On the Diameter of Compact Homogeneous Riemannian Manifolds

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Introduction

Let $M$ be a compact Riemannian manifold. The diameter $d(M)$ of $M$ is defined to be the maximum of $d(p, q)$, $p, q \in M$, where $d(\cdot, \cdot)$ denotes the distance function on $M$ induced by the Riemannian metric.

The main purpose of this paper is to find a positive constant $d$ such that the diameter $d(M)^2 \geq d$ when the sectional curvature $K \leq 1$.

In this paper we consider the case that the manifold $M$ is homogeneous. In [3] the author proved that $d = n/2$ if the manifold has a big isotropy subgroup. It has been left to study the case that the isotropy subgroup is finite. Hence we shall mainly study invariant metrics on a Lie group and prove that the number $d > 0.23$ if the sectional curvature $K \neq 0$ (Theorem 5.1).

§ 1. Fixed Points of Isometries

Let $M$ be a compact $C^\infty$ manifold with a Riemannian metric $g$. Let $d_g(\cdot, \cdot)$ denote the distance function on $M$ induced by $g$. Let $I(M, g)$ denote the group of isometries of $(M, g)$. Let $p$ be a point of $M$. We denote by $I_p(M, g)$ the isotropy subgroup of $I(M, g)$, i.e., $I_p(M, g) = \{a \in I(M, g); ap = p\}$. Let $A$ be a connected subgroup of $I_p(M, g)$. Put $F(A) = \{x \in M; Ax = x\}$. Then it is easy to see that $F(A)$ is a disjoint union of closed totally geodesic submanifolds of $M$. For a curve $c: [0, 1] \to M$, we denote by $\text{length}_g(c)$ the length of $c$ with respect to the metric $g$.

**Lemma 1.1.** Let $A$ be a connected subgroup of $I_p(M, g)$ with $\dim A \geq 1$. 

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Then $F(A) \subseteq M$. Let $\gamma : [0, 1] \rightarrow M$ be a geodesic starting from a point of $F(A)$ in the normal direction to $F(A)$. Assume that the sectional curvature $K_g \leq k$ ($k > 0$) and $\text{length}_g(\gamma) \leq \pi/2\sqrt{k}$. Then $d_g(F(A), \gamma(1)) = \text{length}_g(\gamma)$, i.e., the injectivity radius of $F(A)$ is not less than $\pi/2\sqrt{k}$.

**Proof.** In case that $k = 1$ and $A$ is the identity component of $I_p(M, g)$, this is Proposition 4.2 in [3]. The proof in it is still valid for the case that $A$ is a connected subgroup of $I_p(M, g)$ without any change. Hence we obtain

$$\text{length}_g(\gamma) = \frac{1}{\sqrt{k}} \text{length}_{k_g}(\gamma) = \frac{1}{k} d_{k_g}(F(A), \gamma(1)) = d_g(F(A), \gamma(1)),$$

since $K_{k_g} = K_g/k \leq 1$ and $\text{length}_{k_g}(\gamma) = \sqrt{k} \text{length}_g(\gamma) \leq \pi/2$. Q.E.D.

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**§ 2. The Length of a Killing Vector Field**

Let $(M, g)$ be a compact Riemannian manifold as in Section 1.

**Theorem 2.1.** Let $\xi$ be a non-trivial Killing vector field on the Riemannian manifold $(M, g)$. Put $\alpha = \max_{x \in M} g(\xi, \xi)_x$, and $\mathcal{F} = \{x \in M; g(\xi, \xi)_x = \alpha\}$. Assume that the sectional curvature $K_g \leq 1$ and $\beta = \max_{x \in M} d_g(x, \mathcal{F}) < \pi/2$. Then for any point $p$ of $M$ we obtain

(i) $g(\xi, \xi)_p \geq \alpha \cos^2 \beta$,

(ii) $\| \text{grad } g(\xi, \xi) \|_p \leq 2\alpha \sin \beta$,  

where $\| \|$ denotes $g(\ , \ , )^{1/2}$.

In order to prove the theorem, we provide the following propositions.

**Proposition 2.2.** Let $f$ be a positive differentiable function defined in the interval $(s_1, s_2)$ such that $-\pi/2 < s_1 < 0 < s_2 < \pi/2$, $\max f = f(0)$ and $f''(s) \geq -f(s)$. Then $f(s) \geq f(0) \cos s$.

