$-map

$$ u: X \wedge X' \rightarrow \Sigma^{r,s} $$

showing that $X'$ is an $(r,s)$-dual of $X$.

**Proof.** The proof is almost the same as [11], p. 180. As $X$ and $X'$ are closed in $\Sigma^{r,s+1}$ and $X \cap X' = \emptyset$, $\Sigma^{s+1} - \phi X - \phi X' \neq \emptyset$. We choose a point $a \in \Sigma^{s+1} - \phi X - \phi X'$. Then $\Sigma^{s+1} - \{a\} \approx R^{r,s+1}$ and we have an embedding of $X$ and $X'$ as disjoint $\tau$-subsets of $R^{r,s+1}$. We define a $\tau$-map (which doesn't preserve the base points)

$$ v: X \times X' \rightarrow \Sigma^{r,s} $$

by $v(x, x') = (x - x')/\|x - x'\|$, where $\|$ denotes the standard $\tau$-invariant norm in $R^{r,s+1}$. Suppose we obtained a pointed $\tau$-map $u: X \wedge X' \rightarrow \Sigma^{r,s}$ such that $v$ is $\tau$-homotopic to the composite $X \times X' \rightarrow X \wedge X' \rightarrow \Sigma^{r,s}$, then by [11], page
181, \( \psi u \) and \( \phi u \) are duality map, and \( u \) is a duality \( \tau \)-map. To show the existence of the \( \tau \)-map \( u \) we use the following

**Lemma 2.5.** Let \( X \) be a \( \tau \)-complex and \( Y \) a pointed \( \tau \)-space. If any map of \( X \) to \( Y \) and any map of \( \phi X \) to \( \phi Y \) are null-homotopic, then any \( \tau \)-map is \( \tau \)-homotopic to the constant map.

The proof is the same as [3], Proposition 11.1.

By [11], page 181, any map of \( X \lor X' \) to \( \Sigma^{r+s} \) and any map of \( \phi X \lor \phi X' \) to \( \Sigma^{s} \) are null-homotopic. Then any \( \tau \)-map of \( X \lor X' \) to \( \Sigma^{r+s} \) is \( \tau \)-homotopic to the constant map. In particular, by \( \tau \)-homotopy extension property of \( \tau \)-complex pair \((X \times X', X \lor X')\) \( v \) is \( \tau \)-homotopic to a \( \tau \)-map which sends \( X \lor X' \) to the base point of \( \Sigma^{r+s} \). Thus we obtained the required \((r, s)\)-duality \( \tau \)-map \( u: X \lor X' \to \Sigma^{r+s} \).

**Proposition 2.6.** Let \( X \in \mathcal{C}_{\mathcal{F}_{\bullet}} \). Then there exists an \((r, s)\)-dual \( X' \in \mathcal{C}_{\mathcal{F}_{\bullet}} \) of \( X \) for some \((r, s)\).

**Proof.** For a finite \( \tau \)-complex \( X \) there is a finite simplicial \( \tau \)-complex \( K \) having the \( \tau \)-homotopy type of \( X \) [2], Section 3. \( K \) can be embedded equivariantly to a simplicial \( \tau \)-complex \( \Sigma^{r,s+1} \) for some \((r, s)\). Take the \( \tau \)-subcomplex \( X' \subset \Sigma^{r,s+1} \) complementary to \( K \) as an \((r, s)\)-dual of \( X \) in a similar way to [10]. (We can assume \( X' \in \mathcal{C}_{\mathcal{F}_{\bullet}} \) by replacing \( \Sigma^{r,s+1} \) with \( \Sigma^{r,s+2} \), if necessary. Then \( X' \) is replaced by \( \Sigma X' \).) Then, by Proposition 2.4 there is a duality \( \tau \)-map \( K \land X' \to \Sigma^{r,s} \), and replacing \( K \) to \( X \) by the \( \tau \)-homotopy equivalence we complete the proof.

**Theorem 2.7.** For any finite pointed \( \tau \)-complex \( X \) there exists an equivariant \( S \)-dual of \( X \).

**Proof.** For any \( X \in \mathcal{C}_{\mathcal{F}_{\bullet}} \), \( \Sigma X \) belongs to \( \mathcal{C}_{\mathcal{F}_{\bullet}} \). Then the theorem follows the above proposition.

The following theorem is an equivariant version of Atiyah [4], Proposition (3.2), and the proof is the same as [4].

**Theorem 2.8.** Let \( M \) be a compact smooth \( \tau \)-manifold, and \( i: (M, \partial M) \to (B^{r+s}, S^{r+s}) \) an embedding such that \( i(M) \) is transversal to \( S^{r+s} \) and \( B^{s} - \phi(i(M)) \neq \emptyset \). Let \( v \) be the normal bundle of \( i \). Then the Thom complex \( T(v) \) of \( v \) is an equivariant \( S \)-dual of \( M/\partial M \). (If \( \partial M = \emptyset \), \( M/\partial \) denotes \( M \cup \{pt\} \) as usual.)
Remark 2.9. Any compact smooth \( \tau \)-manifold can be embedded equivariantly to \( B^{r,s} \) transversally to \( S^{r,s} \) for some \((r, s)\), cf., [7], (10.3) and [14], Corollary 1.10.

Remark 2.10. If \( M/\partial M \in C_{\mathcal{A}} \), then there is an \((r, s)\)-duality \( \tau \)-map \( M/\partial M \setminus T(v) \to \Sigma^{r,s} \) by Proposition 2.4.

Proposition 2.11. \( S^{1,0} \) is an equivariant \( S \)-dual of itself.

Proof. \( S^{1,0} \) is a 0-dim compact smooth \( \tau \)-manifold. Then the above theorem implies the result.

§3. Suspensions and Duality

In this section we discuss relations among duality and suspensions \( \sigma^{*,*} \), \( \sigma_{*,*} \) and \( \sigma(*, *) \) (defined below).

First we describe slant products \( / \) in \( \tau \)-cohomology, [3], (13.15).

From now on we often use the notation \( \Sigma^R, R=(r, s) \), to denote \( \Sigma^{r,s} \), for simplicity.

