Abbas Bahri

A remark on the bifurcation diagrams of superlinear elliptic equations

Dedicated to Antonio Ambrosetti on his sixtieth birthday

Received October 27, 2005

Abstract. We prove a formula relating the index of a solution and the rotation number of a certain complex vector along bifurcation diagrams.

We consider a deformation $\Omega_t$ of domains via uniform dilation. For the sake of simplicity, we will consider only the case of starshaped domains.

On $\Omega_t$, we consider the partial differential equation

$$
\begin{align*}
-\Delta u &= g(u), \\
|u|_{\partial \Omega_t} &= 0.
\end{align*}
$$

(1)

where $g(u)$ is “superlinear” and “subcritical”, i.e., $g: \mathbb{R} \to \mathbb{R}$ and

$$
\lim_{|s| \to \infty} \frac{g(s)}{s} = +\infty, \quad |g(s)| \leq C(1 + |s|^q) \quad \text{with} \quad q < \frac{n + 2}{n - 2} \quad (n \geq 3).
$$

We assume that $g$ is $C^\infty$ for the sake of simplicity.

For a generic shape of domains $\Omega_t$, we may assume that the solution set $(t, u_t), t \in (0, \infty)$, is a one-dimensional manifold having possibly infinitely many connected components.

A natural question is: Does every connected component span over $t \in (0, \infty)$? Are there infinitely many components in the solution set spanning over $(0, \infty)$?

Both questions are reformulations of the following conjecture:

Conjecture. For any given $t_0$, (1) has infinitely many solutions.

A. Bahri: Department of Mathematics, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, USA; e-mail: abahri@math.rutgers.edu
A related problem is the following. Let $0 < a < b$ be given. Are the connected components for $t \in [a, b]$ compact? i.e., assuming that we are considering a branch of solutions $u_t, t \in [a, b],$ of

$$\begin{cases} -\Delta u_t = g(u_t), \\ u_t|_{\partial \Omega_1} = 0, \end{cases}$$

is the Morse index of $u_t$ bounded on a given connected component for $t \in [a, b]$?

Indeed, by the results of X. F. Yang \[2\] and Harrabi–Rebhi–Selmi \[1\], a bound on the Morse index of $u_t$ is equivalent to a bound on $\|u_t\|_\infty$ for $t \in [a, b]$ under the additional assumptions:

(i) $g(u) \sim \frac{c_+(u^+)^{p_+} - c_-(u^-)^{p_-}}{|u| \to \infty}$, $1 < p_+, p_- < \frac{(n+2)}{(n-2)}$,

(ii) $g'(u) \sim \frac{p_+ c_+(u^+)^{p_+ - 1} - p_- c_-(u^-)^{p_- - 1}}{|u| \to \infty}$.

Let us consider such a connected component:

![Connected Component Diagram](image)

For values of $t$ such as $t = t_0$, (1) degenerates at $u_{t_0}$ and the Morse index of $u_t$ changes.

Picking up two points $(t_1, u_{t_1})$ and $(t_2, u_{t_2})$ on $C$, we would like to relate the Morse index of $u_{t_2}$ to the Morse index of $u_{t_1}$.

We introduce the vector ($C$ is parametrized by $s$):

$$V(s) = \int_{\Omega_{i(s)}} |\nabla u_{t(s)}|^2 + i \int_{\Omega_{i(s)}} G(u_{t(s)}) \quad \text{with} \quad G(u) = \int_0^u g(x)dx.$$  

We claim that:

**Theorem 1.** $\dot{V}(s)$ is never zero on $C$ generically on $\Omega_1$ and

$$\text{Morse index } (u_{t_2}) - \text{Morse index } (u_{t_1}) = \text{algebraic number of times } \dot{V}(s) \text{ crosses the y-axis.}$$

**Proof.** Let us differentiate \[1\] with respect to $s$. We derive

$$\begin{cases} -\Delta h = g'(u)h, \\ h + tr(\sigma) \frac{\partial u}{\partial r}(\sigma, tr(\sigma))|_{\partial \Omega_1} = 0. \end{cases} \quad (*)$$

with $\partial \Omega_1$ parametrized by $(\sigma, r(\sigma)), \sigma \in S^{n-1}$. 
Indeed, the Dirichlet boundary condition reads \( u_t(\sigma, tr(\sigma)) = 0 \) and we derive our boundary condition after differentiation.