**Proof.** Let $\varepsilon$ be a positive number. Put $f_\varepsilon(s) = (f(0) + \varepsilon) \cos s/(f(s) + \varepsilon)$. Then we obtain

$$f'_\varepsilon(s) = -\frac{(f(0) + \varepsilon)(f(s) + \varepsilon) \sin s - (f(0) + \varepsilon)f''(s) \cos s}{(f(s) + \varepsilon)^2},$$

$$f''_\varepsilon(s) = -\frac{(f(0) + \varepsilon)(f(s) + \varepsilon + f''(s)) \cos s - 2f'_\varepsilon(s)f'(s)}{(f(s) + \varepsilon)^2}.$$
Since \( f'(0) = 0 \) and \( f''(0) < 0 \), \( f_\varepsilon \) is maximal at 0. If \( f'(s_0) = 0 \) for some \( s_0 \), then it follows that

\[
f''_\varepsilon(s_0) = -\frac{(f(0) + \varepsilon)(f(s_0) + \varepsilon + f''(s_0)) \cos s_0}{(f(s_0) + \varepsilon)^2} < 0.
\]

Hence every critical point of \( f_\varepsilon \) in the interval \((s_1, s_2)\) is maximal, which implies that \( f_\varepsilon \) has no critical points in \((s_1, s_2)\) except at 0. Therefore we obtain \( f_\varepsilon(s) \leq f(0) = 1 \), i.e., \((f(0) + \varepsilon) \cos s \leq f(s) + \varepsilon\). By \( \varepsilon \) passing to 0, the assertion is implied.

**Proposition 2.3.** Let \( f: \mathbb{R} \to \mathbb{R} \) be a positive differentiable function such that \( f''(s) \geq -f(s) \geq -\alpha \), where \( \alpha \) is a positive number. Then \( f'(s) \leq \sqrt{2\alpha(x - f(s))} \).

**Proof.** Since \( f''(s) \geq -\alpha \), we have for any \( t > s \)

\[
-\frac{1}{2}(t-s)^2\alpha \leq \int_s^t \left( f''(\sigma) d\sigma \right) d\tau = f(t) - f(s) - (t - s)f'(s).
\]

It implies

\[
f''(s) \leq \frac{1}{2}(t-s)\alpha + \frac{\alpha - f(s)}{t-s}.
\]

Putting \( t - s = \sqrt{2(x - f(s))/\alpha} \), we obtain

\[
f'(s) \leq \sqrt{2\alpha(x - f(s))}.
\]

**Proof of Theorem 2.1.** Let \( \gamma: \mathbb{R} \to M \) be a geodesic with \( \|\dot{\gamma}\| = 1 \). Since \( \zeta \) is a Killing vector field, it satisfies

\[
(\gamma, E) = (g(\nabla_\dot{\gamma} \zeta, \zeta)_{(t)} = g(P_{\dot{\gamma}} \zeta, P_{\dot{\gamma}} \zeta)_{(t)} - g(R(\dot{\gamma}, \zeta) \zeta, \dot{\gamma})_{(t)},
\]

where \( \dot{\gamma} \) denotes the velocity vector of \( \gamma \) and \( R \) is the curvature tensor of the Riemannian connection of \( g \).

(i) Put \( f(s) = \|\dot{\gamma}(t)\| \) and \( F = \{s; f(s) = 0\} \). We define \( E_{\gamma(s) = \zeta_{\gamma(s)}}/f(s) \) for \( s \notin F \). Then from (2.1) we obtain

\[
f(s)f''(s) = (f(s))^2g(P_{\dot{\gamma}} E, P_{\dot{\gamma}} E) - (f(s))^2g(R(\dot{\gamma}, E)E, \dot{\gamma}).
\]

Since the sectional curvature \( K_{gs} \leq 1 \), we obtain

\[
f''(s) \geq -f(s) \quad \text{for} \quad s \notin F.
\]
There is a point \( q \) in \( \mathcal{S} \) such that \( d_g(p, q) = d_g(p, \mathcal{S}) \). Let \( \gamma: [0, s_0] \to M \) be a minimal geodesic from \( q \) to \( p \), i.e., \( d_g(q, p) = s_0 < \pi/2 \). First we show that \( f(s) \neq 0 \) (\( s \in [0, s_0] \)). Suppose that \( F \cap [0, s_0] \neq \emptyset \). Put \( \inf F \cap [0, s_0] = s_1 \). Then \( s_1 > 0 \) and \( s_1 \in F \). From (2.2) and Proposition 2.2, we obtain \( f(s) \geq f(0) \cos s \) (\( s \in [0, s_1] \)). Hence it follows that

\[
f(s_1) \geq f(0) \cos s_1 \geq f(0) \cos s_0 > 0,
\]

which contradicts \( s_1 \in F \). Therefore we obtain \( F \cap [0, s_0] = \emptyset \). Hence (i) follows from Proposition 2.2.