Let \( E=\{E_p, e^E_p\}, F=\{F_q, e^F_q\} \) and \( G=\{G_r, e^G_r\} \) be \( \tau \)-spectra, and \( \mu=\{\mu_{p,q}: E_p \wedge F_q \to G_{p+q}\} \) be a \( \tau \)-pairing, [3], Section 8. A slant product

\[
/ : [X \wedge X', E \wedge Z]^{r,s} \otimes [Y, F \wedge X]_{p,q} \to [Y \wedge X', G \wedge Z]_{p-r,q-s}
\]

(3.1) (3.2)

is defined as follows: Let \( u: \Sigma^{m-r, m-s} X \wedge X' \to E_m \wedge Z \) and \( f: \Sigma^{n+p, n+q} Y \to F_n \wedge X \) represent elements \( \{u\} \in [X \wedge X', E \wedge Z]^{r,s} \) and \( \{f\} \in [Y, F \wedge X]_{p,q} \), respectively. Define \( \mu_{m,n}(u,f): \Sigma^{m-r+p, m-s+n+q} Y \wedge X' \to G_{m+n} \wedge Z \) to be the composite \( \Sigma^{M-R+N+P} Y \wedge X' \xrightarrow{\Delta} \Sigma^{M-R} \Sigma^{N+P} Y \wedge X' \xrightarrow{1 \wedge f \wedge 1} \Sigma^{M-R} F_n \wedge X \wedge X' \xrightarrow{1 \wedge T} \Sigma^{M-R} X \wedge X' \wedge F_n \wedge Z, M-R=(m-r, m-s), N+P=(n+p, n+q). \) Put \( \mu_{m,n}(u,f)=(\rho)^{n+q} \mu_{m,n}(u,f) \). Then \( \{u\}/\{f\} \in [Y \wedge X', G \wedge Z]_{p-r,q-s} \) is defined by \( \mu_{m,n}(u,f) \). Thus \( \Gamma^P_{p,q} \) coincides with the slant product \( \{u\}/: \{Y, X\}_{p,q} \to \{Y \wedge X', \Sigma^R\}_{p,q} \) for \( X \wedge X' \to \Sigma^R, P=(p, q) \).

Slant products satisfy the compatibility with suspensions: For \( x \in [X \wedge X', E \wedge Z]^{r,s} \) and \( y \in [Y, F \wedge X]_{p,q} \),

\[
(\sigma^{a,b} x)/(\sigma_{a,b} y) = x/y \in [Y \wedge X', G \wedge Z]_{p-r,q-s},
\]

\[
(\sigma_{a,b} x) y = \sigma_{a,b}(x/y) \in [Y \wedge \Sigma^{a,b} X', G \wedge Z]_{p-r-a-q-s-b},
\]

\[
\sigma(a, P) = (-\rho)^{b q} \rho^{(a+b)(p+q)}, \quad A=(a, b), \quad P=(p, q),
\]
\[ (3.3) \quad (\sigma_{a,b}x)/y = \sigma_{a,b}(x/y) \in [Y \wedge X', G \wedge \Sigma^{a,b}Z]_{p-r+a-q-s+b}, \]

\[ (3.4) \quad x/(\sigma_{a,b}y) = \sigma_{a,b}(x/y) \in [\Sigma^{a,b}Y \wedge X', G \wedge Z]_{p-r+a-q-s-b}, \]

where \( \sigma_{a,b} : [X \wedge X', E \wedge Z]_{p-q} \to [X \wedge \Sigma^{a,b}X', E \wedge Z]_{p-a-q-b} \) is defined to be the composite: \([X \wedge X', E \wedge Z]_{p-q} \overset{\sigma_{a,b}}{\to} [\Sigma^{a,b}X \wedge X', E \wedge Z]_{p-a-q-s} \overset{[\Sigma^{a,b}]}{\to} [X \wedge \Sigma^{a,b}X', E \wedge Z]_{p-a-q-b}. \) (These can be shown similarly to [3], § 8.)

We use the notation \( \Gamma^p f(x) \) to denote the slant product

\[ x / : [Y, F \wedge X]_{p,q} \to [Y \wedge X', G \wedge Z]_{p-r+a-q-s}, \quad x \in [X \wedge X', E \wedge Z], \]

Then, for a \( \tau \)-map \( u : X \to X' \to \Sigma^p, \Gamma^p_0 = \Gamma^p(u) \) \((E = F = G = SR)\).

Let \( A = (a, b) \) be a pair of non-negative integers. The suspension isomorphisms

\[ \sigma(A) : \{X, Y\}_{p,q} \cong \{\Sigma^A X, \Sigma^A Y\}_{p,q} \]

of stable \( \tau \)-homotopy groups are defined as follows: If \( f : \Sigma^{N+P} X \to \Sigma^N Y \) represents an element \( \{f\} \in \{X, Y\}_{p,q}, N = (n, n), P = (p, q) \), then \( \sigma(A) \{f\} \) is represented by \( \Sigma^A f \) defined as the composite

\[ \Sigma^{N+P}\Sigma^A X \overset{T^{\wedge A}1}{\sim} \Sigma^A \Sigma^{N+P}X \overset{1 \wedge f}{\sim} \Sigma^A \Sigma^N Y \overset{T^{\wedge A}}{\sim} \Sigma^A \Sigma^N Y. \]

From now on we use the notations \( \sigma^A \) and \( \sigma_A \) to denote \( \sigma_{a,b} \) and \( \sigma_{a,b} \) respectively, for simplicity.

We compare \( \sigma(A) : \{X, Y\}_{p,q} \to \{\Sigma^A X, \Sigma^A Y\}_{p,q} \) with \( \sigma^A \circ \sigma_A \).

As to the definitions of \( \sigma^* \) and \( \sigma_{*,*} \) in \( \tau \)-co-(co-)homology represented by \( \tau \)-spectra we refer to [3], (7.5), (7.6) and (13.3).

**Proposition 3.1.** Let \( A = (a, b) \) and \( C = (c, d) \) be pairs of non-negative integers and \( x \in \{X, Y\}_{p,q} \). Then

\[ \sigma(A)x = \alpha(A, P) \cdot \sigma^A \circ \sigma_A x, \]

\[ \sigma^C \circ \sigma(A) = T^* \circ \sigma(A) \circ \sigma^C \quad \text{and} \quad T^c \circ \sigma_C \circ \sigma(A) = \sigma(A) \circ \sigma_C \]

where \( P = (p, q), \alpha(A, P) = (-1)^{pq} \rho_{a+b}(p+q) \) and \( T^c : \Sigma^C \Sigma^A \cong \Sigma^A \Sigma^C \).

**Proof.** Let \( f : \Sigma^{N+P} X \to \Sigma^N Y \) represent \( x \). Compare \( \Sigma^A f \) with \( \sigma^A \circ \sigma_A f \) (= \( n \)-stage of \( \sigma^A \circ \sigma_A \)). Computing permutations of suspension parameters and difference of conventional signs, we see that \([\Sigma^A f] = \alpha(A, P) \cdot \sigma^A \circ \sigma_A f \in [\Sigma^{N+P} X, \Sigma^N Y]^t\), where \([f] \) denotes the \( \tau \)-homotopy class of \( f \). Then we obtain \( \sigma(A)x = \alpha(A, P) \cdot \sigma^A \circ \sigma_A x \). Similarly we obtain the other results. \( \text{q.e.d.} \)**
Remark 3.2. Let $A$, $C$, $T'$ and $\alpha(\cdot, \cdot)$ be as above. Then
\[ \sigma^A \circ \sigma_C = \sigma_C \circ \sigma^A, \]
\[ T''_\theta \circ \sigma_C \circ A = \alpha(A, C) \cdot \sigma_C \circ \sigma_A \quad \text{and} \quad T''_\theta \circ \sigma_C \circ C = \alpha(A, C) \cdot \sigma_C \circ \sigma^A. \]
(Recall the sign conventions A3) and A4.)