The Morse index changes only when \( \dot{t} \) vanishes, so that we have

\[
\begin{aligned}
-\Delta h &= g'(u)h, \\
h|_{\partial \Omega} &= 0.
\end{aligned}
\]

Observe that, with \( I_t(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \int_{\Omega} G(u) \), we find

\[
\frac{d}{ds} I_t(s)(u^s) = \int_{\Omega} \nabla u^s \nabla h - \int_{\Omega} g(u^s)h + \frac{d}{ds} \left( \int_{\Omega} \frac{|\nabla u^s|^2}{2} d\sigma_x \right) \cos \varphi(\sigma) r(\sigma) dy)
\]

\[
= \int_{\partial \Omega} \frac{\partial u^s}{\partial \nu} h + \frac{i}{2} \int_{\partial \Omega} |\nabla u^s|^2 r(\sigma) \cos \varphi(\sigma) d\sigma_i
\]

\[
= -i \int_{\partial \Omega} \frac{\partial u^s}{\partial \nu} \frac{\partial u^s}{\partial r} r d\sigma_i + \frac{i}{2} \int_{\partial \Omega} \left| \frac{\partial u^s}{\partial \nu} \right|^2 \cos \varphi(\sigma) r(\sigma) d\sigma_i
\]

\[
= -\frac{i}{2} \int_{\partial \Omega} |\nabla u^s|^2 r(\sigma) r(\sigma) \cos \varphi(\sigma) d\sigma_i.
\]

On the other hand, if \( \dot{t}(s_0) = 0 \), we compare \( I_t(u_+) \) and \( I_t(u_-) \), where \( u_+ \) and \( u_- \) are solutions for \( s_0 + k, k > 0 \) small, and \( s_0 - k_1, k_1 > 0 \) small, with

\[
t(s_0 + k) = t(s_0 - k_1).
\]

This will tell us how the Morse index changes as \( s \) increases because whichever of \( I_t(s_0 + k)(u(s_0 + k)) \) or \( I_t(s_0 - k_1)(u(s_0 - k_1)) \) is larger will correspond to the larger index:
when an elimination of a pair of critical points occurs in a variational problem, the highest index critical point is above the lowest one.

We renormalize $\Omega_{(s)}$ near $s = s_0$ so that we will be considering only one $\Omega_{(s_0)} = \Omega_0$ with a functional

$$\tilde{I}_{(s)} = t(s)^{n-2}I_{(s)} \left( \frac{x}{t(s)} \right) \quad (t(s_0) = 1 \text{ for example}).$$

Our critical points $u(s_0 + k)$ and $u(s_0 - k_1)$ change into $\tilde{u}(s_0 + k)$ and $\tilde{u}(s_0 - k_1)$. We know that $\tilde{t}(s_0) = 0$.

The branch $(t(s), \tilde{u}(s))$ is differentiable. With $\tilde{u}(s_0) = h$, the direction of degeneracy, we have

$$\begin{align*}
\tilde{u}(s_0 + k) &= u(s_0) + kh + O(k^2), \\
\tilde{u}(s_0 - k_1) &= u(s_0) - k_1h + O(k_1^2), \\
t(s_0 + k) &= t(s_0 - k_1).
\end{align*}$$

Let $w = \tilde{u}(s_0 + k) - \tilde{u}(s_0 - k_1)$. We expand

$$\begin{align*}
\Delta &= \tilde{I}_{(s_0+k)}(\tilde{u}(s_0 + k)) - \tilde{I}_{(s_0-k_1)}(\tilde{u}(s_0 - k_1)) \\
&= t(s_0 - k_1)^{n-2}(\tilde{I}_{(s_0+k)}(u(s_0 + k)) - \tilde{I}_{(s_0-k_1)}(u(s_0 - k_1))) = t(s_0 - k_1)^{n-2}\Delta, \\
\tilde{\Delta} &= \tilde{I}_{(s_0+k)}(\tilde{u}(s_0 + k)) - \tilde{I}_{(s_0-k_1)}(\tilde{u}(s_0 - k_1)) = \frac{1}{2}t''_{(s_0+k)}(u(s_0 - k_1)) \cdot w \cdot w \\
&+ \frac{1}{5}t'''_{(s_0+k)}(u(s_0 - k_1)) \cdot w \cdot w \cdot w + \frac{1}{4}t^{(4)}(u(s_0 - k_1))w \cdot w \cdot w + O(|w|^{5}_{H^0}).
\end{align*}$$