(ii) Let \( \gamma(0) = p \). Put \( f(s) = \|\xi_{\gamma(s/\sqrt{2})}\|^2 \). Then from (i) we obtain

\[
f(s) \geq \alpha \cos^2 \beta > 0.
\]

On the other hand, from (2.1) and \( K_g \leq 1 \), we obtain

\[
f''(s) \geq -f(s).
\]

Hence it follows from Proposition 2.3 that

\[
\dot{g}(\xi, \xi)_{\gamma(0)} = \sqrt{2f'(s)} \\
\leq 2\sqrt{\alpha f(s)} \\
\leq 2\alpha \sin \beta.
\]

We note that

\[
g(\dot{\gamma}, (\nabla g)(\xi, \xi))_p = \dot{g}(\xi, \xi)_p.
\]

Since we can choose the direction of \( \gamma \) at \( \gamma(0) = p \) arbitrarily, our assertion is clear.

Q. E. D.

§3. The Sectional Curvature of Invariant Metrics on a Lie Group

Let \( G \) be a compact connected Lie group with a left-invariant Riemannian metric \( g \). We denote by \( g \) the tangent space to \( G \) at the identity \( e \). Let \( X \) be a tangent vector to \( G \) at \( e \). We denote by \( X^L \) the left-invariant vector field on \( G \) such that the value \( X^L_e \) of \( X^L \) at \( e \) is \( X \). We also define a right-invariant vector field \( X^R \) similarly. We denote by \( g^L \) the Lie algebra of left-invariant vector fields on \( G \).

A bi-linear form \( U(g): g^L \times g^L \to g^L \) is defined by

\[
2g(U(g)(X^L, Y^L), Z^L) = g(X^L, [Z^L, Y^L]) + g(Y^L, [Z^L, X^L])
\]

\( (X, Y, Z \in g) \). We note that the Riemannian connection \( \nabla \) of \( g \) has an
expression
\[ F_{X^L}Y^L = U(g)(X^L, Y^L) + \frac{1}{2} [X^L, Y^L] \quad (X, Y \in \mathfrak{g}) \]

and the curvature tensor \( R(g) \) of \( F \) satisfies
\[
g(R(g)(X, Y)Y, X) = \|U(g)(X^L, Y^L)e\|^2 - g(U(g)(X^L, X^L)e, U(g)(Y^L, Y^L)e) \\
- \frac{3}{4} \|[X^L, Y^L]e\|^2 - \frac{1}{2} g([X^L, [X^L, Y^L]]e, Y^L) \\
- \frac{1}{2} g([Y^L, [Y^L, X^L]]e, X^L). \]

**Lemma 3.1.** \( U(g)(X^L, Y^L)e = -\frac{1}{2}(\text{grad } g(X^R, Y^R))e \quad (X, Y \in \mathfrak{g}). \)

**Proof.** For a vector \( Z \in \mathfrak{g} \), we obtain
\[
g(U(g)(X^L, Y^L)e, Z) = \frac{1}{2} \{g(F_{X^L}Y^L, Z^L)e + g(F_{Y^L}X^L, Z^L)e\} \\
= -\frac{1}{2} \{g(Y^L, F_{X^L}Z^L)e + g(X^L, F_{Y^L}Z^L)e\} \\
= -\frac{1}{2} \{g(Y^R, F_{X^R}Z^R)e + g(X^R, F_{Y^R}Z^R)e\} \\
= -\frac{1}{2} Zg(Y^R, X^R). \]

Let \( a \) be an element of \( G \). We denote by \( R_a \) the right translation by \( a \).
Let \( dv \) be a bi-invariant volume element on \( G \) with \( \int_G dv = 1 \). We define a bi-invariant Riemannian metric \( \tilde{g} \) on \( G \) by
\[
\tilde{g} = \int_{a \in G} R_a^* g \, dv.
\]

Let \( H \) be a finite subgroup of \( G \) such that \( g \) is invariant by the right action of \( H \). Then there is a Riemannian metric on \( G/H \) such that the projection \((G, g) \to G/H \) is a Riemannian covering. We denote the metric also by \( g \). We also define a Riemannian manifold \((G/H, \tilde{g})\) in like manner. The diameter of \((G/H, g)\) (resp. \((G/H, \tilde{g})\)) is denoted by \( d_g(G/H) \) (resp. \( d_{\tilde{g}}(G/H) \)). \( K_g \) denotes the sectional curvature of \((G, g)\).

