Some Remarks on the Modified Korteweg-de Vries Equations

By

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Abstract

In Section 1 we associate certain linear differential operators to modifications of the KdV equation. An interpretation is given to the non-linear transformation of Miura [4] which converts a solution of one of modified KdV equations into that of the KdV equation.

In Section 2 we construct a family of special solutions of another modification of the KdV equation.

1. In this paper we study the modified Korteweg-de Vries (KdV) equations

\[ \dot{v} \pm 6v^2v' + v''' = 0 \]

where \( \dot{v} \) and \( v' \) are \( t \) and \( x \) derivatives of real-valued smooth function \( v = v(x, t) \) \( (-\infty < x, t < \infty) \) respectively. We shall refer to them as equations (1+) and (1-) according to their signs. These equations appear in Zabusky [6] as generalizations of the KdV equation

\[ \dot{u} - 6uu' + u''' = 0 \]

and in Miura [5] where the relation between the solutions of (1) and (2) is discussed. The existence theorem for the initial-value problem of (1) has been proved in Kametaka [2].

Lax [4] has rewritten the KdV equation into an evolution equation for a linear operator: For a complex-valued smooth function \( u(x) \), let \( L_u \) be the one dimensional Schrödinger operator

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\[ L_u = -D^2 + u \]

and put

\[ B_u = -4D^3 + 3uD + 3Du \]

where \( D \) stands for the \( x \) differentiation. Then the operator

\[ [B_u, L_u] = B_uL_u - L_uB_u \]

is the multiplication by the function \( 6uu' - u''' \). So the operator equation for real-valued function \( u(t) = u(t, x) \)

\[ \dot{L}_u(t) = [B_u(t), L_u(t)] \]

is equivalent to the KdV equation.

For the modified KdV equations we can give a similar operator interpretation. For a complex-valued smooth function \( v(x) \), introduce the first order differential operator

\[
L_v = \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix} D + \begin{bmatrix}
0 & v \\
v & 0
\end{bmatrix}
\]

and put

\[
B_v = \begin{bmatrix}
B_{v'} + v^2 & 0 \\
0 & B_{-v'} + v^2
\end{bmatrix}.
\]

Then the operator \([B_v, L_v]\) is the multiplication by the matrix valued function

\[
\begin{bmatrix}
0 & 6v^2v' - v'''
6v^2v' - v''' & 0
\end{bmatrix}.
\]

So for real-valued function \( v(t) = v(x, t) \), the operator evolution equations

\[ \dot{L}_v(t) = [B_v(t), L_v(t)] \]
and

\[ \hat{L}_{i\varepsilon(t)} = [B_{i\varepsilon(t)}, L_{i\varepsilon(t)}] \]

are equivalent to \((1—)\) and \((1+)\) respectively.

Note that we have an operator identity

\( L^2_p = \begin{bmatrix} L_{p'+p^2} & 0 \\ 0 & L_{-p'+p^2} \end{bmatrix} \).

Putting a solution \( v = v(t, x) \) of the equation \((1—)\) into \((3)\), we differentiate \((3)\) with respect to \( t \). Then we have

\[
\begin{bmatrix}
L_{p'+p^2} & 0 \\
0 & L_{-p'+p^2}
\end{bmatrix} = L_v \hat{L}_v + \hat{L}_v L_v
\]

\[ = L_v [B_v, L_v] + [B_v, L_v] L_v \]

\[ = [B_v, L^2_v] \]

and finally

\[ \hat{L}_{\pm p'+p^2} = [B_{\pm p'+p^2}, L_{\pm p'+p^2}] \].

So \( \pm p' + p^2 \) satisfy the KdV equation. This fact has been discovered by Miura \([5]\) by a different consideration.

**2.** In this section we construct a family of special solutions of the modified KdV equation \((1+)\). They are analogous to the \( N \)-tuple wave solutions of the KdV equation, which have been constructed in Gardner, Greene, Kruskal and Miura \([1]\) based on the inverse scattering theory for the Schrödinger equation.

Consider the eigenvalue problem for the operator \( L_{i\varepsilon} : \)

\[ y_2' + ivy_2 = \zeta y_1 \]

\[ -y_1' + ivy_1 = \zeta y_2. \]

Putting \( z_1 = y_1 - iy_2 \) and \( z_2 = y_1 + iy_2 \), we have
This is a special case of the system of first order differential equations

\begin{align}
iz_1' + vz_2 &= \zeta z_1 \\
-iz_2' - vz_1 &= \zeta z_2.
\end{align}

where \(q\) is a complex-valued function and \(q^*\) denotes its complex conjugate.

The inverse scattering theory for (4), namely the problem of the construction of the potential \(q\) from the scattering data, has been discussed by Zakhalov and Shabat [7] and applied to the exact solution of a certain non-linear equation. In what follows we restrict our attention to the case where the fundamental equation of the inverse scattering theory reduces to the system of linear algebraic equations.