Henceforth $\tilde{\sigma}(A)$ denotes the composite
\[ \{X \wedge Y, Z\}_{p,q} \xrightarrow{\sigma(A)} \{\Sigma^A X \wedge Y, \Sigma^A Z\}_{p,q} \xrightarrow{(T' \wedge 1)_\theta} \{X \wedge \Sigma^A Y, \Sigma^A Z\}_{p,q}, \]
$T': X \wedge \Sigma^A Y \rightarrow \Sigma^A X$. Then $\tilde{\sigma}(A) = \alpha(A, P) \cdot \tilde{\sigma} \circ \sigma_A$, and \[ \{u_{A,0}\} = \sigma(A)\{u\}, \{u_{0,A}\} = \tilde{\sigma}(A)\{u\} \text{ for } u: X \wedge X' \rightarrow \Sigma^R. \]

Proposition 3.3. Let $x \in \{X \wedge X', Z\}^{r,s}$ and $A = (a, b)$ be a pair of non-negative integers. Then commutativity holds in each diagram
\[
\begin{array}{c}
\{Y, X\}_{p,q} \xrightarrow{\sigma(A)} \{\Sigma^A Y \wedge X, \Sigma^A Z\}_{p,q} \\
\sigma(A) \downarrow \quad \sigma(A) \\
\{\Sigma^A Y, \Sigma^A X\}_{p,q} \xrightarrow{\sigma(A)(s)} \{\Sigma^A Y \wedge X', \Sigma^A Z\}_{p,q} \\
\end{array}
\]
\[
\begin{array}{c}
\{Y, X\}_{p,q} \xrightarrow{\sigma(A)(s)} \{\Sigma^A Y \wedge X', \Sigma^A Z\}_{p,q} \\
\sigma(A) \downarrow \quad \sigma(A) \\
\{Y \wedge \Sigma^A X', \Sigma^A Z\}_{p,q} \xrightarrow{\sigma(A)} \{Y \wedge X', \Sigma^A Z\}_{p,q} \\
\end{array}
\]
where $P = (p, q)$.

Proof. By Proposition 3.1, (3.1)~(3.4) and Remark 3.2, we see that $\Gamma^P(\sigma(A)x) \circ \sigma(A) = \alpha(A, R) \cdot \alpha(A, P) \cdot (\sigma^A \circ \sigma_A(x)') \sigma^A \circ \sigma^A = \alpha(A, P - R) \cdot (\sigma^A \circ \sigma_A(x)') \sigma^A \circ \sigma^A = \alpha(A, P - R) \cdot \tilde{\sigma} \circ \alpha(A, P) \cdot \sigma^A \circ \sigma_A(x) = \alpha(A, P - R) \cdot \alpha(A, P) \cdot (\sigma^A \circ \sigma_A(x)') \sigma^A \circ \sigma^A = \tilde{\sigma}(A)x, R = (r, s).$ (Remark that $\alpha(A, P - R) = \alpha(A, P) \cdot \alpha(A, R).$) q.e.d.

Next we discuss relations among iterated suspensions and slant products. As is easily seen, for $A = (a, b)$, $A' = (a', b')$, $C = (c, d)$ and $C' = (c', d')$ commutativity holds in each diagram
\[
\begin{array}{c}
\{X, Y\}_{p,q} \xrightarrow{\sigma(C+A)} \{\Sigma^{C+A} X, \Sigma^{C+A} Y\}_{p,q} \\
\sigma(C) \circ \sigma(A) \downarrow \quad \sigma(C) \circ \sigma(A) \\
\{\Sigma^{C+A} X, \Sigma^{C+A} Y\}_{p,q} \xrightarrow{(T \wedge 1)_\theta \circ (T \wedge 1)_\theta} \{\Sigma^{C+A} X, \Sigma^{C+A} Y\}_{p,q} \\
\end{array}
\]
\( \{ \Sigma^{C+A} X \wedge \Sigma^{C'+A'} X', \Sigma^{C+A} \Sigma^{C'+A'} Z \}_{p,q} \)

\[ \xymatrix{ \{ X \wedge X', Z \}_{p,q} \ar[r]_{\sigma(C+A) \otimes (C'+A')} \ar[d]^{(T \wedge T \wedge T)^*} & \{ \Sigma^{C} X \wedge \Sigma^{C'} X', \Sigma^{C} \Sigma^{C'} Z \}_{p,q} \ar[d]^{\sigma(A,C') \cdot (1 + T')^*} \cr \{ \Sigma^{C} X \wedge \Sigma^{C'} X', \Sigma^{C} \Sigma^{C'} Z \}_{p,q} } \]

(3.6) \( T' : \Sigma^A \Sigma^C \cong \Sigma^C \Sigma^A \). Then, by naturality of \( \Gamma \) we obtain

**Proposition 3.4.** Let \( A=(a, b), A'=(a', b'), C=(c, d) \) and \( C'=(c', d') \) be pairs of non-negative integers, \( x \in \{ X \wedge X', Z \}^{r+s} \) and \( P=(p, q) \in \mathbb{Z} \times \mathbb{Z} \). Then commutativity holds in the diagram

\[
\begin{align*}
\{ Y, \Sigma^{C+A} X \}_{p,q} & \xrightarrow{F^p(\sigma(C+A) \otimes (C'+A'))} \{ Y \wedge \Sigma^{C'+A'} X', \Sigma^{C+A} \Sigma^{C'+A'} Z \}_{p-r,q-s} \\
\{ Y, \Sigma^{C} X \}_{p,q} & \xrightarrow{F^p(\sigma(C) \otimes (C') \otimes (A') \otimes (A'))} \{ Y \wedge \Sigma^{C'} X', \Sigma^{C} \Sigma^{C'} Z \}_{p-r,q-s} \\
& \xrightarrow{\sigma(A,C') \cdot (1 + T')^*} \{ Y \wedge \Sigma^{C'} X', \Sigma^{C} \Sigma^{C'} Z \}_{p-r,q-s} 
\end{align*}
\]

**Remark 3.5.** Let \( E \) be a \( \tau \)-spectrum. Define \( \sigma(A) : [X, E \wedge Y]_{p,q} \rightarrow [\Sigma^4 X, E \wedge \Sigma^4 Y]_{p,q} \). \( A=(a, b), a \geq 0, b \geq 0 \), as follows: If \( f : \Sigma^{m+p} X \rightarrow E_m \wedge Y \) represents \( \{ f \} \in [X, E \wedge Y]_{p,q} \), then \( \sigma(A) \{ f \} \) is represented by the composite:

\[
\Sigma^{m+p+\Sigma^4} X \cong \Sigma^4 \Sigma^{m+p} X \cong \Sigma^4 E_m \wedge Y \cong E_m \wedge \Sigma^4 Y.
\]

This is a generalization of the suspension isomorphism \( \sigma(A) \) of stable \( \tau \)-homotopy groups, and Propositions 3.1, 3.3, and 3.4 hold when we replace \( \{ , \} \) by \( \{ , \} \).