**Lemma 3.2.** Assume that \( K_g \leq 1 \) and \( d_g(G/H) < \pi/2 \). Then for any \( X (\in \mathfrak{g} \neq 0) \)
\[
\cos^2 d_g(G/H) \leq \frac{\tilde{g}(X, X)}{g(X, X)} \leq (\cos d_{\tilde{g}}(G/H))^{-2}.
\]
Proof. By definition we obtain
\[ \tilde{g}(X, X) = \int_{x \in G} g(X^R, X^R)_{a} \, dv. \]

Since \( X^R \) is a Killing vector field on \((G, g)\) and \( g(X^R, X^R) \) is constant on each right orbit \( aH \) of \( H \), it follows from Theorem 2.1 that
\[
g(X, X) \cos^2 d_g(G/H) = g(X^R, X^R)_{x} \cos^2 d_g(G/H) \\
\leq \max_{x \in G} g(X^R, X^R)_{x} \cos^2 d_g(G/H) \\
\leq g(X^R, X^R)_{a} \quad (\forall a \in G) \\
\leq \max_{x \in G} g(X^R, X^R)_{x} \\
\leq g(X^R, X^R)_{e}(\cos d_g(G/H))^{-2} \\
= g(X, X)(\cos d_g(G/H))^{-2}.
\]

Hence the assertion is clear. Q. E. D.

Since both metrics \( g \) and \( \tilde{g} \) on \( G \) are left-invariant, from Lemma 3.2 we obtain

**Lemma 3.3.** If \( K_g \leq 1 \) and \( d_g(G/H) < \pi/2 \), then
\[
\cos d_g(G/H) \leq d_g(G/H) \leq (\cos d_g(G/H))^{-1}.
\]

**Lemma 3.4.** Let \( a \) be an element of \( G \). Let \( X, Y \in g \) be linearly independent vectors such that \( g(X, X) = g(Y, Y) = 1 \). Assume that \( K_g \leq 1 \) and \( d_g(G/H) < \pi/2 \). Then
\[
\begin{align*}
(i) & \quad (R_{a}^{*} g)(U(R_{a}^{*} g)(X^L, X^L)_{e}, U(R_{a}^{*} g)(X^L, X^L)_{e})^{1/2} \\
& \leq (\cos d_g(G/H))^{-2} \sin d_g(G/H), \\
(ii) & \quad (R_{a}^{*} g)(R(R_{a}^{*} g)(X, Y), X) \leq (\cos d_g(G/H))^{-4}.
\end{align*}
\]

Proof. Since we have
\[
\tilde{g}(X, X) = \int_{x \in G} g(X^R, X^R)_{x} \, dv = 1,
\]
there is a point \( p \) in \( G \) such that \( g(X^R, X^R)_{p} = 1 \). Since \( X^R \) is a Killing vector field on \((G, g)\) such that \( g(X^R, X^R) \) is constant on each right orbit \( xH \) of \( H \) \( (x \in G) \), it follows from Theorem 2.1 that
\[
\max_{x \in G} g(X^R, X^R)_{x} \leq g(X^R, X^R)_{p}(\cos d_g(G/H))^{-2} = (\cos d_g(G/H))^{-2}.
\]

Similarly we obtain
\[
\max_{x \in G} g(Y^R, Y^R) \leq (\cos d_g(G/H))^2.
\]

(i) Since \((G/H, g)\) and \((G/aHa^{-1}, R^*_a g)\) are isometric, \(K_{R^*_a g} \leq 1\) and 
\(d_{R^*_a g}(G/aHa^{-1}) = d_g(G/H) < \pi/2\). Since \(X^R\) is a Killing vector field also on 
\((G, R^*_a g)\) such that \((R^*_a g)(X^R, X^R)\) is constant on each right orbit of \(aHa^{-1}\),
it follows from Lemma 3.1 and Theorem 2.1 that

\[
(R^*_a g)(U(R^*_a g)(X^L, X^L)_e, U(R^*_a g)(X^L, X^L)_e)^{1/2}
= \frac{1}{2}(R^*_a g)((\text{grad}_{R^*_a g} (R^*_a g)(X^R, X^R)), (\text{grad}_{R^*_a g} (R^*_a g)(X^R, X^R))_e)^{1/2}
\leq \max_{x \in G} (R^*_a g)(X^R, X^R)_x \sin d_g(G/H)
= \max_{x \in G} g(X^R, X^R)_x \sin d_g(G/H)
\implies (\cos d_g(G/H))^{-2} \sin d_g(G/H).
\]