Let \(\zeta_1, \ldots, \zeta_N\) be complex numbers different from each other in the upper half-plane and \(c_1, \ldots, c_N\) be any complex numbers. Put

\[\lambda_j = c_j^{1/2} \exp(i\zeta_j x)\]

and consider a system of linear equations for \(\psi_{1j}, \psi_{2j} (j=1, \ldots, N)\):

\begin{align}
(5a) & \quad \psi_{1j} + \sum_k \lambda_j \bar{\lambda}_k (\zeta_j - \zeta_k^*)^{-1} \psi_{2k} = 0 \\
(5b) & \quad -\sum_k \lambda_j \bar{\lambda}_k (\zeta_j^* - \zeta_k)^{-1} \psi_{1k} + \psi_{2j} = \lambda_j^* 
\end{align}

(The sums are taken from 1 to \(N\) throughout the present paper). Then this system of equations has a non-singular coefficient matrix. Put

\[q(x) = -2i \sum \bar{\lambda}_k \bar{\psi}_{2k}^* \psi_{2k}^* \]  

Then for each \(j\), the pair \((\psi_{1j}, \psi_{2j})\) satisfies the differential equations

\begin{align}
(6) & \quad i\psi_{1j}' - iq\psi_{2j} = \zeta_j \psi_{1j} \\
& \quad -i\psi_{2j}' - iq^\ast \psi_{1j} = \zeta_j \psi_{2j}.
\end{align}
We give a proof of these facts in Appendix.

Multiply $\psi_{2j}^*$ on (5b) and take the summation over $j$. Then we have another expression for $q(x)$:

$$q(x) = 2i \sum_j (\psi_{1j}^2 - \psi_{2j}^2).$$

We pose further restriction on the system (5): Let $M$ be a non-negative integer such that $2M \leq N$. Let $\sigma$ be the permutation among integers between 1 and $N$ defined by

$$\sigma(j) = \begin{cases} j + 1 & j \text{ odd} \leq 2M \\ j - 1 & j \text{ even} \leq 2M \\ j & j > 2M. \end{cases}$$

We assume that $\zeta_{\sigma(j)} = -\zeta_j^*$ and $c_{\sigma(j)} = c_j^*(1 \leq j \leq N)$.

Now let $c_j$ depend on $t$ as

$$c_j(t) = c_j(0) \exp(8i\zeta_j^2t)$$

and put

$$\lambda_j = \lambda_j(x, t) = c_j(t)^{1/2} \exp(i\zeta_j x).$$

**Theorem.** Let $\psi_{1j}(x, t)$ and $\psi_{2j}(x, t)$ be the solution of the system (5) for $\lambda_j = \lambda_j(x, t)$ defined above and put

$$q(x, t) = -2i \sum_j \lambda_j^*(x, t) \psi_{2j}^*(x, t).$$

Then $v(x, t) = -iq(x, t)$ is real-valued and satisfies the modified KdV equation (1+).

**Proof.** Put $\phi_{1j} = i\lambda_j \psi_{1j}$ and $\phi_{2j} = \lambda_j \psi_{2j}$. Then the system (5) is rewritten as

\begin{align*}
(7a) & \quad \lambda_j^2 \phi_{1j} + i \sum_k (\zeta_j - \zeta_k^*)^{-1} \phi_{2k}^* = 0 \\
(7b) & \quad i \sum_k (\zeta_j^* - \zeta_k)^{-1} \phi_{1k} + \lambda_j^2 \phi_{2j}^* = 1.
\end{align*}
It is easy to verify that \( \phi_{1j}^* \) and \( \phi_{2j}^* \) satisfy the same equation as \( \phi_{1j} \) and \( \phi_{2j} \). By the uniqueness of solution we have \( \phi_{1j}^* = \phi_{1j} \) and \( \phi_{2j}^* = \phi_{2j} \). The function \( v(x, t) \) is thus real-valued.

Eliminating \( \phi_{1j} \) from (7), we have a system of linear equations for \( \phi_{2j}^* \):

\[
\sum_{i} \alpha_{ji} \phi_{2i}^* = 1
\]

where

\[
\alpha_{ji} = \alpha_{ji}(x, t) = \sum_{k} \lambda^2_\delta_\zeta^k_j (\zeta^k_j - \zeta^k_j)^{-1} (\zeta_k - \zeta^k_j)^{-1} + \lambda^2_j \delta_{ji}
\]

(\( \delta_{ji} \) is Kronecker’s delta). Now we differentiate (8) with respect to \( t \) and obtain a system of linear equations for \( \phi_{2j} \):

\[
\sum_{i} \alpha_{ji} \phi_{2i}^* = \tau_j
\]

where

\[
\tau_j = -8i \sum_{k} \zeta^k_j \lambda^2_\delta_\zeta^k_j (\zeta^k_j - \zeta^k_j)^{-1} (\zeta_k - \zeta^k_j)^{-1} \phi_{2i}^* - 8i \zeta^k_j \lambda^2_j \phi_{2j}^*.
\]

Let \( \beta_{jk} = \beta_{jk}(x, t) \) be the element of the inverse matrix of the matrix \( (\alpha_{jk}) \). Then we have

\[
\phi_{2j}^* = \sum_{k} \beta_{jk} \phi_{2k} = \sum_{k} \beta_{jk} \phi_{2j} = \sum_{k} \beta_{jk} \tau_k.
\]

