Appendix to V. Mathai and J. Rosenberg’s paper
“A noncommutative sigma-model”

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This short note is an appendix to [6].

Let $\theta \in \mathbb{R}$. Denote by $A_\theta$ the rotation C*-algebra generated by unitaries $U$ and $V$ subject to $UV = e^{2\pi i \theta} VU$, and by $A_\theta^\infty$ its canonical smooth subalgebra. Denote by $\text{Tr}$ the canonical faithful tracial state on $A_\theta$ determined by $\text{Tr}(U^m V^n) = \delta_{m,0} \delta_{n,0}$ for all $m, n \in \mathbb{Z}$. Denote by $\delta_1$ and $\delta_2$ the unbounded closed $*$-derivations of $A_\theta$ defined on some dense subalgebras of $A_\theta$ and determined by $\delta_1(U) = 2\pi i U$, $\delta_1(V) = 0$, and $\delta_2(U) = 0$, $\delta_2(V) = 2\pi i V$. The energy [9], $E(u)$, of a unitary $u$ in $A_\theta$ is defined as

$$E(u) = \frac{1}{2} \text{Tr}(\delta_1(u)^* \delta_1(u) + \delta_2(u)^* \delta_2(u))$$

(1)

when $u$ belongs to the domains of $\delta_1$ and $\delta_2$, and $\infty$ otherwise.

Rosenberg has the following conjecture [9], Conjecture 5.4, p. 108.

Conjecture 1. For any $m, n \in \mathbb{Z}$, in the connected component of $U^m V^n$ in the unitary group of $A_\theta^\infty$, the functional $E$ takes its minimal value exactly at the scalar multiples of $U^m V^n$.

For a $*$-endomorphism $\varphi$ of $A_\theta^\infty$, its energy [6], $\mathcal{L}(\varphi)$, is defined as $2E(\varphi(U)) + 2E(\varphi(V))$. Mathai and Rosenberg’s Conjecture 3.1 in [6] about the minimal value of $\mathcal{L}(\varphi)$ follows directly from Conjecture 1.

Denote by $H$ the Hilbert space associated to the GNS representation of $A_\theta$ for $\text{Tr}$, and denote by $\| \cdot \|_2$ its norm. We shall identify $A_\theta$ as a subspace of $H$ as usual. Then (1) can be rewritten as

$$E(u) = \frac{1}{2} (\|\delta_1(u)\|_2^2 + \|\delta_2(u)\|_2^2).$$

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Now we prove Conjecture 1, and hence also prove Conjecture 3.1 of [6].

**Theorem 2.** Let \( \theta \in \mathbb{R} \) and \( m, n \in \mathbb{Z} \). Let \( u \in A_\theta \) be a unitary whose class in \( K_1(A_\theta) \) is the same as that of \( U^mV^n \). Then \( E(u) \geq E(U^mV^n) \), and “=” holds if and only if \( u \) is a scalar multiple of \( U^mV^n \).

**Proof.** We may assume that \( u \) belongs to the domains of \( \delta_1 \) and \( \delta_2 \). Set \( a_j = u^* \delta_j(u) \) for \( j = 1, 2 \). For any closed \(*\)-derivation \( \delta \) defined on a dense subset of a unital \( C^* \)-algebra \( A \) and any tracial state \( \tau \) of \( A \) vanishing on the range of \( \delta \), if unitaries \( v_1 \) and \( v_2 \) in the domain of \( \delta \) have the same class in \( K_1(A) \), then \( \tau(v_1^* \delta(v_1)) = \tau(v_2^* \delta(v_2)) \) [7], p. 281. Thus

\[
\text{Tr}(a_j) = \text{Tr}((U^mV^n)^* \delta_j(U^mV^n)) = \begin{cases} 2\pi im & \text{if } j = 1, \\ 2\pi in & \text{if } j = 2. \end{cases}
\]

We have

\[
\|\delta_j(u)\|_2^2 = \|a_j\|_2^2 = \|\text{Tr}(a_j)\|_2^2 + \|a_j - \text{Tr}(a_j)\|_2^2 \geq \|\text{Tr}(a_j)\|_2^2 = \begin{cases} 4\pi^2m^2 & \text{if } j = 1, \\ 4\pi^2n^2 & \text{if } j = 2, \end{cases}
\]

and “=” holds if and only if \( a_j = \text{Tr}(a_j) \). It follows that \( E(u) \geq 2\pi^2(m^2 + n^2) \), and “=” holds if and only if \( \delta_1(u) = 2\pi imu \) and \( \delta_2(u) = 2\pi inu \). Now the theorem follows from the fact that the elements \( a \) in \( A_\theta \) satisfying \( \delta_1(a) = 2\pi ima \) and \( \delta_2(a) = 2\pi ina \) are exactly the scalar multiples of \( U^mV^n \).