(ii) Since \(g\) and \(R^*_a g\) are isometric, we obtain

\[
1 \geq K_{R^*_a g}(X, Y) = \frac{(R^*_a g)(R(R^*_a g)(X, Y), Y)}{(R^*_a g)(X, X)(R^*_a g)(Y, Y) - (R^*_a g)(X, Y)^2}.
\]

Hence

\[
(R^*_a g)(R(R^*_a g)(X, Y), Y) \leq (R^*_a g)(X, X)(R^*_a g)(Y, Y)
= g(X^R, X^R)_a g(Y^R, Y^R)_a
\leq (\cos d_g(G/H))^{-4}.
\]

**Theorem 3.5.** Assume that the sectional curvature \(K_g \leq 1\) and the diameter \(d_g(G/H) < \pi/2\). Then the sectional curvature \(K_g\) of \(\check{g}\) satisfies

\[
K_g \leq (\cos d_g(G/H))^{-4}(1 + \sin^2 d_g(G/H)).
\]

**Proof.** Since \(\check{g}\) is bi-invariant, \(U(\check{g}) \equiv 0\) (cf. Lemma 3.1). We take vectors 
\(X, Y (\in g)\) such that \(\check{g}(X, X) = \check{g}(Y, Y) = 1\) and \(\check{g}(X, Y) = 0\). Then from 
Lemma 3.4 we obtain

\[
K_g(X, Y) = \frac{\check{g}(R(\check{g})(X, Y), Y)}{\check{g}(X, X) \check{g}(Y, Y) - \check{g}(X, Y)^2}
= - \frac{3}{4} \check{g}([X^L, Y^L]_e, [X^L, Y^L]_e)
- \frac{1}{2} \check{g}([X^L, [X^L, Y^L]]_e, Y^L) - \frac{1}{2} \check{g}([Y^L, [Y^L, X^L]]_e, X^L)
= \int_{a \in G} \left\{ - \frac{3}{4} (R^*_a g)([X^L, Y^L]_e, [X^L, Y^L]_e)
- \frac{1}{2} (R^*_a g)([X^L, [X^L, Y^L]]_e, Y^L) \right\}
\]
§ 4. Bi-invariant Metrics and Finite Subgroups of a Lie Group

Let $G$ be a compact connected Lie group as in Section 3. Let $\exp$ denote the usual exponential mapping from $\mathfrak{g}$ to $G$, i.e., for a tangent vector $X \in \mathfrak{g}$, $\gamma(t) = \exp tX$ ($t \in \mathbb{R}$) is a one-parameter subgroup of $G$ such that $\gamma(0) = X$. The usual bracket operation is defined by

$$[X, Y] = \frac{d}{dt} (\text{Ad}(\exp tX)Y)_{t=0}$$

$X, Y \in \mathfrak{g}$. Let $\tilde{g}$ be a bi-invariant Riemannian metric on $G$ with sectional curvature $K_g \leq k$ ($k > 0$). We note the mapping $\exp : \mathfrak{g} \to G$ coincides with the usual exponential mapping of the Riemannian manifold $(G, \tilde{g})$ because the metric $\tilde{g}$ is bi-invariant. For non-zero vectors $X$ and $Y$ of $\mathfrak{g}$ we denote by $\varkappa(X, Y)$ the angle which $X$ and $Y$ make. $\| \|$ denotes $\tilde{g}(\ , \ )^{1/2}$.

**Lemma 4.1.** Let $X$ and $Y$ be non-zero vectors of $\mathfrak{g}$. We have

$$\varkappa(\text{Ad}(\exp Y)X, X) \leq \frac{\|[Y, X]\|}{\|X\|}.$$

**Proof.** Since the metric $\tilde{g}$ is bi-invariant and $\frac{d}{dt} \text{Ad}(\exp tY)X_{|t=0} = [Y, X]$, we see $\|\text{Ad}(\exp tY)X\| = \|X\|$ and $\left\|\frac{d}{dt} \text{Ad}(\exp tY)X\right\| = \|[Y, X]\|$. Hence it follows that

$$\varkappa(\text{Ad}(\exp Y)X, X) \leq \frac{1}{\|X\|} \int_0^1 \left\|\frac{d}{dt} \text{Ad}(\exp tY)X\right\| dt = \frac{\|[Y, X]\|}{\|X\|}.$$  

Q. E. D.

**Lemma 4.2.** Let $X$ and $Y$ be non-zero vectors of $\mathfrak{g}$ such that $\exp tX$ ($0 \leq t \leq 1$) and $\exp tY$ ($0 \leq t \leq 1$) are minimal geodesics. Suppose that $\|X\| = d_g(e, \exp X) < \pi/\sqrt{k}$ and $\|[X, Y]\|/\|X\| < \pi/3$. Then
Remark. The similar estimate is found in [1].