Using these relations and (7a), we have a formula for the \( t \)-derivative of \( v \):

\[
v' = 16i \sum_{j} (-\zeta^3_j \psi_{1j}^2 + \zeta^3_j \psi_{2j}^2).
\]

We differentiate

\[
v = 2 \sum_{j} (\psi_{1j}^2 - \psi_{2j}^2)
\]

successively with respect to \( x \) and obtain the formulas for \( x \)-derivatives of \( v \):

\[
v' = 4i \sum_{j} (-\zeta_j \psi_{1j}^2 + \zeta_j \psi_{2j}^2)
\]
\[ v'' = -2v^3 + 8 \sum_j (-\zeta_j^3 \psi_{1j}^2 + \zeta_j^4 \psi_{2j}^2) \]
\[ v''' = -6v^2 v' + 16i \sum_j (\zeta_j^3 \psi_{1j}^2 - \zeta_j^4 \psi_{2j}^2). \]

Beside the relation (6), we have used the relations
\[
\text{Re}(\sum_j \zeta_j \psi_{1j} \psi_{2j}) = 0 \\
\sum_j \zeta_j \psi_{1j} \psi_{2j} = -8^{-1} v \\
\text{Re}(\sum_j \zeta_j^2 \psi_{1j} \psi_{2j}) = 0
\]
to derive these formulas. Q.E.D.

If \( N = 1 \), then \( \zeta_1 = i \eta (\eta > 0) \) and \( c = c_1(0) \) is real. We have thus solutions
\[
v(x, t) = (\text{sgn } c) s(x - 4\eta^2 t - \delta, \eta)
\]
where
\[
s(x, \eta) = -2\eta \text{ sech}(2\eta x)
\]
and
\[
\delta = \delta(c, \eta) = (2\eta)^{-1} \log \left( \frac{|c|}{2\eta} \right).
\]
These solutions coincide with the soliton solutions known to exist for the generalized KdV equations (see Zabusky [6]).

Now let \( N = 2 \) and \( M = 0 \). Then \( \zeta_j = i \eta_j, \) \( 0 < \eta_1 < \eta_2 \) and \( c_j = c_j(0) \) are real. The solutions decompose into two solitons as \( t \to \pm \infty \):
\[
v(x, t) - \sum_{j=1}^{2} (\text{sgn } c_j) s(x - 4\eta_j^2 t - \delta_j^\pm, \eta_j) \to 0
\]
where
\[
\delta_1^+ = \delta(c_1, \eta_1) + \eta_1^{-1} \log(\eta_2 - \eta_1)(\eta_2 + \eta_1)^{-1} \\
\delta_2^+ = \delta(c_2, \eta_2) \\
\delta_1^- = \delta(c_1, \eta_1) \\
\delta_2^- = \delta(c_2, \eta_2) + \eta_2^{-1} \log(\eta_2 - \eta_1)(\eta_2 + \eta_1)^{-1}.
\]
More generally in the case $M=0$ (i.e. all of $\zeta_j$ are purely imaginary) the corresponding solutions seem to decompose into solitons as $t \to \pm \infty$.

Appendix. The following arguments are quite similar to that of Kay and Moses [3] where the construction of reflectionless potential for Schrödinger equation has been discussed.

The $N \times N$ matrix $A = (i(\zeta_j - \zeta_k^*)^{-1})$ is positive definite because of the identity

$$i(\zeta_j - \zeta_k^*)^{-1} = \int_0^\infty \exp(i\zeta_j t) \exp(i\zeta_k^* t) dt.$$  

Eliminating $\psi_{1j}$ from (5), we have a system of $N$ linear equations for $\psi_{2j}^*$:

$$\sum_i b_{ij} \psi_{2i}^* + \psi_{2j}^* = \lambda_j^*,$$  

where

$$b_{ij} = \sum_i \lambda_j^* \lambda_k^* (\zeta_j^* - \zeta_k^*)^{-1} (\zeta_k - \zeta_k^*)^{-1}.$$  

Putting $B = (b_{jk})$, we have

$$\det B = \lambda_1 \lambda_2 \cdots \lambda_N |A|^2,$$  

so $\det B$ is positive. Any principal minor of $B$ is also positive because it is expressed as the sum of the determinant of the matrices having the same form as $B$. Now the characteristic polynomial of $B$ is

$$\det (B + \lambda I) = \lambda^N + a_1 \lambda^{N-1} + \cdots + a_N,$$  

where $a_j$ is positive, being the sum of the principal minors of $B$ of order $j$. If we set $\lambda = 1$, we see that the matrix $B + I$ is invertible and so is the coefficient matrix of (5).

Differentiating the equations (5) with respect to $x$, we see that $2N$ functions

$$\psi_{1j}^* + i\zeta_j \psi_{1j} - q \psi_{2j} \quad \psi_{2j}^* - i\zeta_j \psi_{2j} + q^* \psi_{1j}.$$  


satisfy the homogeneous system of equations associated with (5) and therefore vanish.

References