Next we discuss composition of stable \( \tau \)-maps. Let \( x \in \{ X, Y \}_{p+q}, y \in \{ Y, Z \}_{r+s} \). We define \( y \circ x \in \{ X, Z \}_{p+q+r+s} \) as follows: Let \( f : \Sigma^{m+p} X \rightarrow \Sigma^m Y, g : \Sigma^{n+R} Y \rightarrow \Sigma^N Z \) represent \( x, y \) respectively, \( M=(m, m), P=(p, q), N=(n, n), R=(r, s) \). Define \( \xi(f, g) \) by the composite:

\[
\Sigma M \Sigma^N Y \xrightarrow{T^1} \Sigma^{m+p} \Sigma^N Y \xrightarrow{T^2} \Sigma^{m+n+p} Y \xrightarrow{T^3} \Sigma^{m+n} Z \cong \Sigma^N Z
\]

where \( T^1 = \rho^{m+p}(n+s) T^1 \), \( T^2 = \rho^{m+n} T^1 \), \( T^3 = \rho^{n+r} T^1 \), \( T = (T \wedge T \wedge T) \), \( k = k(m) = m(m-1)/2 \), \( e = e_{m+n-1} \). Then \( y \circ x \) is represented by \( \rho^k \xi(f, g) \). Note that the diagram

\[
\xymatrix{ \Sigma^N Z \ar[r]^{T^2} & \Sigma^{m+n} Z \ar[d]^%T^3 \\
\epsilon(N+M) T^1 \ar[u]_{\epsilon(N+M) T^1} & \Sigma^M \Sigma^N Y } \]
is $\tau$-homotopy commutative for large $m, n$, we see that $y \circ x = y/x$.  By (3.1)-(3.4) and Proposition 3.3, we also see the following compatibility with suspensions:

For $x \in \{X, Y\}_{p+q}$, $y \in \{Y, Z\}_{r+s}$ and $A = (a, b)$, $a \geq 0$, $b \geq 0$,

\[ (\sigma_A y) \circ x = \sigma_A (y \circ x), \quad (\sigma_A y) \circ (\sigma_A x) = y \circ x, \quad (\sigma_A y) \circ (\sigma_A x) = \sigma(A) (y \circ x). \]

(3.7)

For a $\tau$-spectrum $E = \{E_n\}$ we define $\tau$-pairings $\nu = \{v'_{m,n}\} : SR \wedge E \to E$ and $\nu' = \{v_{m,n}\} : E \wedge SR \to E$ by $v_{m,n} = \tilde{e} \circ T : E_m \wedge E_n \to E_{m+n}$ and $v'_{m,n} = (-\rho)^{m+n} v_{h, m \to T'} : E_m \wedge E_n \to E_{m+n}$. Let $x \in \{X, Z\}_{r+s}$. Then, using $\nu$, we have a map $x_* : [Y, E \wedge X]_{p+q} \to [Y, E \wedge Z]_{p+r,q+s}$ defined by $x_*(y) = x/y$ for $y \in [Y, E \wedge X]_{p+q}$. Let $f : \Sigma^{M+R} X \to \Sigma^{M} Z$ represent $x$, $M = (m, m)$, $R = (r, s)$, then $\rho^k \sigma_{-M} f_* \circ \sigma_{M+R} : [Y, E \wedge X]_{p+q} \to [Y, E \wedge Z]_{p+r,q+s}$, $k = m(m-1)/2$, coincides with $x_*$, i.e., for any representative $f$ of $x$

\[ x_* = \rho^k \sigma_{-M} f_* \circ \sigma_{M+R}, \quad k = k(m) = m(m-1)/2. \]

We use $1_X$ also to denote the identity map of $X$. Note that $1_X \in \{X, X\}_{0,0}$ is represented by $\rho^1 \Sigma M X$, see [3], Section 8, Example 1. Similarly let $y \in \{Y, X\}_{p+q}$, then, using $\nu'$, we have a map $y_* : [Y, E \wedge Z]_{r,s} \to [Y, E \wedge Z]_{p+r,q+s}$ defined by $y_*(x) = x/y$ for $x \in [X, E \wedge Z]_{r,s}$ and $y_*(y) = \rho^k \sigma_{-M} g_* \circ \sigma_{M}$ for $g : \Sigma^{M+R} Y \to \Sigma^{M} X$, $\{g\} = y$.

Let $x \in \{X, Y\}_{p+q}$, $z \in \{Z, W\}_{r,s}$. Then $x \wedge 1_Z \in \{X \wedge Z, Y \wedge Z\}_{p+q}$, $1_Y \wedge z \in \{Y \wedge Z, Y \wedge W\}_{r,s}$, $x \wedge 1_W \in \{X \wedge W, Y \wedge W\}_{p+q}$ and $1_X \wedge z \in \{X \wedge Z, X \wedge W\}_{r,s}$, where $x \wedge 1_Z$ is represented by $f \wedge 1_Z$ for a representative $f$ of $x$, and $1_Y \wedge z$ is represented by the composite: $\Sigma^{N+R} Y \wedge Z \Rightarrow \Sigma^{N+R} Z \wedge Y \Rightarrow \Sigma^{N} Z \wedge Y \Rightarrow \Sigma^{N} Y \wedge Z$ for a representative $g : \Sigma^{N+R} Z \to \Sigma^{N} Z$ of $z$. Then, by definition, we see easily that

\[ (1_Y \wedge z) \circ (x \wedge 1_Z) = \alpha(P, R) \cdot (x \wedge 1_W) \circ (1_X \wedge z) \in \{X \wedge Z, Y \wedge W\}_{p+R,q+s} \]

We also see the following

**Proposition 3.6.** Let $x \in \{X \wedge X', \Sigma^{0,0}\}_{r,s}$, $y \in \{Y, X\}_{p+q}$ and $z \in \{Z, Y\}_{p+q}$. Then $z \wedge 1_X \in \{Z \wedge X', Y \wedge X\}_{p+q}$ and

\[ f(x) \circ (y \circ z) = \alpha(P, R) \cdot (x \wedge 1_X) \circ (z \wedge 1_Y) = (f(x,y)) \circ (z \wedge 1_Y). \]

In particular, $y \wedge 1_X \in \{Y \wedge X', X \wedge X\}_{p+q}$ and

\[ f(x) y = x/y = (x/1_X, X') \circ (y \wedge 1_X) = x \circ (y \wedge 1_X). \]
§ 4. S-Duals of Stable \( \tau \)-Maps