Proof. From Lemma 4.1, we obtain \( \zeta(\text{Ad}(\exp Y)X, X) < \pi/3 \). It implies \( \|X\| > \|X - \text{Ad}(\exp Y)X\| \). Let us define a curve \( \gamma: [0, 1] \rightarrow g \) by \( \gamma(t) = tX + (1 - t) \text{Ad}(\exp Y)X \). We have the sectional curvature \( K_g \geq 0 \) since the metric \( \tilde{g} \) is bi-invariant. From Rauch’s comparison theorem we easily see that

\[
d_g(e, \exp X) = \|X\|
\geq \|X - \text{Ad}(\exp Y)X\|
= \text{length}(\gamma)
\geq \text{length}(\exp \circ \gamma)
\geq d_g(\exp X, \exp Y \exp X \exp Y^{-1}).
\]

Q. E. D.

Lemma 4.3. Let \( c(\neq e) \) be an element of the center of \( G \). If \( G \) is semi-simple, then \( d_g(e, c) \geq \pi/\sqrt{k} \).

Proof. Let \( \gamma: [0, 1] \rightarrow G \) be a minimal geodesic from \( e \) to \( c \). Then \( \gamma \) is expressed as \( \gamma(t) = \exp tY \) for some \( X \in g \). Since \( G \) is semi-simple, the orbit \( \text{Ad}(G)X (\subseteq g) \) of \( X \) by the adjoint action of \( G \) is at least of one dimension. Since \( \exp \text{Ad}(G)X = c \), \( c \) is conjugate to \( e \) along \( \gamma \). Hence the assertion follows from the Morse-Shoenberg theorem. Q. E. D.

Let \( x \) be a point of \( G \). \( C(x) \) denotes the cut locus of \( x \) with respect to the metric \( \tilde{g} \).

Lemma 4.4. If \( G \) is semi-simple, then \( d_g(e, C(e)) \geq \pi/2\sqrt{k} \).

Proof. Since the metric \( \tilde{g} \) is bi-invariant, the isotropy subgroup \( I_e(G, \tilde{g}) \) at \( e \) contains the inner automorphisms \( \text{Ad}(G) \) of \( G \). Since the fixed points \( F(\text{Ad}(G)) \) is the center of \( G \) and since the center consists of finite points, the assertion follows from Lemma 1.1. Q. E. D.

Let \( H \) be a finite subgroup of \( G \). Let \( h \) be an element of \( H \backslash \{e\} \) which is the closest to \( e \). Put \( Z(h) = \{x \in G; xh = hx\} \).

Lemma 4.5. If \( G \) is semi-simple and \( d_g(e, h) < \pi/2\sqrt{k} \), then \( Z(h) \subseteq G \) and \( \max_{x \in G} d_g(x, Z(h)) \geq \pi/2\sqrt{k} \).

Proof. From Lemma 4.4, we see that the minimal geodesic from \( e \) to \( h \) is unique. We denote the geodesic by \( \exp tX (0 \leq t \leq 1) \), where \( X \in g \). Since for any \( x \in Z(h) \) the inner automorphism by \( x \) fixes the endpoints of the geodesic and
since the minimal geodesic connecting $e$ and $h$ is unique, we obtain $x(\exp tX)x^{-1} = \exp tX$ ($0 \leq t \leq 1$), which implies that

$$Z(h) = \{x \in G; (\exp tX)x(\exp tX)^{-1} = x \ (t \in \mathbb{R})\} = \mathcal{F}(\{L_{\exp tX}(R_{\exp tX})^{-1}; t \in \mathbb{R}\}).$$

Since $G$ is semi-simple, the assertion follows from Lemma 1.1. Q. E. D.

**Lemma 4.6.** Suppose that $H \subseteq Z(h)$. Then for any $a \in H \setminus Z(h)$ $d_g(e, a) \geq \pi/6\sqrt{k}$.

**Proof.** Let $\gamma, \delta: [0, 1] \to G$ be minimal geodesics from $e$ to $h$ and $a$ respectively. Then there are $X$ and $Y$ in $g$ such that $\gamma(t) = \exp tX$ and $\delta(t) = \exp tY$. If $\|[X, Y]\|/\|X\| < \pi/3$, Lemma 4.2 implies that

$$d_g(e, h) > d_g(h, aha^{-1}) = d_g(e, aha^{-1}h^{-1}),$$

which contradicts the choice of $h$. Hence we obtain $\|[X, Y]\|/\|X\| \geq \pi/3$