We know that

$$w = (k + k_1)h + O(k^2 + k_1^2) = (k + k_1)h + O((k + k_1)^2).$$

Thus,

$$\tilde{\Delta} = \frac{1}{2}t''_{(s_0+k)}(u(s_0 - k_1)) \cdot h \cdot h(k + k_1)^2 + \frac{1}{2}t''_{(s_0+k)}(u(s_0 - k_1)) \cdot h \cdot O((k + k_1)^2)(k + k_1)$$

$$+ O((k + k_1)^4) + \frac{1}{5}t'''_{(s_0+k)}(u(s_0 - k_1)) \cdot h \cdot h(k + k_1)^3$$

$$+ O((k + k_1)^4) + \frac{1}{2}t''_{(s_0+k)}(u(s_0 - k_1)) \cdot h \cdot h(k + k_1)^2 \left. \right|_{t = \tilde{t}(s_0)}$$

$$+ O((k + k_1)^3) + \frac{1}{5}t'''_{(s_0+k)}(u(s_0 - k_1)) \cdot h \cdot h(k + k_1)^3 + O((k + k_1)^4)$$

$$= \frac{1}{2}t''_{(s_0)}(u(s_0)) \cdot h \cdot h(k + k_1)^2 + \frac{1}{2}t''_{(s_0+k)}(u(s_0 - k_1)) \cdot h \cdot h(k + k_1)^2 \left. \right|_{t = \tilde{t}(s_0)}$$

$$+ O((k + k_1)^3) + \frac{1}{5}t'''_{(s_0+k)}(u(s_0 - k_1)) \cdot h \cdot h(k + k_1)^3 + O((k + k_1)^4).$$

On the other hand,

$$t(s_0 + k) = t(s_0) + \frac{1}{2}t''(s_0)k^2 + O(k^3), \quad t(s_0 - k_1) = t(s_0) + \frac{1}{2}t''(s_0)k_1^2 + O(k_1^3).$$
so that, since \( t(s_0 + k) = t(s_0) - k_1 \),

\[
k = k_1(1 + o(1)).
\]

Thus

\[
\tilde{\Delta} = -\frac{1}{12} \int_{t(s_0)}^{(3)}(u(s_0)) \cdot h \cdot h \cdot h(k + k_1)^3 + O((k + k_1)^4).
\]

We set \( t(s_0) = 1 \) so that

\[
\tilde{\Delta} = \frac{1}{12} \int g''(u(s_0))h^3(k + k_1)^3 + O((k + k_1)^4).
\]

Differentiating \((*)\), we derive (at \( s_0 \))

\[
\begin{cases}
-\Delta \dot{h} + g'(u)\dot{h} = g''(u)\dot{h}^2, \\
\dot{h} + r(\sigma)^2(t(s_0) \partial u \partial r)_{\sigma = 0} = \partial u \partial \sigma |_{\sigma = 0} = 0.
\end{cases}
\]

Thus,

\[
\int g''(u)\dot{h}^3 = \int_{\Omega}(-\Delta \dot{h} + g'(u)\dot{h})h = \int \nabla \dot{h} \nabla h - \int g'(u)\dot{h}^2
\]

\[
= \int_{\partial \Omega} \dot{h} \frac{\partial u}{\partial v} - \int_{\Omega} (\Delta h + g'(u)h) \dot{h} = \int_{\partial \Omega} \dot{h} \frac{\partial h}{\partial v}
\]

\[
= -\tilde{t}(s_0) \int_{\partial \Omega} \frac{\partial u}{\partial v} \frac{\partial \dot{h}}{\partial \sigma} r(\sigma) \mathrm{d}\sigma = -\tilde{t}(s_0) \int_{\partial \Omega} \frac{\partial u(s_0)}{\partial v} \frac{\partial \dot{h}}{\partial \sigma} \mathrm{d}\sigma.
\]

On the other hand, at every \( t \),

\[
\int_{\partial \Omega} \left| \frac{\partial u}{\partial v} \right|^2 x \cdot v \mathrm{d}\sigma = c_n \left( \int_{\Omega} g(u)u - \frac{n-2}{2n} G(u) \right).
\]

Differentiating and applying at \( s = s_0 \), we find \( \tilde{t}(s_0) = 0 \)

\[
2 \int_{\partial \Omega} \frac{\partial h}{\partial v} \frac{\partial u}{\partial v} x \cdot v \mathrm{d}\sigma = c_n \int_{\Omega} \left( \frac{n+2}{2n} g(u)h + g'(u)u \dot{h} \right) = \tilde{c}_n \int_{\Omega} g(u)h.
\]

Thus, at \( s_0 \),

\[
\int g''(u)\dot{h}^3 = -\frac{\tilde{c}_n}{2} \tilde{t}(s_0) \int_{\Omega} g(u)h.
\]

We see that the sign of \( \int g''(u)\dot{h}^3 \) depends on \( \tilde{t}(s_0) \) and on \( \int_{\Omega} g(u)h \). Thus, the change of the Morse index at the crossing of \( t(s_0) \) depends on the convexity of \( t(s) \) and on the sign of \( \int_{\Omega} g(u)h \). This is directly related to the rotation of \( \tilde{I}(s) + i \int_{\Omega} g(u)h \), which in turn relates directly to \( \dot{I} + i \int G \), hence to \( \int |\nabla u|^2 + i \int G \). Theorem 1 follows. \( \square \)
References