Let \( X, X' \in \mathcal{C}\mathcal{F}_\mathcal{F} \) and \( R = (r, s) \in \mathbb{Z} \times \mathbb{Z} \). An element \( x \in \{X \land X', \Sigma^{0,0}\}^{r,s} \) is called to be an \( R \)-\textit{duality} when the maps

\[
\begin{align*}
\Gamma^{*, *}(x) : \{\Sigma^{0,0}, X\}^{*, *}_{r,s} &\to \{X', \Sigma^{0,0}\}^{*, *}_{r-r, s-s}, \\
\Gamma^{*, *}(\overline{x}) : \{\Sigma^{0,0}, X'\}^{*, *}_{r,s} &\to \{X, \Sigma^{0,0}\}^{*, *}_{r-r, s-s}
\end{align*}
\]

are isomorphisms, Wirthmüller [16] (cf., Remark 2.3), where \( \overline{x} \) denotes \( T'^\ast x \in \{X' \land X, \Sigma^{0,0}\}^{r,s} \), \( T' : X' \land X \cong X \land X' \). We also call an element \( x \in \{X \land X', \Sigma^{a,b}\}^{r,s} \) to be a \textit{duality} if \( \Gamma^{*, *}(x) \) and \( \Gamma^{*, *}(\overline{x}) \) are isomorphisms. Then \( \Gamma^{*, *}(x) : \{Y, X\}^{*, *}_{r,s} \to \{Y \land X', \Sigma^{a,b}\}^{*, *}_{r-r, s-s} \) and \( \Gamma^{*, *}(\overline{x}) : \{Y', X'\}^{*, *}_{r,s} \to \{Y \land X, \Sigma^{a,b}\}^{*, *}_{r-r, s-s} \) are isomorphisms for any \( Y \in \mathcal{C}\mathcal{F}_\mathcal{F} \) (Theorem 1.3), and an \( R \)-\textit{duality} \( \tau \)-map gives an \( R \)-\textit{duality} \( \{u\} \in \{X \land X', \Sigma^{a,b}\}^{0,0} \) (Theorem 2.1).

Let \( x \in \{X \land X', \Sigma^{a,b}\}^{r,s} \) and \( y \in \{Y \land Y', \Sigma^{a,b}\}^{r',s'} \) be dualities. The duality isomorphism

\[
D(x, y) : \{X, Y\}^{p,q}_{r,s} \to \{Y', X'\}^{p+r-r', q+s-s'}_{r,s'}
\]

is defined to be the composite

\[
\begin{align*}
\{X, Y\}^{p,q}_{r,s} \xrightarrow{\Gamma^{*, *}(y)} \{X \land Y', \Sigma^{a,b}\}^{p-r', q-s' -s} \xrightarrow{T'^*} \{Y' \land X, \Sigma^{a,b}\}^{p-r', q-s' -s} \\
\{Y', X'\}^{p+r-r', q+s-s'}_{r,s'} \xrightarrow{\Gamma^{*, *}(\overline{x})} \{Y \land Y', \Sigma^{a,b}\}^{p+r-r', q+s-s'}_{r,s'}
\end{align*}
\]

Clearly \( D(x, y)^{-1} = D(y, \overline{x}) \), and \( D(\sigma_{-a-b}x, \sigma_{-a-b}y) = D(x, y) \) by (3.3).

Using the results in Section 3, we discuss compatibility of duality isomorphisms \( D \) with suspensions.

**Proposition 4.1.** Let \( x \in \{X \land X', \Sigma^{0,0}\}^{r,s} \) and \( y \in \{Y \land Y', \Sigma^{0,0}\}^{r',s'} \) be dualities and \( A = (a, b) \) a pair of non-negative integers. Then commutativity holds in each diagram

\[
\begin{align*}
\{X, Y\}^{p,q}_{r,s} \xrightarrow{D(x,y)} \{Y', X'\}^{p+r-r', q+s-s'}_{r,s'} &\quad \xrightarrow{D(\sigma(A)x, \sigma(A)y)} \{Y', X'\}^{p+r-r', q+s-s'}_{r,s'} \\
\{X, Y\}^{p,q}_{r,s} \xrightarrow{D(x,y)} \{Y', X'\}^{p+r-r', q+s-s'}_{r,s'} &\quad \xrightarrow{\sigma(A)} \{Y', X'\}^{p+r-r', q+s-s'}_{r,s'}
\end{align*}
\]
Proof. Clearly $\tilde{\sigma}(A)x = \tilde{\sigma}(A)\tilde{x}$. Let $T''': Y' \wedge \Sigma^A X \cong \Sigma^A X \wedge Y', T': Y' \wedge X \cong X \wedge Y'$. Then $T''\circ \sigma(A) = \tilde{\sigma}(A)\circ T''$. By Proposition 3.3 $D(\sigma(A)x, \sigma(A)y) \circ \sigma(A) = \Gamma(\tilde{\sigma}(A))^{-1}\circ T''\circ \Gamma(\sigma(A)y) \circ \sigma(A) = \Gamma(\tilde{\sigma}(A))^{-1}\circ T''\circ \Gamma(\sigma(A)y) = \Gamma(\tilde{\sigma}(A)x)^{-1}\circ \tilde{\sigma}(A)\circ T''\circ \Gamma(\sigma(A)y) = D(x, y)$. Similarly we obtain the other.

q.e.d.

Let $x$, $y$ and $A = (a, b)$ be as above, and $A' = (a', b')$, $C = (c, d)$, $C' = (c', d')$ be pairs of non-negative integers. By Proposition 4.1 we have

\begin{equation}
D(\sigma(A)\tilde{\sigma}(A')x, \sigma(A)\tilde{\sigma}(A')y) \circ \sigma(A) = \sigma(A') \circ D(x, y).
\end{equation}

Similarly to Proposition 4.1, by (3.5) and Proposition 3.4 we see that

\begin{equation}
D(\sigma(C, A)x, \sigma(C, A)y) \circ \sigma(C) = D(\sigma(C + A)x, \sigma(C + A)y) \circ \sigma(C + A) = D(x, y),
\end{equation}

\begin{equation}
D(\sigma(C', A')x, \sigma(C', A')y) = (T \wedge 1)^* \circ (T \wedge 1)^* \circ D(\sigma(C' + A')x, \sigma(C' + A')y) = (T \wedge 1)^* \circ (T \wedge 1)^* \circ \sigma(C' + A') \circ D(x, y) = \sigma(C', A') \circ D(x, y),
\end{equation}

\begin{equation}
D(\sigma(C, C', A, A')x, \sigma(C, C', A, A')y) \circ \sigma(C) = \sigma(C') \circ D(\sigma(A, A')x, \sigma(A, A')y)
\end{equation}

and

\begin{equation}
D(\sigma(C, C', A, A')x, \sigma(C, C', A, A')y) = D(\sigma(C, A, C', A')x, \sigma(C, A, C', A')y) = D(\sigma(C, A, C', A')x, \sigma(C, A, C', A')y),
\end{equation}

where $\sigma(C, A)$, $\sigma(C', A')$, $\sigma(A, A')$ and $\sigma(C, C', A, A')$ denote $\sigma(C) \circ \sigma(A)$, $\tilde{\sigma}(C') \circ \tilde{\sigma}(A')$ and $\sigma(C) \circ \tilde{\sigma}(C') \circ \sigma(A) \circ \tilde{\sigma}(A')$ respectively. Thus we obtain