On the other hand, we have

$$k \geq K_g(X, Y) = \frac{1}{4} \frac{[X, Y]^2}{\|X\|^2\|Y\|^2 - \bar{g}(X, Y)^2}.$$

Hence it follows that

$$k \geq \frac{1}{4} \frac{[X, Y]^2}{\|X\|^2\|Y\|^2} \geq \frac{\pi^2}{36\|Y\|^2},$$

which implies

$$d_g(e, a) = \|Y\| \geq \frac{\pi}{6\sqrt{k}}.$$

Q. E. D.

**Theorem 4.7.** Assume that the group $G$ is not abelian. Then the following (i), (ii) and (iii) hold.

(i) $d_g(G) \geq \frac{\pi}{2\sqrt{k}}$.

(ii) If $G$ is simply connected, then $d_g(G) \geq \frac{\pi}{\sqrt{k}}$.

(iii) $\max_{x \in G} d_g(x, H) = d_g(G/H) \geq \frac{\pi}{12\sqrt{k}}$.

**Proof.** Let $Z$ be the identity component of the center of $G$. We put $G' = G/Z$. Then $G'$ is semi-simple and if $G$ is simply connected, so is $G'$. The metric $\bar{g}$ on $G$ induces a Riemannian metric $\bar{g}'$ on $G'$ so that the projection
\(\pi: G \to G'\) is a Riemannian submersion. Let \(\mathfrak{z}\) denote the tangent space to \(Z\) at \(e\). We take orthonormal vectors \(X\) and \(Y\) in \(g\) such that \(X, Y \perp \mathfrak{z}\). Then from O'Neil's theorem for Riemannian submersion we obtain

\[
K_g(\pi_*X, \pi_*Y) = K_g(X, Y) + \frac{3}{4} \| [X, Y] \|_2^2,
\]

where \([X, Y]_\mathfrak{z}\) denotes the orthogonal projection of \([X, Y]\) to \(\mathfrak{z}\). Since \([X, Y] \perp \mathfrak{z}\), we obtain \(K_g \leq k\). On the other hand it is clear that, for any point \(x\) and \(y\) of \(G\), \(d_g(x, y) \geq d_g(\pi(x), \pi(y))\). Hence we have only to prove the theorem with the assumption that \(G\) is semi-simple. So we suppose \(G\) is semi-simple.

(i) From Lemma 4.4, we obtain

\[
d_g(G) \geq d_g(e, C(e)) \geq \frac{\pi}{2\sqrt{k}}.
\]

(ii) Corollary 5.12 in [2] states that the cut locus and first conjugate locus coincide. The assertion follows easily from the Morse-Shoenberg theorem.

(iii) If \(H = \{e\}\), the inequality follows from (i). Hence we assume that \(H \nsubseteq \{e\}\). Let \(h\) be an element of \(H \setminus \{e\}\) which is the closest to \(e\). Let \(m\) be the middle point of a minimal geodesic from \(e\) to \(h\). Then it is easy to see that \(d_g(m, H) = d_g(m, e) = \frac{1}{2} d_g(e, h)\). If \(d_g(e, h) \geq \pi/2\sqrt{k}\), then \(\max_{x \in \mathcal{G}} d_g(x, H) \geq d_g(m, H) \geq \pi/4\sqrt{k}\). Hence we may assume that \(d_g(e, h) < \pi/2\sqrt{k}\). If \(H \subset Z(h)\), the inequality follows from Lemma 4.5. Therefore we suppose that \(H \nsubseteq Z(h)\). Let \(a\) be an element of \(H \setminus Z(h)\) which is the closest to \(e\). Let \(m'\) be the middle point of a minimal geodesic from \(e\) to \(a\). Then we have \(d_g(m', H) = d_g(m', e)\). (In fact, suppose that there is an element \(b \in H\) with \(d_g(m', b) < d_g(m', e)\). Then it follows from \(d_g(e, b) \leq d_g(e, m') + d_g(m', b) < d_g(e, a)\) that \(b \in Z(h)\). Hence we obtain \(a \in H \setminus Z(h)\) and \(d_g(e, ab^{-1}) = d_g(b, a) \leq d_g(b, m') + d_g(m', a) < d_g(e, a)\), which means that \(a\) is not the closest to \(e\) of \(H \setminus Z(h)\).) Therefore the inequality follows from Lemma 4.6.

Q.E.D.

§ 5. Diameter Estimate

**Theorem 5.1.** Let \((M, g)\) be a compact homogeneous Riemannian manifold with the sectional curvature \(K_g \leq 1\) and \(K_g \neq 0\). Then the diameter \(d_g(M)\) of \((M, g)\) is not less than a positive constant \(d > 0.23\).

