A short proof of Morley’s theorem

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We present a proof of the following:

**Morley’s theorem** (1899) In any triangle, the three points of intersection of the adjacent angle trisectors form an equilateral triangle.

**Proof.** Let \( \alpha, \beta, \gamma \) be arbitrary positive angles with \( \alpha + \beta + \gamma = 60^\circ \). For any angle \( \eta \) we put \( \eta' := \eta + 60^\circ \).

Let \( \triangle DEF \) be an equilateral triangle, and \( A \) [resp. \( B, C \)] be the point lying opposite to \( D \) [resp. \( E, F \)] with respect to \( EF \) [resp. \( FD, DE \)] and satisfying \( \angle AFE = \beta', \angle AEF = \gamma' \) [resp. \( \angle BDF = \gamma', \angle BFD = \alpha'; \angle CED = \alpha', \angle CDE = \beta' \)]. Then \( \angle EAF = 180^\circ - (\beta' + \gamma') = \alpha \), and similarly \( \angle FBD = \beta \), \( \angle DCE = \gamma \). By symmetry it is enough to show that \( \angle BAF = \alpha \) and \( \angle ABF = \beta \) as well.

The perpendiculars from \( F \) to \( AE \) and \( BD \) have the same length \( s \). If the perpendicular from \( F \) to \( AB \) has length \( h < s \), then \( \angle BAF < \alpha \) and \( \angle ABF < \beta \). If, on the other hand, \( h > s \), then \( \angle BAF > \alpha \) and \( \angle ABF > \beta \). Since

\[
\angle BAF + \angle ABF = \alpha' + \beta' + 60^\circ - 180^\circ = \alpha + \beta,
\]

we see that necessarily \( h = s \) and \( \angle BAF = \alpha, \angle ABF = \beta \). \( \square \)

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