**Proposition 4.2.** Let $x \in \{X \wedge X', \Sigma^{0,0}_{*} \} \tau^{*}$ and $y \in \{Y \wedge Y', \Sigma^{0,0}_{*} \} \tau'^{*}$ be dualities, and $A = (a, b)$, $A' = (a', b')$, $C = (c, d)$ and $C' = (c', a')$ be pairs of non-negative integers. Then the following diagram is commutative:

\begin{align*}
\sigma(A) & \quad \{X, Y\} \quad \sigma(C + A) \\
\{\Sigma^A X, \Sigma^A Y\} & \quad \{Y', X'\} \quad \sigma(C' + A')
\end{align*}

\begin{align*}
\sigma(A) & \quad \{X, Y\} \quad \sigma(C + A) \\
\{\Sigma^A X, \Sigma^A Y\} & \quad \{Y', X'\} \quad \sigma(C' + A')
\end{align*}

\begin{align*}
\sigma(C) & \quad \{\Sigma^{C + A} X, \Sigma^{C + A} Y\} \quad \sigma(C') \\
\{\Sigma^{C + A} X, \Sigma^{C + A} Y\} & \quad \{\Sigma^{C' + A'} Y, \Sigma^{C' + A'} X'\} \quad \sigma(C')
\end{align*}

\begin{align*}
\sigma(C) & \quad \{\Sigma^{C + A} X, \Sigma^{C + A} Y\} \quad \sigma(C') \\
\{\Sigma^{C + A} X, \Sigma^{C + A} Y\} & \quad \{\Sigma^{C' + A'} Y, \Sigma^{C' + A'} X'\} \quad \sigma(C')
\end{align*}

\begin{align*}
\sigma(C) & \quad \{\Sigma^{C + A} X, \Sigma^{C + A} Y\} \quad \sigma(C') \\
\{\Sigma^{C + A} X, \Sigma^{C + A} Y\} & \quad \{\Sigma^{C' + A'} Y, \Sigma^{C' + A'} X'\} \quad \sigma(C')
\end{align*}

\begin{align*}
\sigma(C) & \quad \{\Sigma^{C + A} X, \Sigma^{C + A} Y\} \quad \sigma(C') \\
\{\Sigma^{C + A} X, \Sigma^{C + A} Y\} & \quad \{\Sigma^{C' + A'} Y, \Sigma^{C' + A'} X'\} \quad \sigma(C')
\end{align*}

\begin{align*}
\sigma(C) & \quad \{\Sigma^{C + A} X, \Sigma^{C + A} Y\} \quad \sigma(C') \\
\{\Sigma^{C + A} X, \Sigma^{C + A} Y\} & \quad \{\Sigma^{C' + A'} Y, \Sigma^{C' + A'} X'\} \quad \sigma(C')
\end{align*}

\begin{align*}
\sigma(C) & \quad \{\Sigma^{C + A} X, \Sigma^{C + A} Y\} \quad \sigma(C') \\
\{\Sigma^{C + A} X, \Sigma^{C + A} Y\} & \quad \{\Sigma^{C' + A'} Y, \Sigma^{C' + A'} X'\} \quad \sigma(C')
\end{align*}
where $T = (T \wedge 1)^* = (T \wedge 1)_*$, etc. are denoted by $\{,\}$, and $D(\sigma(\cdot))$ denotes $D(\sigma(\cdot)x, \sigma(\cdot)y)$.

From (3.1)–(3.4) and the definition of $D(x, y)$ we obtain

**Proposition 4.3.** Let $x \in \{X \wedge X', \Sigma^0,0\}^{r,s'}$ and $y \in \{Y \wedge Y', \Sigma^0,0\}^{r',s'}$ be dualities, $z \in \{X, Y\}_{p,q}$ and $A = (a, b)$ be a pair of non-negative integers. Then $D(x, y)$ satisfies the compatibility with suspensions $\sigma^A$ and $\sigma_A$:

$D(\sigma^A x, y) \circ \sigma^A z = \alpha(A, P + R - R') \cdot D(x, y) z \in \{Y', X'\}_{p + r - r', q + s - s'}$,

$D(\sigma_A x, y) z = \alpha(A, P) \circ D(x, y) z \in \{\Sigma^4 Y', X'\}_{p + r - r', q + s - s'}$.

where $P = (p, q)$, $R = (r, s)$, $R' = (r', s')$ and $a(A, C) = (-p)^b \rho(a \circ c)(b \circ d)$ for $C = (c, d)$.

Next we see the relation of compositions of stable $T$-maps and their duals.

**Proposition 4.4.** Let $x \in \{X \wedge X', \Sigma^0,0\}^{r,s}$, $y \in \{Y \wedge Y', \Sigma^0,0\}^{r',s'}$ and $z \in \{Z \wedge Z', \Sigma^0,0\}^{r,s''}$ be dualities, and $u \in \{X, Y\}_{p,q}$, $v \in \{Y, Z\}_{p',q'}$. Then $v \circ u \in \{X, Z\}_{p + p' + r - r', q + q' + s - s''}$ and

$D(x, z)(v \circ u) = \alpha(P, P' + R' - R'') \cdot (D(x, y) u) \circ (D(y, z) v) \in \{Z', X'\}_{p + p' + r - r', q + q' + s - s''}$,

where $P = (p, q)$, $P' = (p', q')$, $R' = (r', s')$, $R'' = (r'', s'')$ and $\alpha(P, C) = (-\rho)^{q} \rho(a \circ c)(b \circ d)$ for $C = (c, d)$.

**Proof.** Let $T_1 : X \wedge Y' \cong Y', T_2 : Y \wedge Z' \cong Z', T_3 : X \wedge Z' \cong X', T_4 : Y \wedge Y' \cong Z' \wedge Y$, and $u' = D(x, y) u$, $v' = D(y, z) v$. Then, by definition, $y/u = T_2^u(\vec{x}/u')$ and $z/v = T_4^v(\vec{y}/v')$. By Proposition 3.6, we see that $T(z)(v \circ u) = z/(v \circ u) = (z/v) \circ (u \wedge 1_{Z'}) = (T_2^u(\vec{x}/u')) \circ (u \wedge 1_{Z'}) = (T_4^v(\vec{y}/v')) \circ (u \wedge 1_{Z'}) = y \circ (u \wedge 1_{Z'})$, and $T_3^s \circ \Gamma(\vec{x}) (u' \circ v') = T_3^s(\vec{x}/(u' \circ v')) = (T_4^s(\vec{x}/u')) \circ (1_{X} \wedge v') = (y/v) \circ (1_{X} \wedge v') = (y/v) \circ (1_{X} \wedge v')$. Then, by (3.9), we complete the proof.