**Proof.** Let \(p\) be a point of \(M\). If \(\dim I_g(M, g) \geq 1\), then from Lemma 1.1 we obtain
\[ d_g(M) \geq \max_{x \in M} d_g(x, F(I_p^0(M, g))) \geq \frac{\pi}{2}, \]

where \( I_p^0(M, g) \) denotes the identity component of \( I_p(M, g) \) (see also [3]).

Hence we assume that \( \dim I_p(M, g) = 0 \). We denote by \( G \) the identity component of \( I(M, g) \) and put \( H = G \cap I_p(M, g) \). Since the projection \( G \to G/H = M \) is a covering, \( g \) induces a left-invariant Riemannian metric on \( G \) such that the projection is a Riemannian covering. We denote the metric also by \( g \). It is invariant by the inner automorphism by \( H \). We define a Riemannian metric \( \tilde{g} \) as in Section 3. Suppose that \( d_g(G/H) < \frac{\pi}{2} \). Then from Theorem 3.5 we obtain

\[ K_{\tilde{g}} \leq (\cos d_g(G/H))^{-\frac{4}{3}}(1 + \sin^2 d_g(G/H)) \]

Since \( K_{\tilde{g}} \neq 0 \), it is easily seen that \( G \) is not abelian. Hence it follows from Lemma 3.3 and Theorem 4.7 that

\[ \frac{\pi}{12\sqrt{\cos d_g(G/H)^{-\frac{4}{3}}(1 + \sin^2 d_g(G/H))}} \leq d_g(G/H) \leq d_g(G/H)(\cos d_g(G/H))^{-1}. \]

We put

\[ d = \inf \left\{ t \geq 0; \, \frac{\pi}{12} \leq t(\cos t)^{-3}(1 + \sin^2 t)^{1/2} \right\}. \]

Then \( \pi/2 > d > 0.23 \) and \( d_g(M) \geq d \). Q. E. D.

**Theorem 5.2.** Let \((M, g)\) be a simply connected compact homogeneous Riemannian manifold with sectional curvature \( K_g \leq 1 \) and \( K_g \neq 0 \). Then the diameter \( d_g(M) \) of \((M, g)\) is not less than a positive constant \( d_0 \) (> 0.81).

**Proof.** As in the proof of Theorem 5.1, we may assume that \( \dim I_p(M, g) = 0 \). We define \( G \) and \( H \) as in the proof of Theorem 5.1. Since \( M \) is simply connected, \( H = \{e\} \) and \( G = M \). Hence it follows from Lemma 3.3 and (i) of Theorem 4.7 that we can replace (5.1) by

\[ \frac{\pi}{\sqrt{\cos d_g(G)^{-\frac{4}{3}}(1 + \sin^2 d_g(G))}} \leq d_g(G)(\cos d_g(G))^{-1}. \]

We put

\[ d_0 = \inf \left\{ t \geq 0; \, \pi \leq t(\cos t)^{-3}(1 + \sin^2 t)^{1/2} \right\}. \]

Then \( \pi/2 > d_0 > 0.81 \) and \( d_g(M) = d_g(G) \geq d_0 \). Q. E. D.

**Theorem 5.3.** Let \( G \) be a compact connected Lie group with a left-
invariant metric \( g \). Assume that the sectional curvature \( K_g \leq 1 \) and \( G \) is not abelian. Then the diameter \( d_g(G) \) of \((G, g)\) is not less than a positive constant \( d_1 \) (> 0.66).

Proof. We define a metric \( \tilde{g} \) as in Section 3. We may assume that \( d_g(G) < \pi/2 \). From Theorem 3.5, we obtain

\[
K_g \leq (\cos d_g(G))^{-4}(1 + \sin^2 d_g(G)).
\]

Hence it follows from Lemma 3.3 and Theorem 4.7 that

\[
(5.3) \quad \frac{\pi}{2} \sqrt{(\cos d_g(G))^{-4}(1 + \sin^2 d_g(G))} \leq d_g(G) \leq d_g(G)(\cos d_g(G))^{-1}.
\]

We put

\[
d_1 = \inf \left\{ t > 0; \frac{\pi}{2} \leq t(\cos t)^{-3}(1 + \sin^2 t)^{1/2} \right\}.
\]

Then

\[
\frac{\pi}{2} > d_1 > 0.66 \quad \text{and} \quad d_g(G) \geq d_1. \quad \text{Q.E.D.}
\]

References