When \( \theta \in \mathbb{R} \) is irrational, the \( C^* \)-algebra \( A_\theta \) is simple [10], Theorem 3.7, has real rank zero [1], Theorem 1.5, and is an \( A\mathbb{T} \)-algebra [5], Theorem 4. It is a result of Elliott that for any pair of \( A\mathbb{T} \)-algebras with real rank zero, every homomorphism between their graded \( K \)-groups preserving the graded dimension range is induced by a \(*\)-homomorphism between them [4], Theorem 7.3. The graded dimension range of a unital simple \( A\mathbb{T} \)-algebra \( A \) is the subset \( \{(g_0, g_1) \in K_0(A) \oplus K_1(A) : 0 \preceq g_0 \leq [1_A]_0 \} \cup \{(0, 0)\} \) of the graded \( K \)-group \( K_0(A) \oplus K_1(A) \) [8], p. 51. It follows that, when \( \theta \) is irrational, for any group endomorphism \( \psi \) of \( K_1(A_\theta) \), there is a unital \(*\)-endomorphism \( \varphi \) of \( A_\theta \) inducing \( \psi \) on \( K_1(A_\theta) \). It is an open question when one can choose \( \varphi \) to be smooth in the sense of preserving \( A_\theta^\infty \), though it was shown in [2], [3] that if \( \theta \) is irrational and \( \varphi \) restricts to a \(*\)-automorphism of \( A_\theta^\infty \), then \( \psi \) must be an automorphism of the rank-two free abelian group \( \mathbb{Q} \) with determinant 1. When \( \psi \) is the zero endomorphism, from Theorem 2 one might guess that \( \mathcal{L}(\varphi) \) could be arbitrarily small. It is somehow surprising, as we show now, that in fact there is a
common positive lower bound for $\mathcal{L}(\varphi)$ for all $0 < \theta < 1$. This answers a question Rosenberg raised at the Noncommutative Geometry workshop at Oberwolfach in September 2009.

**Theorem 3.** Suppose that $0 < \theta < 1$. For any unital $\ast$-endomorphism $\varphi$ of $A_\theta$, one has $\mathcal{L}(\varphi) \geq 4(3 - \sqrt{5})\pi^2$.

Theorem 3 is a direct consequence of the following lemma.

**Lemma 4.** Let $\theta \in \mathbb{R}$ and let $u, v$ be unitaries in $A_\theta$ with $uv = \lambda vu$ for some $\lambda \in \mathbb{C} \setminus \{1\}$. Then $E(u) + E(v) \geq 2(3 - \sqrt{5})\pi^2$.

**Proof.** We have

$$\text{Tr}(uv) = \text{Tr}(\lambda vu) = \lambda \text{Tr}(uv),$$

and hence $\text{Tr}(uv) = 0$. Thus

$$-\text{Tr}(u) \text{Tr}(v) = \text{Tr}(uv - \text{Tr}(u) \text{Tr}(v)) = \text{Tr}((u - \text{Tr}(u))v + \text{Tr}(u)(v - \text{Tr}(v))) = \text{Tr}((u - \text{Tr}(u))v).$$

We may assume that both $u$ and $v$ belong to the domains of $\delta_1$ and $\delta_2$. For any $m, n \in \mathbb{Z}$, denote by $a_{m,n}$ the Fourier coefficient $\langle u, U^m V^n \rangle$ of $u$. Then $a_{0,0} = \text{Tr}(u)$ and

$$(2\pi)^2 \|u - \text{Tr}(u)\|_2^2 = \sum_{m,n \in \mathbb{Z}, m^2 + n^2 > 0} |2\pi a_{m,n}|^2 \leq \sum_{m,n \in \mathbb{Z}, m^2 + n^2 > 0} |2\pi a_{m,n}|^2 (m^2 + n^2) = \|\delta_1(u)\|_2^2 + \|\delta_2(u)\|_2^2 = 2E(u).$$

Thus

$$|\text{Tr}(u)|^2 = \|\text{Tr}(u)\|_2^2 = \|u\|_2^2 - \|u - \text{Tr}(u)\|_2^2 \geq 1 - \frac{1}{2\pi^2} E(u)$$

and

$$|\text{Tr}((u - \text{Tr}(u))v)| \leq \|(u - \text{Tr}(u))v\|_2 = \|u - \text{Tr}(u)\|_2 \leq \left(\frac{1}{2\pi^2} E(u)\right)^{1/2}. $$

Similarly, $|\text{Tr}(v)|^2 \geq 1 - \frac{1}{2\pi^2} E(v)$.

Write $\frac{1}{2\pi^2} E(u)$ and $\frac{1}{2\pi^2} E(v)$ as $t$ and $s$, respectively. We just need to show that $t + s \geq 3 - \sqrt{5}$. If $t \geq 1$ or $s \geq 1$, then this is trivial. Thus we may assume that $1 - t, 1 - s > 0$. Then

$$(1 - t)(1 - s) \leq |\text{Tr}(u) \text{Tr}(v)|^2 \leq t.$$
Equivalently, \( t(1 - s) \geq 1 - (t + s) \). Without loss of generality, we may assume \( s \geq t \). Write \( t + s \) as \( w \). Then

\[
t(1 - w/2) \geq t(1 - s) \geq 1 - (t + s) = 1 - w,
\]

and hence

\[
w = t + s \geq \frac{1 - w}{1 - w/2} + \frac{w}{2}.
\]

It follows that \( w^2 - 6w + 4 \leq 0 \). Thus \( w \geq 3 - \sqrt{5} \). \qed

References


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