We reduce duality $T$-maps to stable $T$-maps in $\{X \wedge X', \Sigma^0,0\}^{r,s}$. For $X \in \mathcal{F}_G^e$ there exists an S-dual $X'$ by Theorem 2.7. Then we can choose the duality $T$-map having the form $u' : \Sigma^0,1X \wedge X' \rightarrow \Sigma^0,1\Sigma^r$. In fact, let $u' : \Sigma^0,1X \wedge X' \rightarrow \Sigma^r$ be a duality $T$-map obtained by Propositions 2.4 and 2.6, then $s \geq 0$ from the construction. We define $u$ by $\rho^s T \circ u' : \Sigma^0,1X \wedge X' \rightarrow \Sigma^0,1\Sigma^r$. This is the required one. For this $u$ we define an $(r, s)$-duality $\langle u \rangle \in \{X \wedge X', \Sigma^0,0\}^{r,s}$ by
(4.6) \[ \langle u \rangle = \sigma_{r-s}(0, 1)^{-1}\{u\}. \]

For a \( \tau \)-map \( u : X \wedge X' \rightarrow \Sigma R, \ R = (r, s) \), and \( u_{p,p'} : \Sigma^p X \wedge \Sigma^{p'} X' \rightarrow \Sigma^p \Sigma^{p'} \Sigma^R \), we see easily that \( \{ u_{p,p'} \} = \sigma(P) \circ \delta(P)^{-1}\{u\} \). Thus, for a duality \( \tau \)-map \( u_{p,p'} : \Sigma^p X \wedge \Sigma^{p'} X' \rightarrow \Sigma^p \Sigma^{p'} \Sigma^R \), we observe that \( \langle u_{p,p'} \rangle = \sigma_{-p} \circ \delta(P)^{-1} \circ \sigma(0, 1)^{-1}\{u_{p,p'}\} = \langle u \rangle \) is an \( R \)-duality.

**Theorem 4.5.** Let \( X' \) and \( X'' \) be \( S \)-duals of \( X \in \mathcal{C} \mathcal{F} \) so that \( x \in \{ X \wedge X', \Sigma^{0,0}\}^{r,s} \) and \( x' \in \{ X \wedge X'', \Sigma^{0,0}\}^{r',s'} \) are dualities. Then a stable \( \tau \)-homotopy equivalence \( \{ f \} \in \{ X', X''\}^{r-r',s-s}, f : \Sigma^{n+r'-r,s'-s} X' \rightarrow \Sigma^{n,s} X'' \), is canonically determined by \( \{ f \} = D(x', x)1_X \) for large \( n \).

**Proof.** Put \( v = D(x, x')1_X \in \{ X'', X'\}^{r-r',s-s} \). Then, by Proposition 4.4 we see that \( v \) is the inverse of \( \{ f \} \). q.e.d.

§5. **Duality between \( \tau \)-Cohomology and Homology**

First we see the following \( \tau \)-cohomology version of [16], Proposition 1.2, and the proof is the same as [16]. (Use Comparison Theorem 1.3.‘)

**Proposition 5.1.** Let \( x \in \{ X \wedge X', \Sigma^{0,0}\}^{r,s} \) be an \((r, s)\)-duality, and \( Y \in \mathcal{C} \mathcal{F} \), \( Z \in \mathcal{C} \mathcal{H} \). Then

\[ \Gamma^p(x) : \{ Y, Z \wedge X \}_{p,q} \rightarrow \{ Y \wedge X', Z \}_{p-r, q-s} \]

is a \( \Lambda \)-isomorphism for each \( P = (p, q) \in \mathbb{Z} \times \mathbb{Z} \).

Let \( E = \{ E_n ; n \in \mathbb{Z} \} \) be a \( \tau \)-spectrum. A decomposition

\[ [X, E]_{p,q} = \lim_n \{ \Sigma^{n+p} X, E_n \}^r = \lim_m \{ \Sigma^{M+N+p} X, \Sigma^M E_n \}^r, \]

\( P = (p, q), M = (m, m), N = (n, n), \) implies

\( (5.1) \]

\[ [X, E]_{p,q} = \lim_n \{ \Sigma^{n+p} X, E_n \}_{0,0} \]

(cf., [3], the proof of Proposition 13.5). Then we obtain the following

**Theorem 5.2.** Let \( x \in \{ X \wedge X', \Sigma^{0,0}\}^{r,s} \) be an \((r, s)\)-duality, \( E = \{ E_n ; n \in \mathbb{Z} \} \) a \( \tau \)-spectrum, \( Y \in \mathcal{C} \mathcal{F} \) and \( P = (p, q) \in \mathbb{Z} \times \mathbb{Z} \). Then there exists a duality isomorphism

\[ \Gamma^p(x, E) : [Y, E \wedge X]_{p,q} \cong [Y \wedge X', E]_{p-r, q-s}. \]

**Proof.** By Proposition 5.1 we obtain the isomorphism

\[ \sigma_{r,s} \circ \Gamma^p(x) : \{ \Sigma^{n+p} Y, E_n \}_{0,0} \cong \{ \Sigma^{n+p} Y, E_n \wedge \Sigma^R \}_{0,0}, \]
$N = (n, n)$, $R = (r, s)$. Then, taking the direct limit and by (5.1) we obtain the isomorphism to be the composite

$$[Y, E \wedge X]_{p,q} \cong \lim_{\to} [Y \wedge X', E \wedge \Sigma^p]_{p,q} \cong [Y \wedge X', E]_{r-s}. \quad \text{q.e.d.}$$

As $S^1_{s} \cong \{0, 0\}$ is an $S$-dual of itself and there is a $(0, 0)$-duality, Proposition 2.11, putting $Y = \Sigma^0$, we obtain

**Corollary 5.3.** Let $x \in \{X \wedge X', \Sigma^0\}^{r,s}$ be an $(r, s)$-duality and $E$ a $\tau$-spectrum. Then

$$\Gamma^p(x, E): \tilde{h}_{p,q}(X; E) \cong \tilde{h}^{r-p,s-q}(X'; E).$$

In particular

$$\Gamma^p(x, E): h_{p,q}(S^1_{s}; E) \cong h^{r-p,s-q}(S^1_{s}; E).$$

Let $h_{*,*} = \{h_{p,q}, (p, q) \in \mathbb{Z} \times \mathbb{Z}\}$ be a reduced $\tau$-homology theory on $\mathscr{F}$. For each $X \in \mathscr{F}$, there is an $S$-dual $X'$ of $X$ and a duality $x \in \{X \wedge X', \Sigma^0\}^{r,s}$ by Theorem 2.7 and (4.6). Put

$$h_{p,q}(X) = \tilde{h}^{r-p,s-q}(X').$$

for each $(p, q) \in \mathbb{Z} \times \mathbb{Z}$. By Theorem 4.5, $h_{p,q}(X)$ is uniquely determined up to canonical isomorphisms. Let $f: X \rightarrow Y$ be a $\tau$-map in $\mathscr{F}$. Choose an $S$-dual $Y'$ and a duality $y \in \{Y \wedge Y', \Sigma^0\}^{r,s}$. Then we see that $D(x, y): \{X, Y\}_{p,q} \cong \{Y', X\}_{p+r-s'-q-s-s'}$. Put $u = D(x, y) \{f\}$. Define

$$f^*: \tilde{h}^{p,q}(Y) \rightarrow \tilde{h}^{p,q}(X)$$

by the following: Let $f': \Sigma^{M+P+R-R'} Y' \rightarrow \Sigma^M X'$ represent $u$, $M = (m, m)$, $P = (p, q)$, $R = (r, s)$, $R' = (r', s')$. Then $f^*$ is defined by $\rho^k \sigma_{-M} f^* \sigma_{M+P+R-R'}$, $k = k(m)$, cf. (3.8). This definition is independent of the choice of representatives of $u$. Suspensions $\sigma_{a,b}$ are defined by $\sigma(a, b)x$ and (5.2).

In order to show that $\tilde{h}^{p,q}(Z) \rightarrow \tilde{h}^{p,q}(Y) \rightarrow \tilde{h}^{p,q}(X)$ is exact for a $\tau$-cofibration sequence $X \rightarrow Y \rightarrow Z$, we use the following $\tau$-cohomology version of [16], Proposition 4.1, and [12], Theorem (6.10).

**Proposition 5.4.** Let $x \in \{X \wedge X', \Sigma^0\}^{r,s}$ and $y \in \{Y \wedge Y', \Sigma^0\}^{r,s}$ be $(r, s)$-dualities. Let $f: X \rightarrow Y$ and $f': Y' \rightarrow X'$ be $\tau$-maps such that $\{f\} = D(x, y) \{f\}$. Then there exists an $(r, s + 1)$-duality $w \in \{C_f \wedge C_{f'}, \Sigma^0\}^{r,s}$ compatible with the $\tau$-cofibration sequence of $f$ and $f'$, i.e.,
Let \( x \in \{ X \wedge X', \Sigma^0 \}^{r,s} \) and \( y \in \{ Y \wedge Y', \Sigma^0 \}^{r',s'} \) be given dualities. Let \( m = \max(r, r', s, s') \), \( M = (m, m) \). Then \( \tilde{\sigma}^{M-R_X} \) and \( \tilde{\sigma}^{M-R_Y} \), \( R=(r, s) \), \( R'=(r', s') \), are both \( (m, m) \)-dualities. And choose a representative \( f' \) of \( D(\tilde{\sigma}^{M-R_X}, \tilde{\sigma}^{M-R_Y}) \). Then we can apply the above Proposition to a given \( f: X \to Y \), and by (5.2), (5.3) we see that \( \tilde{h}^{p,q}(Z) \xrightarrow{\varrho} \tilde{h}^{p,q}(Y) \xrightarrow{f'} \tilde{h}^{p,q}(X) \) is exact for the cofibration sequence \( X \xrightarrow{f} Y \xrightarrow{\varrho} Z \). Thus we obtain

**Proposition 5.5.** By the above Definitions (5.2) and (5.3) \( \tilde{h}^{*,*} = \{ \tilde{h}^{p,q}, (p, q) \in \mathbb{Z} \times \mathbb{Z} \} \) is a reduced \( \tau \)-cohomology theory on \( \mathscr{F}_\tau^\circ \).

By [2], Theorem 3.4 for \( G=\mathbb{Z}/2\mathbb{Z} \) any reduced \( \tau \)-cohomology theory on \( \mathscr{F}_\tau^\circ \) is represented by an \( \Omega-\tau \)-spectrum \( E \), i.e.,

\[
\tilde{h}^{p,q}(X) \cong \tilde{h}^{p,q}(X; E), \quad X \in \mathscr{F}_\tau^\circ, \quad (p, q) \in \mathbb{Z} \times \mathbb{Z}.
\]

Then, by (5.2) we obtain a sequence of natural isomorphisms:

\[
\tilde{h}_{p,q}(X) \cong \tilde{h}_{r-p,s-q}(X') \cong \tilde{h}_{r-p,s-q}(X'; E) \cong \tilde{h}_{p,q}(X; E).
\]

Thus we obtain an equivariant version of G. W. Whitehead [15] as follows.

**Theorem 5.6.** A reduced \( \tau \)-homology theory on \( \mathscr{F}_\tau^\circ \) is represented by a suitable \( \Omega-\tau \)-spectrum.

**Corollary 5.7.** A reduced \( \tau \)-homology theory on \( \mathscr{F}_\tau^\circ \) is represented by a suitable \( \Omega-\tau \)-spectrum.

**Proof.** If an \( \Omega-\tau \)-spectrum \( E \) represents \( \tilde{h}_{*,*} \mid_{\mathscr{F}_\tau^\circ} \), then \( E \) represents \( \tilde{h}_{*,*} \) by Theorem 1.3'. Thus the corollary follows from the above theorem.

q.e.d.

§ 6. Atiyah-Poincaré Duality in \( \tau \)-(Co-) Homology

In this section we discuss Atiyah-Poincaré-type duality for real-complex orientable \( \tau \)-cohomology theories [1].

A compact smooth \( \tau \)-manifold is called a weakly real-complex manifold if the normal bundle \( v \) of an equivariant embedding \( (M, \partial M) \to (B^a, S^a) \) trans-
versal to $S^{a,b}$ is a real-complex vector bundle ($=\text{Real vector bundle in the sense of Atiyah}\ [5]$) for some $(a, b)$. Let $h^{*,*}$ be a real-complex orientable $\tau$-cohomology theory and $M$ be a weakly real-complex manifold with $r$-dimensional normal real-complex vector bundle $\nu$ of an embedding $(M, \partial M) \to (B^{a,b}, S^{a,b})$, $r \geq 1$. Then there is the Thom isomorphism

$$\Phi: h^{*,*}(M) \cong h^{*,*}(T(\nu)).$$

On the other hand, by Theorem 2.8 and Corollary 5.3 we obtain the duality isomorphism

$$D: h^{a-r-p,b-r-q}(M/\partial M) \cong h^{p+r,a+q}(T(\nu)),$$

where $h^{*,*}$ is represented by a $\tau$-spectrum. If $\dim M = m + n$ and $\dim \phi M = n$, then $a - r = m$ and $b - r = n$. Combining these isomorphisms we obtain the following

**Theorem 6.1.** Let $h^{*,*}$ be a real-complex orientable $\tau$-cohomology theory and $M$ a weakly real-complex manifold such that $\dim M = m + n$ and $\dim \phi M = n$. Then there exists a duality isomorphism

$$D_M = D^{-1} \circ \Phi: h^{p,q}(M) \cong h_{m-p,n-q}(M, \partial M)$$

for every $(p, q) \in \mathbb{Z} \times \mathbb{Z}$.

Typical example of real-complex orientable $\tau$-cohomology theory is $\mathbb{MR}^{*,*}$, [8]. Thus we obtain

**Corollary 6.2.** Let $M$ be a weakly real-complex manifold such that $\dim M = m + n$ and $\dim \phi M = n$. Then there exists a duality isomorphism

$$D_M: \mathbb{MR}^{p,q}(M) \cong \mathbb{MR}_{m-p,n-q}(M, \partial M)$$

for every $(p, q) \in \mathbb{Z} \times \mathbb{Z}$.

**References**


